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A Hybrid Approach of the Variational Iteration Method and Adomian Decomposition Method for Solving Fractional Integro-Differential Equations

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Abstract

In this study, we propose a hybrid analytical technique that integrates the Adomian Decomposition Method (ADM) with the Variational Iteration Method (VIM) to solve both linear and nonlinear integro-differential equations of integer and fractional orders. This approach extends and refines the Odibat Decomposition Method (ODM) by addressing key limitations inherent in ADM and VIM—specifically, the reliance on linearization, Adomian polynomials, and Lagrange multipliers. By circumventing these computational complexities, the proposed method enables the direct and efficient construction of series solutions with improved convergence properties. The hybrid scheme is designed for broader applicability and enhanced computational simplicity, making it a powerful tool for analyzing complex integro-differential systems. Its effectiveness and robustness are demonstrated through a range of illustrative examples, confirming the method's capability to provide accurate analytical approximations with minimal computational overhead.

Keywords: Integro differential equations; Fractional calculus; Adomian Decomposition Method; Variational Iteration Method; Series solution; Analytical methods; Nonlinear equations.

Introduction

Ordinary differential equations (ODEs), fractional differential equations, fractional integral equations, and fractional integro-differential equations play vital roles in modeling complex systems across engineering and applied sciences. Over recent decades, significant progress has been made in the development of analytical and approximate methods for solving such equations. Notable techniques include the Adomian Decomposition Method (ADM) (Tate & Dinde, 2019; Odibat, 2019), the Variational Iteration Method (VIM) (Wazwaz, 2009), the Homotopy Perturbation Method (HPM) (Momani & Odibat, 2007), the Differential Transform Method (DTM) (Rashidi et al., 2020; Bhalekar & Daftardar-Gejji, 2012), and the Laplace Decomposition Method (Yang, 2013).

Despite their utility, these methods face certain limitations. For example, the ADM requires the computation of Adomian polynomials for nonlinear terms (Hemeda, 2018), while the VIM depends on determining suitable Lagrange multipliers, which can lead to poor convergence due to integration by parts (Ziane & Cherif, 2018). To address these challenges, we propose a modified hybrid method that avoids both linearization and the use of Adomian polynomials and Lagrange multipliers. Furthermore, this method is extended to fractional-order equations and examined in the limit as the fractional order approaches an integer.

In this seminar, we present and analyze this hybrid technique for solving both linear and nonlinear fractional integro-differential equations of various forms.

$$D^{\alpha}u(x) = p(t)u(x) + g(x) + \lambda \int_{0}^{x} (x,t)F(u(t))dt \quad \ldots (1)$$

and

$$D^{\alpha}u(x) = p(t)u(x) + g(x) + \lambda \int_{0}^{1} (x,t)F(u(t))dt \quad \ldots (2)$$

for $x \epsilon [0,1]$, with the initial conditions

$u^i = \delta_i$, $\quad i = 0, 1, 2, \ldots$

Where g , p , and k are known functions, $u(x)$ is the unknown function, D^{α} is the Caputo fractional differential operator of order α Eqn. (1) and (2) arise in the mathematical modeling of various physical phenomena such as heat conduction in materials, conduction, convection and radiation problems (Caputo, 1967)

Basic Definition and Properties

Definition 1

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). e.g $D^{2.5}, D^{\pi}, D^{i+1}, D^{\frac{1}{2}}$, $J^{1.5}, J^{\pi}, J^{i+2}, J^{\frac{1}{2}}$ e.t.c

Definition 2

Riemann-Liouville fractional integration of order α is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (x-t)^{\alpha-1}\, dt, \qquad \alpha > 0 , \quad x > 0 \qquad \ldots (3)$$

Definition 3

Riemann – Liouville fractional derivative denoted D^{α} is defined as

$$D^{\alpha} f(x) = \frac{d^m}{dx^m} (I^{m-\alpha} f(x)) \qquad \ldots (4)$$

Definition 4

The Caputo Factional Derivative of order α is defined as

$$D^{\alpha} f(x) = I^{m-\alpha} \left(\frac{d^m}{dx^m} f(x) \right) \qquad \ldots (5)$$

Where m is a positive integer with the property that $m - 1 < \alpha < m$.

Definition 5

In this work, we will define the absolute error as:

Absolute Error= $|u(x) - u_n(x)|: 0 \leq x \leq 1.$

Where $u(x)$ is the exact solution and $u_n(x)$ will be the approximate solution from the method

Hence, we have the following properties:

1. $J^\alpha J^v f = J^{\alpha+v} f, \alpha, v > 0, f \in C_\mu, \mu > 0$
2. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}\ ,\ \alpha > 0\ , \gamma > -1\ ,\ x > 0$
3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0) \frac{x^k}{k!}\ ,\ x > 0\ ,\ m - 1 < \alpha \leq m$
4. $D^\alpha J^\alpha f(x) = f(x),\ x > 0\ ,\ m - 1 < \alpha \leq m$
5. $D^\alpha C = 0,\ where\ C\ is\ a\ constant.$
6. $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}\ ,\ \beta \epsilon N_0\ ,\ \beta \geq \alpha$

Where α is an integer and N_0 are natural numbers

7. The n-fold integral formula

$$\int_a^{x_1} \int_a^{x_2} \dots \int_a^{x_n} u(x_n) dx_n \dots dx_2 dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} u(t)\, dt \qquad \dots(6)$$

Analysis of the Hybrid Method

The Hybrid method for the solution of Nonlinear Fractional Volterra Integro-Differential Equation

To illustrate the hybrid concepts, we consider Eq. (1) with given initial conditions. According to the hybrid method, we first construct the correction functional owing to the VIM and ODM as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}(D^\alpha u(x) - p(t)u(x) - g(x) - \lambda\int_0^t (x,r)F(u(r))dr\right]dt \quad \ldots(7)$$

Now, we rewrite the correctional function in Adomian recursive relations as follow:

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda\int_0^r (x,r)F(u_k(r))dr\right)\right]dt \quad \ldots(8)$$

$n \geq 0$.

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots(9)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\left(D^\alpha[u_0(x)] - p(t)u_0(x) - g(t) - \lambda\int_0^t (x,r)F(u_0(r))dr\right)\right]dt \quad \ldots(10)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \dots (11)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \dots (12)$$

…

$$u_n(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{n-1} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \dots (13)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \dots (14)$$

The Hybrid method for the solution of Nonlinear Fractional Fredholm Integro-Differential Equation

Secondly, we consider the nonlinear fractional integro-differential equation (2). According to the hybrid method, we construct the correction functional owing to the ODM void of linearization and Adomian's polynomials as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}(D^\alpha u(x) - p(t)u(x) - g(x) - \lambda\int_0^1 (x,t)F(u(t))dt\right]dt \qquad \dots(15)$$

Now, we rewrite the correctional function in Adomian recursive relations as follow:

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda\int_0^1 (x,t)F(u_k(t))dt\right)\right]dt \qquad \dots(16)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \qquad \dots(17)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\left(D^\alpha[u_0(x)] - p(t)u_0(x) - g(t) - \lambda\int_0^1 (x,t)F(u_0(t))dt\right)\right]dt \qquad \dots(18)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \dots (19)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \dots (20)$$

…

$$u_n(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{n-1} \left(D^\alpha[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \dots (21)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \dots (22)$$

The Hybrid method for the solution of Linear Fractional Volterra Integro-Differential Equation

To illustrate the hybrid concepts on linear fractional Volterra Integro-Differential Equation, we consider Eq. (1) with $F(u(t)) = u(t)$, and with given initial conditions. According to the hybrid method, we first construct the correction functional by applying the integral operator I^α to both sides of Eq. (1) owing to the VIM and ODM as follows:

By properties of fractional integral and derivatives we have:

$$D^{\alpha} f(x) = I^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right) \quad \dots (23)$$

$$I^{\alpha} D^{\alpha} u(x) = u(x) - \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \dots (24)$$

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda \int_0^x (x,t) \sum_{k=0}^{m-1} u_k(t)\, dt\right] \quad \dots (25)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

Is obtained from the given initial condition

From the above equation the iterate are determined by the following recursive way

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \dots (26)$$

$$u_{n+1}(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda \int_0^x (x,t) \sum_{k=0}^{m-1} u_k(t)\, dt\right] \quad \dots (27)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_1(x) = I^\alpha\left[p(t)u_0(t) + g(x) + \lambda\int_0^x (x,t)u_0(t)dt\right] \quad \dots (28)$$

$$u_2(x) = I^\alpha\left[p(t)\sum_{k=0}^{1} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{1} u_k(t)\,dt\right] \quad \dots (29)$$

$$u_3(x) = I^\alpha\left[p(t)\sum_{k=0}^{2} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{2} u_k(t)\,dt\right] \quad \dots (30)$$

$\dots$

$$u_n(x) = I^\alpha\left[p(t)\sum_{k=0}^{n-1} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{n-1} u_k(t)\,dt\right] \quad \dots (31)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \dots (32)$$

The Hybrid method for the solution of linear Fractional Fredholm Integro-Differential Equation

Lastly, we consider the linear fractional integro-differential equation where $F(u(t))$ in Eqn. (2) becomes $u(t)$. According to the hybrid method, we apply I^{α} to both sides of the Eqn. (2), we obtain

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \dots(33)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

Is obtained from the given initial condition

From the above equation the iterate are determined by the following recursive way

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_{n+1}(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \dots(34)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_1(x) = I^{\alpha}\left[p(t)u_0(t) + g(x) + \lambda\int_0^1 (x,t)u_0(t)dt\right] \quad \ldots(35)$$

$$u_2(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{1} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{1} u_k(t)\,dt\right] \quad \ldots(36)$$

$$u_3(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{2} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{2} u_k(t)\,dt\right] \quad \ldots(37)$$

$\ldots$

$$u_n(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{n-1} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{n-1} u_k(t)\,dt\right] \quad \ldots(38)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \ldots(39)$$

RESULTS AND DISCUSSION

Linear and Nonlinear FFIDE.

In order to show the effectiveness of the Hybrid method for solving integro-differential equations of fractional order, in this section, we consider linear and nonlinear first order FFIDE.

Example 1: Here, we consider the linear fractional FIDE (Wang et al., 2021).

$$D^{\alpha}u(x) = 1 - \frac{x}{3} + \int_0^1 x\, t u(t) dt, \quad x, \alpha \epsilon [0,1]. \quad \dots (40)$$

With exact solution

$$u(x) = x$$

In view of Eqn. (33), the Eqn. (40) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + I^{\alpha}\left[1 - \frac{x}{3} + \int_0^1 xt \sum_{k=0}^{m-1} u_k(t)\, dt\right] \quad \dots (41)$$

Now, we rewrite Eqn. (41) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} = 0 + I^{\alpha}[1] = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \quad \dots (42)$$

$$u_{n+1}(x) = I^{\alpha}\left[-\frac{x}{3} + \int_0^1 xt \sum_{k=0}^{m-1} u_k(t)\, dt\right] \quad \dots (43)$$

$n \geq 0.$

$$u_1(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xtu_0(t)dt\right] = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)dt\right]$$

$$= 0 \qquad \dots (44)$$

$$= I^\alpha\left[-\frac{x}{3} + \frac{x}{(\alpha+2)\Gamma(1+\alpha)}\right] = 0$$

$$u_2(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{1} u_k(t)\,dt\right]$$

$$= 0 \qquad \dots (45)$$

$$u_3(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{2} u_k(t)\,dt\right]$$

$$= 0 \qquad \dots (46)$$

…

$$u_{n+1}(x) = 0\ , \qquad n > 0$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{x^\alpha}{\Gamma(1+\alpha)} + 0 + 0 + \cdots = \frac{x^\alpha}{\Gamma(1+\alpha)} \qquad \dots (47)$$

TABLE 1: The numerical results for various α (Example 1)

X	EXACT	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0	0	0	0	0	0
0.1	0.1	0.1	0.13089729	0.170165429	0.219588076
0.2	0.2	0.2	0.24426298	0.296275221	0.356721882
0.3	0.3	0.3	0.351835603	0.409796587	0.473798446

X	EXACT	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0.4	0.4	0.4	0.455810839	0.515845121	0.579496408
0.5	0.5	0.5	0.557190444	0.616662213	0.677466395
0.6	0.6	0.6	0.656548451	0.7134973	0.769687847
0.7	0.7	0.7	0.754256205	0.807141664	0.857387963
0.8	0.8	0.8	0.850573101	0.89813852	0.941394691
0.9	0.9	0.9	0.945690256	0.986882258	1.022300353
1	1	1	1.039754134	1.073671274	1.100547405

Table 1 shows the numerical solution for various values of α. It is observed that the results obtained for α=1 coincides with the exact solution, whereas for other values of α, the approximate solution is in good agreement with the exact solution

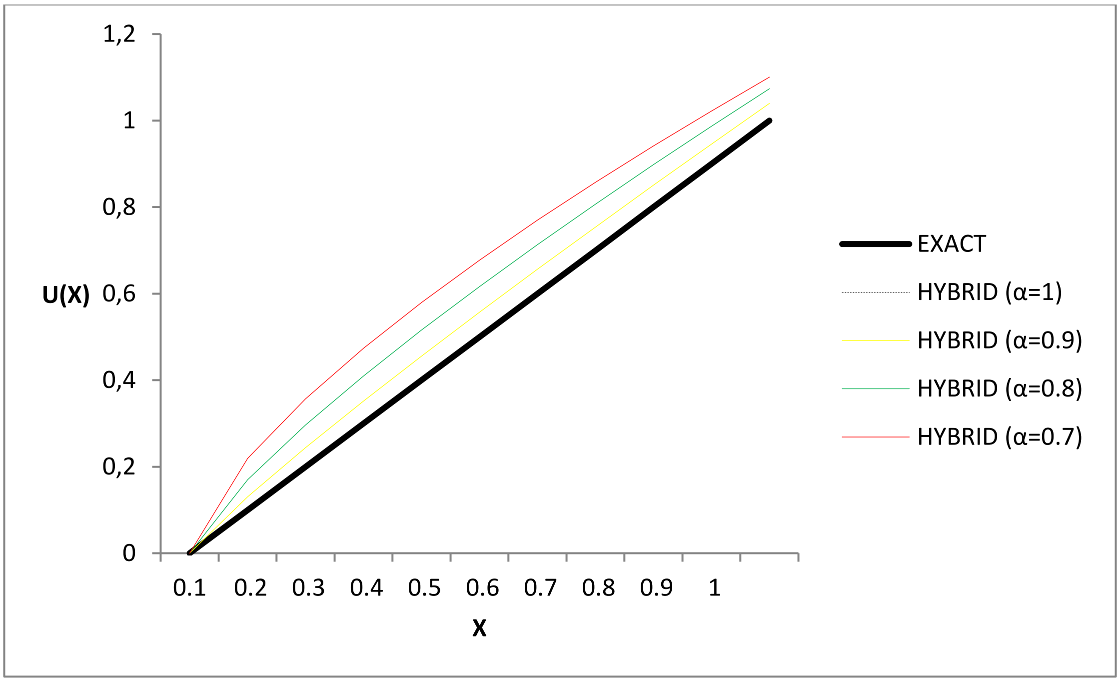

FIGURE 1: The plots of approximate solutions for fractional order α corresponding to example 1 are shown. From this figure, it is obvious that when α=1, the exact and approximate solution coincide.

Example 2: Here, we consider the nonlinear fractional FIDE. (Wang et al., 2021) and (Okai J. O, 2020)

$$D^{\alpha}u(x) = 1 - \frac{x}{4} + \int_0^1 x\,t[u(t)]^2 dt \qquad \ldots (48)$$

For $0 < \alpha \leq 1$ and with initial condition

$u(0) = 0.$

The exact solution

$u(x) = x$

In view of Eqn. (33), the Eqn. (48) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{\alpha}\left[1 - \frac{x}{4} + \int_0^1 xt \sum_{k=0}^{m-1} [u_k(t)]^2 dt\right] \qquad \ldots (49)$$

Now, we rewrite Eqn. (49) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^{\alpha}[1] = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \qquad \ldots (50)$$

$$u_{n+1}(x) = I^{\alpha}\left[-\frac{x}{4} + \int_0^1 xt \sum_{k=0}^{m-1} [u_k(t)]^2 dt\right] \qquad \ldots (51)$$

$n \geq 0.$

$$u_1(x) = I^{\alpha}\left[-\frac{x}{3} + \int_0^1 xt[u_0(t)]^2 dt\right]$$
$$= 0 \qquad \text{... (52)}$$

$$u_2(x) = I^{\alpha}\left[-\frac{x}{3} + \int_0^1 xt \sum_{k=0}^{1}[u_k(t)]^2 dt\right]$$
$$= 0 \qquad \text{... (53)}$$

$$u_3(x) = I^{\alpha}\left[-\frac{x}{3} + \int_0^1 xt \sum_{k=0}^{2}[u_k(t)]^2 dt\right]$$
$$= 0 \qquad \text{... (54)}$$

…

$$u_{n+1}(x) = 0\ , \qquad n > 0$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{x^{\alpha}}{\Gamma(1+\alpha)} + 0 + 0 + \cdots = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \qquad \text{... (55)}$$

TABLE 2: The numerical results for various α (Example 2)

X	EXACT	HYBRID (α=1)	HYBRID (α=0.3)	HYBRID (α=0.6)	HYBRID (α=0.9)
0	0	0	0	0	0
0.1	0.1	0.1	0.558444121	0.281124038	0.13089729
0.2	0.2	0.2	0.687525359	0.426104362	0.24426298
0.3	0.3	0.3	0.776454658	0.543463943	0.351835603
0.4	0.4	0.4	0.846443005	0.64585344	0.455810839
0.5	0.5	0.5	0.905046148	0.738380103	0.557190444
0.6	0.6	0.6	0.955927814	0.823737302	0.656548451
0.7	0.7	0.7	1.001173014	0.903559604	0.754256205

X	EXACT	HYBRID (α=1)	HYBRID (α=0.3)	HYBRID (α=0.6)	HYBRID (α=0.9)
0.8	0.8	0.8	1.042093577	0.978930759	0.850573101
0.9	0.9	0.9	1.079574147	1.050614737	0.945690256
1	1	1	1.114242509	1.119174954	1.039754134

The results for different values of *a* are depicted in Table 2. It is also observed the method produced the exact solution for the fractional order α=1, whereas the accuracy is decreased with decreasing the value of *a*.

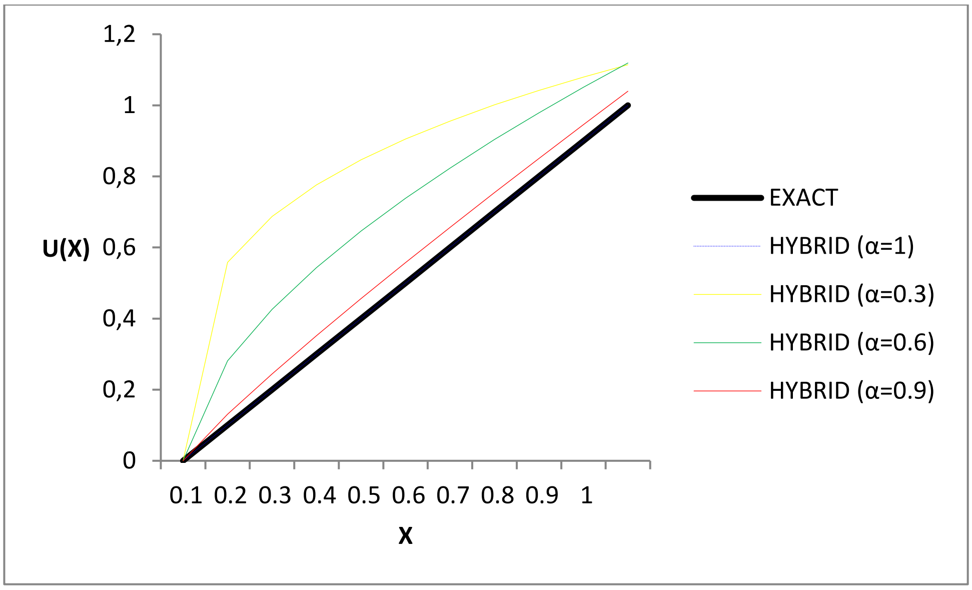

FIGURE 2: The plots of approximate solutions for fractional order α corresponding to example 2 are shown. From this figure, it is obvious that when α=1, the exact and approximate solution coincides.

TABLE 3: Comparison between the exact solution and solution obtain through the Hybrid method, LDM, LT and the ADM for Example 2 (α=1)

X	EXACT	HYBRID	LDM (Yang, 2013)	LT (Wang et al.,2021)	ADM (Okai J. O, 2020)
0	0	0	0	0	0
0.1	0.1	0.1	0.09875	0.0999	0.0996875
0.2	0.2	0.2	0.19500	0.1998	0.1987500
0.3	0.3	0.3	0.28875	0.2997	0.2971875
0.4	0.4	0.4	0.38000	0.3996	0.3950000

X	EXACT	HYBRID	LDM (Yang, 2013)	LT (Wang et al.,2021)	ADM (Okai J. O, 2020)
0.5	0.5	0.5	0.46875	0.4995	0.4921875
0.6	0.6	0.6	0.55500	0.5994	0.5887500
0.7	0.7	0.7	0.63875	0.6993	0.6846875
0.8	0.8	0.8	0.72000	0.7992	0.7800000
0.9	0.9	0.9	0.79875	0.8991	0.8746875
1	1	1	0.8750	0.999	0.9687500

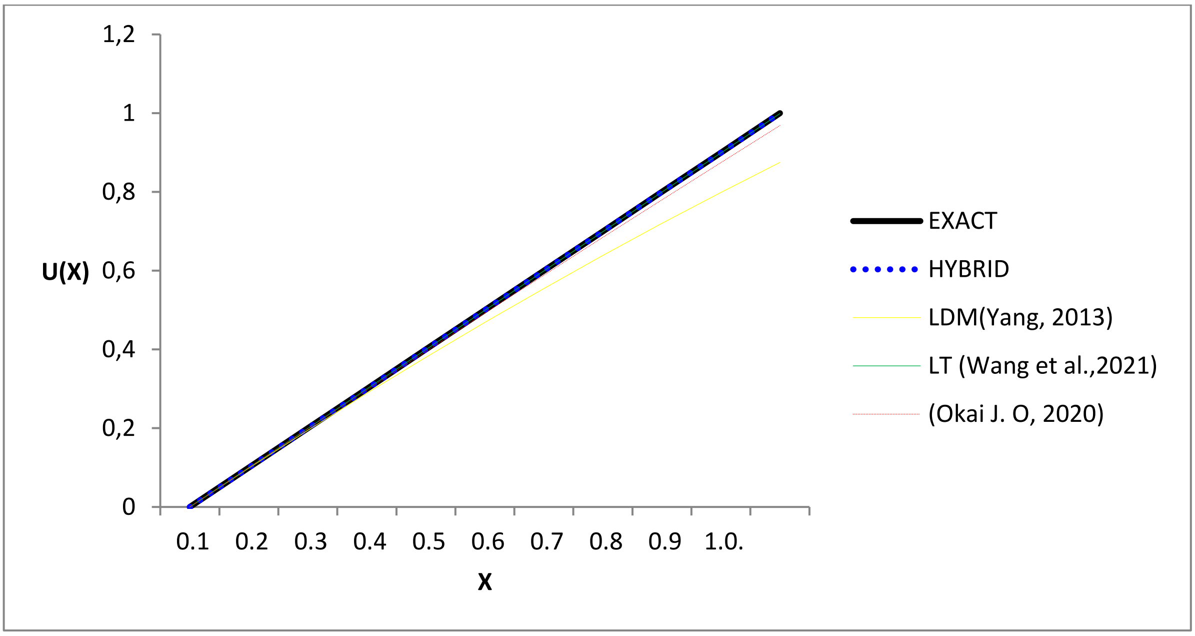

FIGURE 3: The plots of approximate solutions for fractional order $\alpha=1$ corresponding to example 2 are shown for the Hybrid method, LDM, LT, ADM and the exact solution. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

Linear and Nonlinear FVIDE.

In order to show the effectiveness of the Hybrid method for solving integro-differential equations of fractional order, in this section, we consider linear and nonlinear first order FVIDE.

Example 3: Here, we consider the linear fractional VIDE (Yang, 2013), (Ilejimi D.O, 2019) and (Rawashdeh, 2006):

$$D^{3/4}u(x) = \frac{6x^{9/4}}{\Gamma(13/4)} + \left(\frac{-x^2e^x}{5}\right)u(x) + \int_0^x e^x\, tu(t)dt \quad \dots (56)$$

With the initial condition

$$u(0) = 0 \quad \dots (57)$$

and the exact solution is $u(x) = x^3$.

In view of Eqn. (25), the Eqn. (56) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{3/4}\left[\frac{6x^{9/4}}{\Gamma(13/4)} + \left(\frac{-x^2e^x}{5}\right)u(x) + \int_0^x e^x\, tu(t)dt\right] \quad \dots (58)$$

Now, we rewrite Eqn. (58) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^{3/4}\left[\frac{6x^{9/4}}{\Gamma(13/4)}\right] = x^3 \quad \dots (59)$$

$$u_{n+1}(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)\sum_{k=0}^{m-1} u_k(x) + \int_0^x e^x t \sum_{k=0}^{m-1} u_k(t)\, dt\right] \quad \dots (60)$$

$n \geq 0$.

$$u_1(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)u_0(x) + \int_0^x e^x t u_0(x)dt\right]$$

$$= 0 \quad \dots (61)$$

$$u_2(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)\sum_{k=0}^{1} u_k(x) + \int_0^x e^x t \sum_{k=0}^{1} u_k(t)\,dt\right]$$

$$= 0 \quad \dots (62)$$

$u_{n+1}(x) = 0\ , \qquad n > 1$

So that the analytic solution is

$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$

$u(x) \cong$

$x^3 \quad \dots (63)$

TABLE 4: Comparison between the exact solution and solution obtain through the Hybrid method, Collocation method and the Picards method for Example 3 (α=$^3/_4$)

X	EXACT	HYBRID	Collocation method	Picard's
0	0	0	0	0
0.1	0.001	0.001	0.0008	0.00092
0.2	0.008	0.008	0.0064	0.00736
0.3	0.027	0.027	0.0216	0.02484
0.4	0.064	0.064	0.0512	0.05888
0.5	0.125	0.125	0.1	0.115
0.6	0.216	0.216	0.1728	0.19872
0.7	0.343	0.343	0.2744	0.31556
0.8	0.512	0.512	0.4096	0.47104
0.9	0.729	0.729	0.5832	0.67068
1	1	1	0.8	0.92

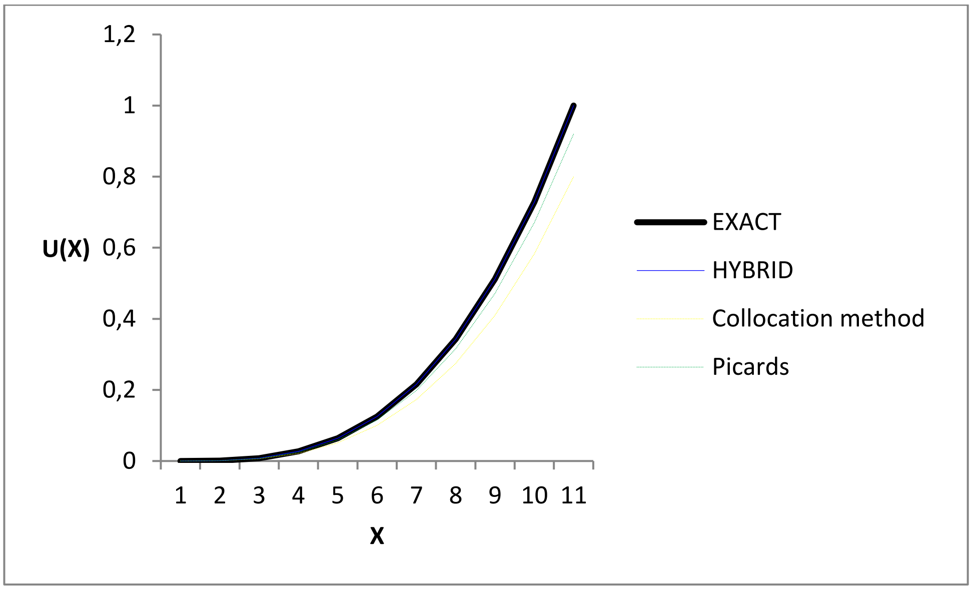

FIGURE 4: The plots of approximate solutions the Hybrid method, Collocation method, the Picards method and the exact solution for Example 3 are shown. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

Example 4: Here, we consider the nonlinear FVIDE (Yang, 2013) and (Ilejimi D.O, 2019)

$$D^{6/5}u(x) = \frac{5x^{4/5}}{2\Gamma(4/5)} - \frac{x^9}{252} + \int_0^x (x-t)^2\,[u(t)]^3 dt \qquad \dots(64)$$

With the initial condition

$$u(0) = u'(0) = 0 \qquad \dots(65)$$

and the exact solution is $u(x) = x^2$.

In view of Eqn. (25), the Eqn. (64) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{6/5}\left[\frac{5x^{4/5}}{2\Gamma(4/5)} - \frac{x^9}{252} + \int_0^x (x-t)^2\,[u(t)]^3 dt\right] \qquad \dots(66)$$

Now, we rewrite Eqn. (66) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^{6/5}\left[\frac{5x^{4/5}}{2\Gamma(4/5)}\right]$$

$$= x^2 \qquad \dots (67)$$

$$u_{n+1}(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x-t)^2 \sum_{k=0}^{m-1}[u_k(t)]^3\,dt\right] \qquad \dots (68)$$

$n \geq 0.$

$$u_1(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x-t)^2[u_0(t)]^3\,dt\right]$$

$$= 0 \qquad \dots (69)$$

$$u_2(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x-t)^2 \sum_{k=0}^{1}[u_k(t)]^3\,dt\right]$$

$$= 0 \qquad \dots (70)$$

$u_{n+1}(x) = 0\ , \qquad n > 1$

So that the analytic solution is

$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$

$u(x) \cong$

$x^2 \qquad \dots (71)$

TABLE 5: Comparison between the exact solution and solution obtain through the Hybrid method, Collocation method and the Picards method for Example 4 ($\alpha = 6/5$)

X	EXACT	HYBRID	COLLOCATION METHOD	Picard's method
0	0	0	0	0
0.1	0.001	0.001	0.0099	0.0093
0.2	0.008	0.008	0.0396	0.0372
0.3	0.027	0.027	0.0891	0.0837

X	EXACT	HYBRID	COLLOCATION METHOD	Picard's method
0.4	0.064	0.064	0.1584	0.1488
0.5	0.125	0.125	0.2475	0.2325
0.6	0.216	0.216	0.3564	0.3348
0.7	0.343	0.343	0.4851	0.4557
0.8	0.512	0.512	0.6336	0.5952
0.9	0.729	0.729	0.8019	0.7533
1	1	1	0.99	0.93

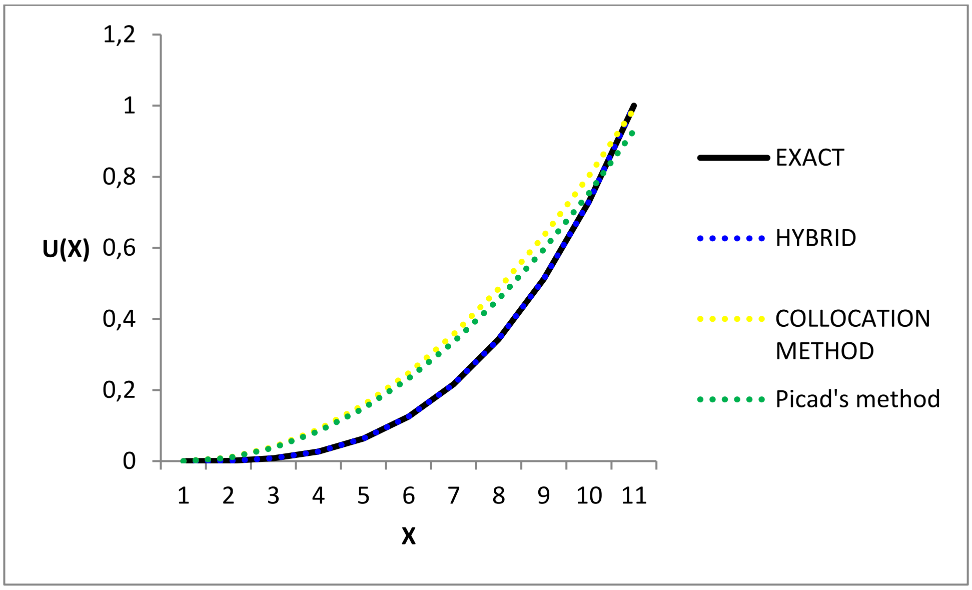

FIGURE 5: The plots of approximate solutions the Hybrid method, Collocation method, the Picards method and the exact solution for Example 4 are shown. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

Discussion

Based on the numerical analysis performed, we observed that the Hybrid method converges well with the exact solution when **α=1 (Example** 1 and 2, see Table 1 and 2), the comparison shows that as $\alpha \rightarrow 1$, the approximate solution tends to x, which is the exact solution of the two equations. The computational procedures is easy to understand and gives a rapidly convergence series solution when compared with LDM, LT, ADM. In Example 3 and 4 the hybrid yields the exact solution in just two iterations (See Table 4 and 5).

Conclusions

From the present work, the following conclusions may be drawn:

1. In these examples, we solved linear fractional integro-differential equations, the Hybrid method gives more accurate results than other methods studied.
2. The work is completely void of linearization.
3. In solving nonlinear fractional integro-differential equations, the Hybrid method gives more accurate results than the collocation method and Adomian decomposition method.
4. From the given nonlinear Examples, it is obvious that the work was carried out without the Adomian's polynomials and the Langrage's multiplier.
5. The results obtained are in good agreement with that of exact solution, and therefore as α approach the integer order under consideration the Hybrid method tends to the exact solution, the Hybrid method described is precise and accurate.
6. The solved problems in this work are all solved for the closed interval [0,1] and with fractional order of integration and differentiation.

Also, from this work, we can recommend the following open problems for future work:

1. Studying the behaviour and solution of real life problems, which may be modeled as fractional integro-differential equations.
2. Studying fractional integro-differential equations resulting from other types of fractional integro-differential operators, linear or nonlinear.
3. Study the singular fractional integro-differential equations with the aid of the Hybrid method.
4. Using the Hybrid method for solving other equations such as the heat equations (PDE).

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