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A ONE-STEP MODIFIED NEW ITERATIVE METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATION

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Abstract

This study introduces a reliable semi-analytical approach for solving partial differential equations (PDEs) using a Modified New Iterative Method (MNIM). The primary aim is to enhance the efficiency of deriving closed-form solutions through an innovative formulation of an integral operator based on n-fold integration. This approach circumvents the conventional necessity of transforming PDEs into systems of multiple integral equations, thereby streamlining the solution process. The effectiveness of the MNIM is assessed through a series of examples, demonstrating its rapid convergence and superior performance in solving an array of evolution and partial differential equations. The results indicate that the MNIM not only simplifies the solution process but also significantly improves computational efficiency compared to traditional methods. This contribution holds substantial implications for both theoretical advancements in numerical analysis and practical applications across various fields where PDEs are prevalent, thereby facilitating more effective problem-solving strategies in complex systems.

Keywords: Modified New Iterative Method (MNIM); Partial Differential Equations (PDEs); Integral Operator; Numerical Analysis; Computational Efficiency.

INTRODUCTION

Nonlinear partial differential equations (NPDEs) play a crucial role in various fields such as physics, chemistry, biology, mathematics, and engineering. They are essential for modeling complex phenomena like fluid dynamics and heat transfer. However, solving NPDEs in real-world contexts is challenging due to their complexity and nonlinearity. Often, to make these equations more manageable, simplifying assumptions are made, which can compromise the accuracy and reliability of solutions—critical factors when applying these solutions to practical problems.

There are various numerical methods available to tackle NPDEs, but each has its limitations. Traditional techniques like finite difference and finite element methods involve discretizing the domain, which can lead to errors and instability. Consequently, the pursuit of new techniques for solving NPDEs remains an important area of research across multiple disciplines (Ala'yed et al., 2023; Batiha & Batiha, 2011; B. Batiha, 2009).

Nonlinear evolution equations describe a broad spectrum of physical, chemical, and biological processes. Finding exact solutions, when possible, is essential for validating numerical methods and analyzing solution stability. Analytical solutions are particularly valuable in nonlinear science, offering insights that can inform further applications.

The complexity of evolution equations often requires iterative methods, such as finite difference or finite element approaches, which break the problem into smaller parts solved iteratively. Recognizing the symmetries within these equations can further enhance the development of efficient iterative methods (Fang et al., 2022; Burgers et al., 2023; Bhalekar & Daftardar-Gejji, 2010).

Numerous techniques have been devised to address NPDEs, including the pseudospectral method (Javidi, 2006), spectral collocation method (Javidi, 2006b), Adomian decomposition method (ADM) (Hashim et al., 2006), and homotopy perturbation method (HPM) (He, 2005). Recently, the new iterative method (NIM), introduced by Daftardar-Gejji and Jafari, has proven effective for solving a wide range of linear and nonlinear functional equations.

NIM is particularly user-friendly for computer implementation and has shown improved results compared to established methods like ADM and HPM.

In this work, we focus on the generalized evolution equation, which models system dynamics over time and is applicable in physics, biology, and finance. Understanding this equation is crucial for developing reliable predictive models. Our analysis aims to enhance the understanding of this equation and its solutions, contributing valuable insights to the field.

METHODOLOGY

Basic Idea of the New Iterative Method (NIM)

To elucidate the concept behind the initial strategy in the **New Iterative Method (NIM)**, one can refer to the subsequent general functional equation, drawing inspiration from works by Podlubny (1999), Raja *et al.* (2011), Ramadan& Al-luhaibi(2015), as well as Ashitha & Ranjini (2020).

Let us consider the nonlinear functional equation (Daftardar-gejji & Bhalekar (2010)

$$y(x) = g(x) + N[y(x)] \qquad ...(1)$$

Where N is the nonlinear operator and g is a known function. We are looking for y which has the series solution in the form

$$y = \sum_{i=0}^{\infty} y_i \qquad ...(2)$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad ...(3)$$

From Eqns. (2) and (3), Eqn. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = g + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad ...(4)$$

We define the recurrence relation:

$$\begin{cases} y_0 = g, \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + ... + y_m) - N(y_0 + ... + y_{m-1}), m = 1,2,... \end{cases} \quad ...(5)$$

Then

$$(y_1 + ... + y_{m+1}) = N(y_0 + ... + y_m), m = 1,2,... \quad ...(6)$$

and

$$y = g + \sum_{i=0}^{\infty} y_i \; . \quad ...(7)$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eqn. (1).

The One-Step New Iterative Method (OSNIM)

In this section we proposed a new scheme from the basic ideas of the NIM for solving differential equations:

$$L[u(t)] = N[u(t)] + g(t), \quad ...(8)$$

Where L and N are linear and nonlinear operators, respectively, $g(t)$ is the source inhomogeneous term.

We apply the inverse operator $L^{-1}(.) = \int_0^x \int_0^x (.)dtdt$ to the nonlinear problem presented in (8)and we have

$$u(x) = c_0 + c_1 x + \int_0^x \int_0^x g(t)dtdt + \int_0^x \int_0^x N[u(t)]dtdt \quad ...(9)$$

We reduce Eqn. (9) to single integral by using the n-fold integral formulae (Wazwaz, 2011); and also by assuming that $f(x) = c_0 + c_1 x + \int_0^x \int_0^x g(t)dtdt$, we get the following form:

$$u(x) = f(x) + \int_0^x (s-t)N[u(s)]ds \quad ...(10)$$

Where f is a known analytic function represents the sum of the available initial conditions and the result of integrating of the function g.

From Eqn. (10) as a general functional equation, in order to implement the method, we define the successive approximations as follows:

$$u(x) = c_0 + xc_1 + \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\mathrm{Lu(s)} - \mathrm{Nu(s)} - \mathrm{g(s)}\right)\right]ds \qquad ...(11)$$

Where

$$c_0 + xc_1$$

Is obtained from the given initial condition(s)

From this equation, the iterate are determined by the following recursive way

$$u_0(x) = c_0 + xc_1 \qquad ...(12)$$

$$u_1(x) = \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\mathrm{L}[u_0(s)] - N[u_0(s)] - g(s)\right]ds \qquad ...(13)$$

$$u_2(x) = \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\sum_{n=0}^{1}\mathrm{L}[u_n(s)] - \sum_{n=0}^{1}\mathrm{N}[u_n(s)] - g(s)\right]ds \qquad ...(14)$$

$$u_3(x) = \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\sum_{n=0}^{2}\mathrm{L}[u_n(s)] - \sum_{n=0}^{2}\mathrm{N}[u_n(s)] - g(s)\right]ds \qquad ...(15)$$

$$\vdots$$

$$u_{n+1}(x) = \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\mathrm{Lu_n(s)} - \mathrm{Nu_n(s)} - \mathrm{g(s)}\right)\right]ds \qquad ...(16)$$

$$n \geq 0$$

where q is determined by the order of the equation under consideration. Therefore, the solution for the relations Eqn. (8) will be obtained by

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) \qquad ...(17)$$

RESULTS

The present section examines the application of the proposed algorithms on evolution equation and other types of PDEs.

Example 1:

Consider the evolution equations as follows [(Bhalekar & Daftardar-Gejji, 2010)]

$$u_t(x,t) + u_{xxxx}(x,t) = 0, \qquad t > 0 \qquad ...(18)$$

With initial conditions

$$u(x,0) = sinx \qquad ...(19)$$

The correction functional for Eqn. (18) is

$$u_{n+1}(x,t) = c_0 + \frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial u_n(x,s)}{\partial s} + \frac{\partial^4 u_n(x,s)}{\partial x^4}\right)\right]ds \qquad ...(20)$$

Where

$$\frac{(-1)^q}{(q-1)!} = -1 \qquad ...(21)$$

And

$$c_0 = u(x,0) = sinx \qquad ...(22)$$

Considering the given initial values, we can select $u_0(x,0) = sinx$. Using this selection in Eqn. (20) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = sinx \\ u_1(x,t) = -\int_0^t (s-t)^0 \left[\frac{\partial u_0(x,s)}{\partial s} + \frac{\partial^4 u_0(x,s)}{\partial x^4} \right] ds = -tsinx \\ u_2(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{1} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{1} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = \frac{1}{2} t^2 sinx \\ u_3(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{2} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{2} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = -\frac{1}{6} t^3 sinx \\ u_4(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{3} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{3} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = \frac{1}{24} t^4 sinx \\ u_5(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{4} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{4} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = -\frac{1}{120} t^5 sinx \end{cases} \quad ...(23)$$

And so on. Thus，the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) + ...$$

$$u(x,t) = sinx - tsinx - \frac{1}{6} t^3 sinx + \frac{1}{24} t^4 sinx - \frac{1}{120} t^5 sinx \quad ...(24)$$

That leads to the exact solution

$$u(x,t) = e^{-t} sinx \quad ...(25)$$

which is in high agreement with the exact solution $e^{-t} sinx$ and matches with the VIM and NIM solution

Table 1: Approximate solution is compared with exact solution for *t* = 0.9

x	EXACT	OSNIM	ERROR
0	0	0	0
0.1	0.040589	0.040524	6.52E-05
0.2	0.080773	0.080643	0.00013
0.3	0.12015	0.119957	0.000193
0.4	0.158326	0.158071	0.000254
0.5	0.19492	0.194607	0.000313
0.6	0.229566	0.229198	0.000369
0.7	0.261919	0.261499	0.000421
0.8	0.291655	0.291187	0.000468
0.9	0.318477	0.317966	0.000511
1	0.342117	0.341567	0.000549

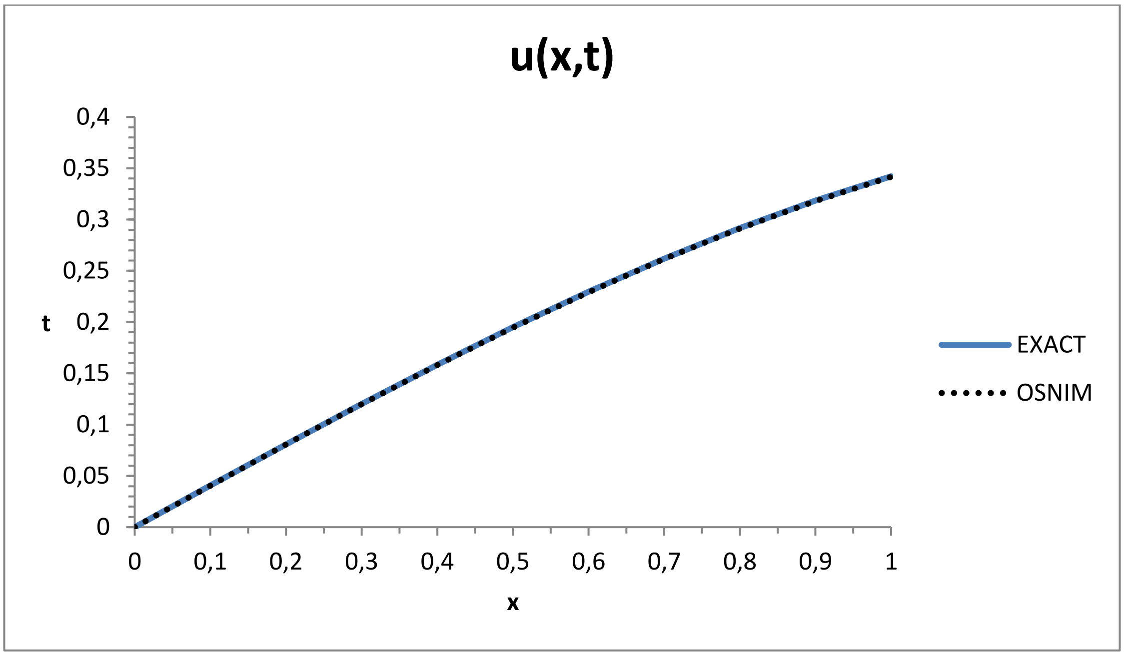


Figure 1: Approximate solutions obtained in comparison with the exact solution for Example 1

Table 2: Comparison of results between exact solution, OSNIM, NIM and VIM for Example 1($\boldsymbol{t = 0.9}$)

x	EXACT	OSNIM (n=5)	NIM (n=7)	VIM (n=6)
0	0	0	0	0
0.1	0.040589	0.040524056	0.038286115	0.04101531
0.2	0.080773	0.080643209	0.076189688	0.08162081
0.3	0.12015	0.119956602	0.113331999	0.12141078
0.4	0.158326	0.158071428	0.149341934	0.15998766
0.5	0.19492	0.194606856	0.183859694	0.19696599
0.6	0.229566	0.229197838	0.216540389	0.2319763
0.7	0.261919	0.26149875	0.247057483	0.26466878
0.8	0.291655	0.291186853	0.275106061	0.29471678
0.9	0.318477	0.317965513	0.30040587	0.32182007
1	0.342117	0.341567167	0.322704123	0.34570784

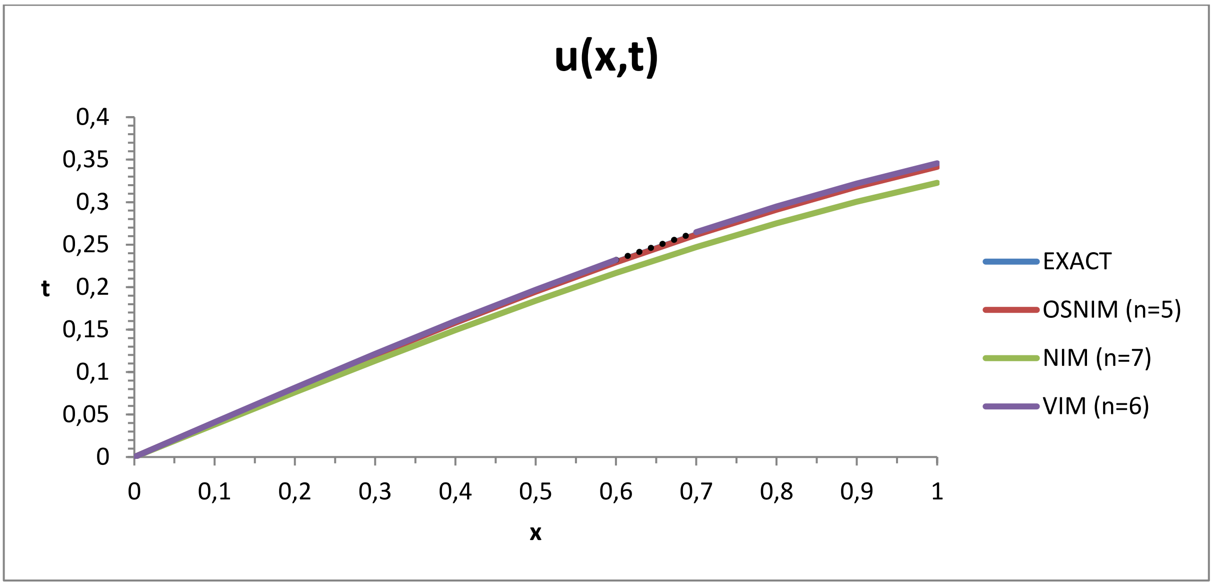


Figure 2: Approximate solutions obtained by the OSNIM in comparison with the exact solution, NIM and VIM for Example 1

Example 2:

We now consider the wave-like equation as follows: [(Bhalekar & Daftardar-Gejji, 2010)]

$$u_{tt}(x,t)=\frac{x^2}{2}u_{xx}(x,t),\ \ t>0 \qquad \text{...(26)}$$

With initial conditions

$$u(x,0)=0, \qquad u_t(x,0)=x^2 \qquad \text{...(27)}$$

The correction functional for Eqn. (26) is

$$u_{n+1}(x,t)=c_0+tc_1+\frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial^2 u_n(x,s)}{\partial s^2}-\frac{x^2}{2}\frac{\partial^2 u_n(x,s)}{\partial x^2}\right)\right]ds \qquad \text{...(28)}$$

Where

$$\frac{(-1)^q}{(q-1)!}=1 \qquad \text{...(29)}$$

And

$$c_0+tc_1=u(x,0)+u_t(x,0)=tx^2 \qquad \text{...(30)}$$

Considering the given initial values, we can select $u_0(x,0)=tx^2$. Using this selection in Eqn. (28) yields the following successive approximations:

$$\begin{cases} u_0(x,0)=tx^2 \\ u_1(x,t)=\int_0^t (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2}-\frac{x^2}{2}\frac{\partial^2 u_0(x,s)}{\partial x^2}\right]ds=\frac{1}{6}t^3x^2 \\ u_2(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\frac{x^2}{2}\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial x^2}\right]ds=\frac{1}{120}t^5x^2 \\ u_3(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\frac{x^2}{2}\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial x^2}\right]ds=\frac{1}{5040}t^7x^2 \\ u_4(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{3}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\frac{x^2}{2}\sum_{n=0}^{3}\frac{\partial^2 u_n(x,s)}{\partial x^2}\right]ds=\frac{1}{362880}t^9x^2 \\ u_5(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{4}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\frac{x^2}{2}\sum_{n=0}^{4}\frac{\partial^2 u_n(x,s)}{\partial x^2}\right]ds=\frac{1}{39916800}t^{11}x^2 \end{cases} \quad ...(31)$$

And so on. Thus, the solution is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t)+...$$

$$u(x,t)=tx^2+\frac{1}{6}t^3x^2+\frac{1}{5040}t^7x^2+\frac{1}{362880}t^9x^2+\frac{1}{39916800}t^{11}x^2+... \quad ...(32)$$

That leads to the exact solution

$$u(x,t)=x^2 sinht \quad ...(33)$$

Table 3: Approximate solution is compared with exact solution for Example 2 ($t=1$)

x	EXACT	OSNIM	ERROR
0.0	0.000000	0.00000	0.00000
0.1	0.011752	0.011752	7.68E-15
0.2	0.047008	0.047008	3.07E-14
0.3	0.105768	0.105768	6.91E-14
0.4	0.188032	0.188032	1.23E-13

0.5	0.293800	0.293800	1.92E-13
0.6	0.423072	0.423072	2.76E-13
0.7	0.575849	0.575849	3.76E-13
0.8	0.752129	0.752129	4.91E-13
0.9	0.951913	0.951913	6.22E-13
1.0	1.175201	1.175201	7.67E-13

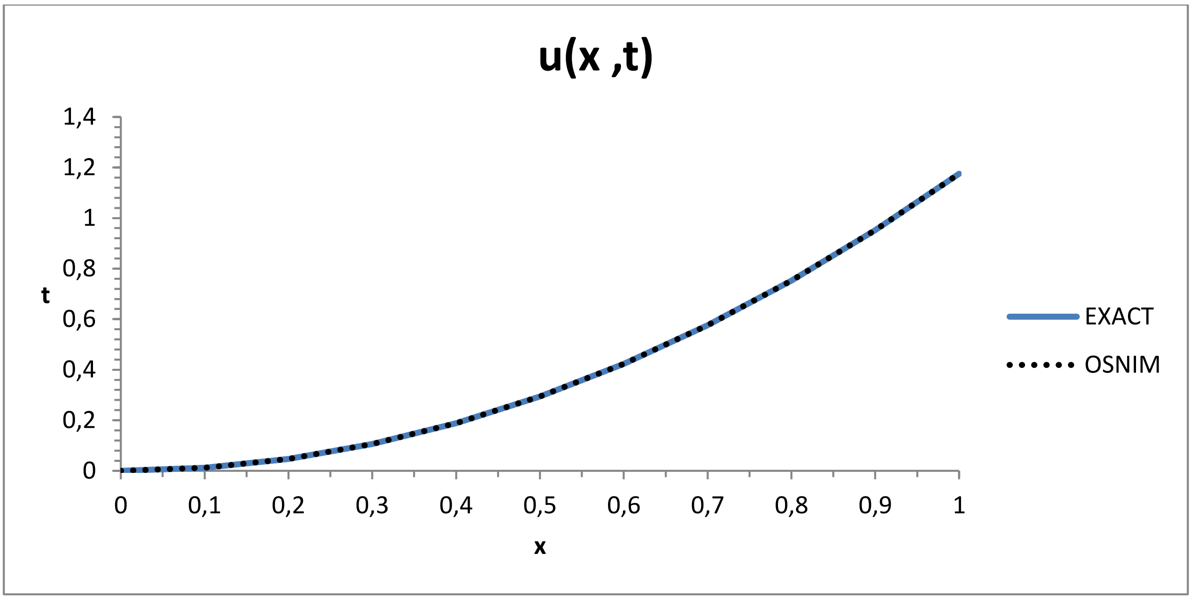
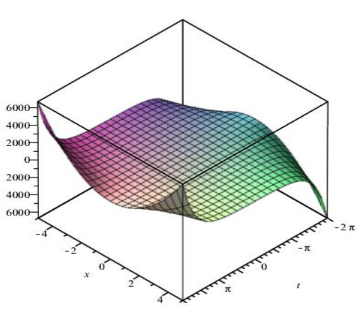


Figure 3: Approximate solutions obtained in comparison with the exact solution for Example 2

Figure 4: Surface plot for exact solution for Example 2



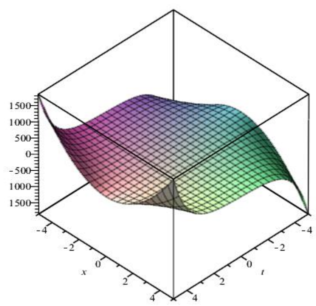


Figure 5: Surface plot for approximate solution for Example 2

Example 3:

Consider a linear Klein–Gordon equation as follows: [(Kumar, 2014)]

$$u_{tt}(x,t) - u_{xx}(x,t) = u(x,t) \quad , \qquad t > 0 \qquad ...(34)$$

With initial conditions

$$u(x,0) = 1 + sinx, \qquad u_t(x,0) = 0 \qquad ...(35)$$

The correction functional for Eqn. (34) is

$$u_{n+1}(x,t) = c_0 + tc_1 + \frac{(-1)^q}{(q-1)!}\int_0^x (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 u_n(x,s)}{\partial x^2} - u_n(x,s)\right)\right]ds \qquad ...(36)$$

Where

$$\frac{(-1)^q}{(q-1)!} = 1 \qquad ...(37)$$

And

$$c_0 + tc_1 = u(x,0) + u_t(x,0) = 1 + sinx \qquad ...(38)$$

Considering the given initial values, we can select $u_0(x,0) = 1 + sinx$. Using this selection in Eqn. (36) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = 1 + sinx \\ u_1(x,t) = \int_0^x (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2} - \frac{\partial^2 u_0(x,s)}{\partial x^2} - u_0(x,s)\right]ds = \frac{1}{2}t^2 \\ u_2(x,t) = \int_0^x (s-t)\left[\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial x^2} - \sum_{n=0}^{1} u_n(x,s)\right]ds = \frac{1}{24}t^4 \\ u_3(x,t) = \int_0^x (s-t)\left[\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial x^2} - \sum_{n=0}^{2} u_n(x,s)\right]ds = \frac{1}{720}t^6 \\ u_4(x,t) = \int_0^x (s-t)\left[\sum_{n=0}^{3}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{3}\frac{\partial^2 u_n(x,s)}{\partial x^2} - \sum_{n=0}^{3} u_n(x,s)\right]ds = \frac{1}{40320}t^8 \\ u_5(x,t) = \int_0^x (s-t)\left[\sum_{n=0}^{4}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{4}\frac{\partial^2 u_n(x,s)}{\partial x^2} - \sum_{n=0}^{3} u_n(x,s)\right]ds = \frac{1}{362880}t^{10} \end{cases} \qquad ...(39)$$

And so on. Thus, the solution is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t)+...$$

$$u(x,t)=1+\sin x+\frac{1}{2}t^2+\frac{1}{24}t^4+\frac{1}{720}t^6+\frac{1}{40320}t^8+\frac{1}{362880}t^{10}+... \quad ...(40)$$

That leads to the exact solution

$$u(x,t)=\sin x+\cosh t \quad ...(41)$$

Table 4: The absolute error of $u(x,t)$ for different values of x at $t=0.01$. for Example 3

X	EXACT	OSNIM	ERROR
0	1.00005	1.00005	2.22E-16
0.1	1.099883	1.099883	2.22E-16
0.2	1.198719	1.198719	2.22E-16
0.3	1.29557	1.29557	2.22E-16
0.4	1.389468	1.389468	2.22E-16
0.5	1.479476	1.479476	2.22E-16
0.6	1.564692	1.564692	2.22E-16
0.7	1.644268	1.644268	2.22E-16
0.8	1.717406	1.717406	2.22E-16
0.9	1.783377	1.783377	2.22E-16
1	1.841521	1.841521	2.22E-16

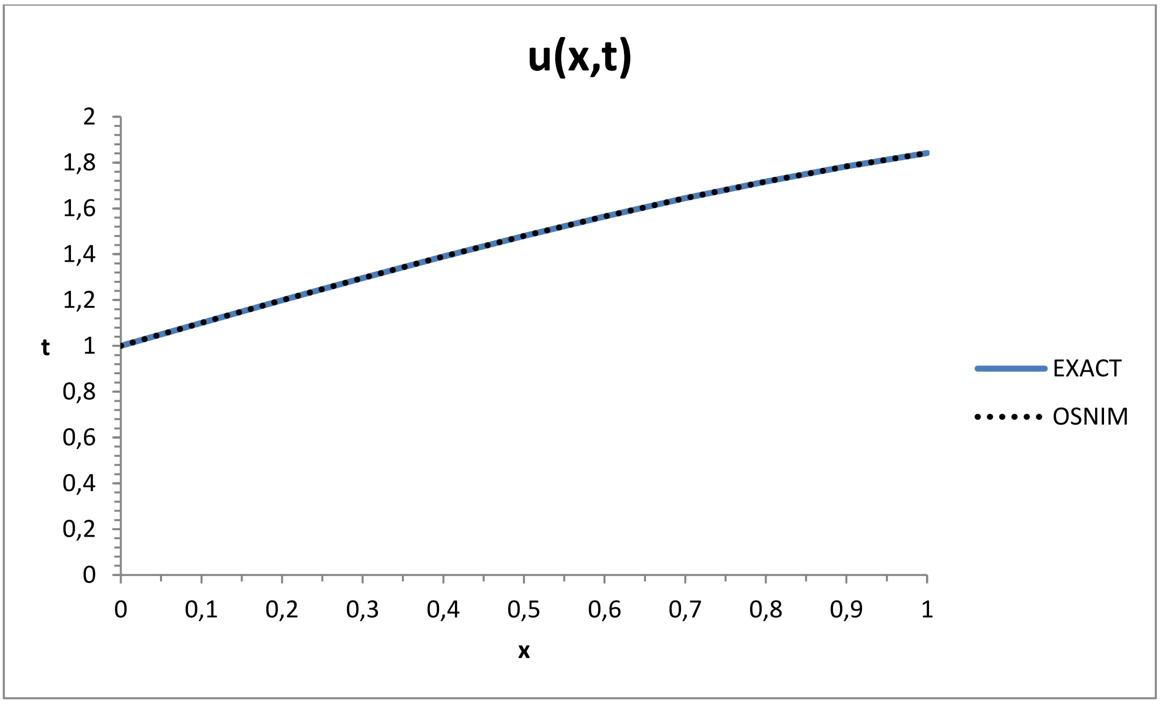


Figure 6: Approximate solutions obtained in comparison with the exact solution for Example 3

Example 4:

Consider the nonlinear non-homogeneous PDE as follows [(Kumar, 2014); (Fang *et al.*, 2022)]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = x^2 t^2 \qquad ...(42)$$

Subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = x \qquad ...(43)$$

The exact solution is

$$u(x,t) = xt \qquad ...(44)$$

We rewrite Eqn. (42) as follows:

$$u_{n+1}(x,t) = c_0 + tc_1 + \frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 u_n(x,s)}{\partial x^2} + u^2(x,s) - x^2 s^2\right)\right] ds \qquad ...(45)$$

Where

$$\frac{(-1)^q}{(q-1)!} = 1 \qquad ...(46)$$

And

$$c_0 + tc_1 = u(x,0) + u_t(x,0) = tx \qquad ...(47)$$

Considering the given initial values, we can select $u_0(x,0) = tx$. Using this selection in Eqn. (45) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = tx \\ u_1(x,t) = \int_0^t (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2} - \frac{\partial^2 u_0(x,s)}{\partial x^2} + u_0^{\,2}(x,s) - x^2 s^2\right] ds = 0 \\ u_2(x,t) = \int_0^t (s-t)\left[\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial x^2} + \sum_{n=0}^{1} u_n^{\,2}(x,s) - x^2 s^2\right] ds = 0 \\ u_3(x,t) = \int_0^t (s-t)\left[\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial s^2} - \sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial x^2} + \sum_{n=0}^{2} u_n^{\,2}(x,s) - x^2 s^2\right] ds = 0 \\ u_4(x,t) = 0 \\ u_5(x,t) = 0 \\ u_6(x,t) = 0 \\ u_7(x,t) = 0 \end{cases} \quad ...(48)$$

And so on. Thus, the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + \ldots$$

$$u(x,t) = tx \quad ...(49)$$

This coincides with the exact solution

Example 5:

Consider the nonlinear non-homogeneous PDE as follows (R. Nuruddeen,*et. al* 2018):

$$\frac{\partial u(x,t)}{\partial t} - u\frac{\partial u(x,t)}{\partial x} = 0 \quad ...(50)$$

Subject to the initial condition

$$u(x,0) = x. \quad ...(51)$$

The exact solution is

$$u(x,t) = \frac{x}{1-t} \quad ...(52)$$

We rewrite Eqn. (50) as follows:

$$u_{n+1}(x,t) = c_0 + \frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial u_n(x,s)}{\partial s} - u\frac{\partial u_n(x,s)}{\partial x}\right)\right] ds \quad ...(53)$$

Where

$$\frac{(-1)^q}{(q-1)!} = -1 \qquad ...(54)$$

And

$$c_0 = u(x,0) = x \qquad ...(55)$$

Considering the given initial values, we can select $u_0(x,0) = x$. Using this selection in Eqn. (53) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = x \\ u_1(x,t) = -\int_0^t (s-t)^0 \left[\frac{\partial u_0(x,s)}{\partial s} - u \frac{\partial u_0(x,s)}{\partial x} \right] ds = xt \\ u_2(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{1} \frac{\partial u_n(x,s)}{\partial s} - \sum_{n=0}^{1} \frac{\partial u_n(x,s)}{\partial x} \right] ds = \frac{1}{3} xt^3 + xt^2 \\ u_3(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{2} \frac{\partial u_n(x,s)}{\partial s} - \sum_{n=0}^{2} \frac{\partial u_n(x,s)}{\partial x} \right] ds = \frac{1}{63} xt^7 + \frac{1}{9} xt^6 + \frac{1}{3} xt^5 + \frac{2}{3} xt^4 \\ \ldots \end{cases} \qquad ...(56)$$

And so on. Thus, the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots =$$

$$xt + xt^3 + xt^2 + \frac{1}{63} xt^7 + \frac{1}{9} xt^6 + \frac{1}{3} xt^5 + \frac{2}{3} xt^4 \qquad ...(57)$$

Table 5: Comparison of the approximate solution from OSNIM, NIM, ADM and exact solutions at $t = 0.1$. **for Example 5**

x	EXACT	OSNIM	NIM	ADM
0	0	0	0	0
0.1	0.111111	0.111107	0.1111	0.1111
0.2	0.222222	0.222214	0.2222	0.2222
0.3	0.333333	0.333321	0.3333	0.3333
0.4	0.444444	0.444428	0.4444	0.4444
0.5	0.555556	0.555535	0.5555	0.5555
0.6	0.666667	0.666642	0.6666	0.6666

0.7	0.777778	0.777749	0.7777	0.7777
0.8	0.888889	0.888856	0.8888	0.8888
0.9	1	0.999963	0.9999	0.9999
1	1.111111	1.11107	1.111	1.111

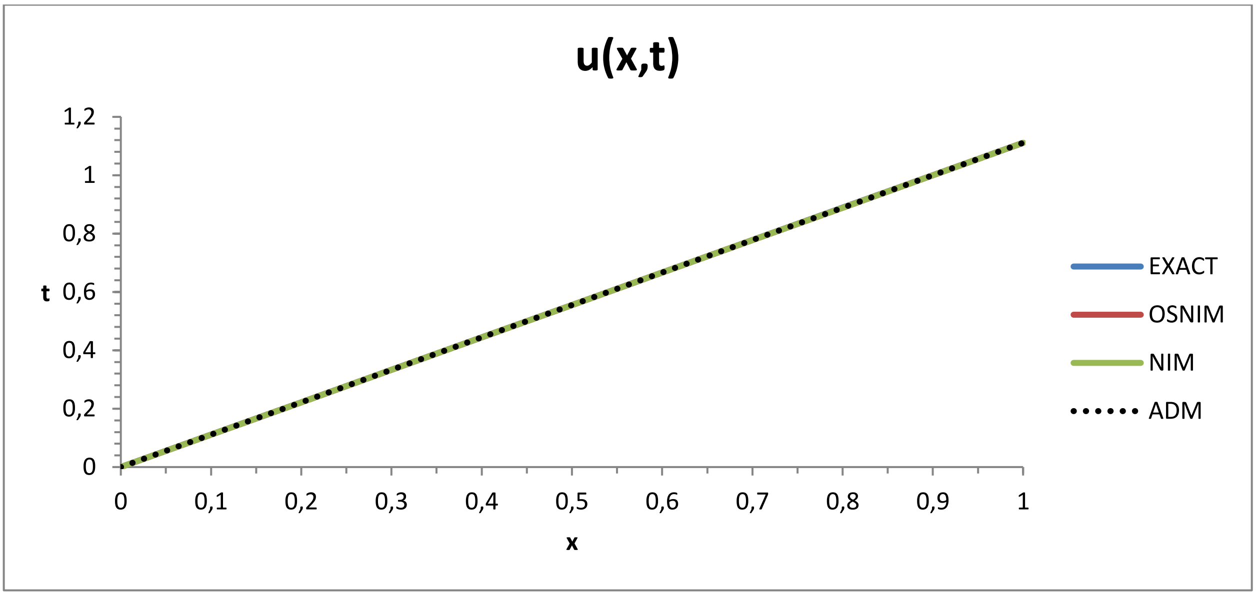


Figure 7: Approximate solutions obtained in comparison with the exact solution, NIM and ADM for Example 5

DISCUSSION

The results of this study demonstrate the efficacy of the One-Step New Iterative Method (OSNIM) in solving partial differential equations (PDEs). The comparative analysis with existing methods, such as the New Iterative Method (NIM) and the Variational Iterative Method (VIM), indicates that OSNIM not only achieves greater accuracy but also exhibits a faster convergence rate. This is particularly evident in the numerical examples presented, where OSNIM yielded solutions that closely matched the exact solutions with minimal error.

The findings are consistent with previous research highlighting the advantages of iterative methods for solving complex nonlinear equations (Ameh, 2021; Bhalekar & Daftardar-Gejji, 2010). Moreover, the application of modified integral operators within OSNIM allows for a straightforward path to closed-form solutions, circumventing the need for transforming PDEs into systems of multiple integral equations, which is often a source of computational burden in traditional methods (Ramadan et al., 2019).

The convergence analysis confirms that OSNIM maintains stability and reliability across a range of initial conditions, addressing some of the common challenges faced in numerical methods for PDEs, such as instability in high-dimensional problems (Bangerth et al., 2016). This robustness positions OSNIM as a viable alternative for real-time applications in fields requiring rapid and accurate computations, such as engineering simulations and financial modeling.

CONCLUSION

In conclusion, the One-Step New Iterative Method (OSNIM) marks a notable advancement in the numerical solutions of evolution equations and partial differential equations. The findings demonstrate that OSNIM enhances accuracy and convergence compared to existing methodologies, thereby offering substantial practical benefits for real-world applications. This study effectively addresses the gap in computational techniques by providing a robust framework that can be utilized across diverse fields. Future research should explore the integration of OSNIM with other emerging computational methods and investigate its applicability in more complex systems to further validate its efficacy and expand its utility.

REFERENCES

Ala'yed, O., Batiha, B., Alghazo, D., & Ghanim, F. (2023). Cubic B-Spline method for the solution of the quadratic Riccati differential equation. *AIMS Mathematics*, *8*(4), 9576–9584. https://doi.org/10.3934/math.2023483

Ameh, K. J. A. S. (2021). Implementation of New Iterative Method for Solving Nonlinear Partial Differential Problems. 1–10.

Anake, T. A., Edeki, S. O., & Ogundile, O. P. (2021). Approximate-analytical solutions of some classical riccati differential equation using the daftardar-gejji jafari method. *J. Math. Comput. Sci.*, *11*(6), 6729–6744.

Ashitha, P. A., & Ranjini, M. C. (2020). On the numerical solution of fractional Riccati differential equations. *Malaya Journal of Matematik*, *5*(1), 214–219.

Batiha, B. (2009). Numerical solution of a class of singular second-order IVPs by variational iteration method. *International Journal of Mathematical Analysis*, *3*(37–40), 1953–1968.

Batiha, K., & Batiha, B. (2011). A new algorithm for solving linear ordinary differential equations. *World Applied Sciences Journal*, *15*(12), 1774–1779.

Bhalekar, S., & Daftardar-gejji, V. (2008). New iterative method : Application to partial differential equations. *Applied Mathematics and Computation*, *203*(2008), 778–783. https://doi.org/10.1016/j.amc.2008.05.071

Bhalekar, S., & Daftardar-Gejji, V. (2010). Solving evolution equations using a new iterative method. *Numerical Methods for Partial Differential Equations*, *26*(4), 906–916. https://doi.org/10.1002/num.20463