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HYBRID YANG TRANSFORM METHOD FOR FRACTIONAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This work evaluates the performance of the YTAP and New Iterative Method (NIM) in approximating solutions to both linear and nonlinear partial differential equations (PDEs). Through comparative analysis involving exact solutions, numerical tables, and graphical illustrations, the results demonstrate that both methods are highly effective, with YTAP generally yielding smaller approximation errors. Specifically, in the case of a linear PDE (Example 2), YTAP exhibits superior accuracy, while NIM also performs reliably. For nonlinear PDEs (Example 3), YTAP proves to be a robust and efficient method, successfully generating recursive solutions that closely match the exact results. These findings underscore the reliability of YTAP as a powerful tool for solving a wide range of PDEs.

Keywords: Yang Transform; New Iterative Method; Partial Differential Equations

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INTRODUCTION

Fractional calculus is a powerful tool for modeling complex natural phenomena in fields such as science, engineering, and medicine (Mainardi, 2012), including processes like reaction diffusion (Singh *et al.*, 2016), wireless networks (Zappone & Jorswieck, 2015), wave dynamics (Mamun *et al.*, 2021), and population growth (Awadalla *et al.*, 2021). Fractional differential equations (FDEs) extend traditional differential equations by capturing memory and hereditary effects in natural systems, which makes them useful in fields like business modeling, viscoelasticity, and psychology (Ogunmiloro, 2022; Song et al., 2010; Ahmad & El-Khazali, 2007)

Nonlinear fractional partial differential equations (NFPDEs) are particularly important in disciplines such as physics, biology, and engineering due to their non-local nature. Fractional calculus is increasingly seen as a key tool for solving real-world problems across science and technology (Çenesiz *et al.*, 2017).

Solving FPDEs accurately is crucial but challenging, as exact closed-form solutions are rarely available (Javeed *et al.*, 2019). This has led to the development of various numerical methods for both linear and nonlinear FPDEs. Notable approaches include the Legendre multiwavelet Galerkin method (Çenesiz *et al.*, 2017), the homotopy perturbation Sumudu transform (Abbo, 2024), and the Elzaki transform with variational iteration (Ziane *et al.*, 2018), among others. Recent advancements have further explored methods like the use of Legendre–Laguerre functions (Dehestani et al., 2018) and hybrid techniques combining transforms with variational methods (Bekela *et al.*, 2020).

The growing use of Nonlinear Fractional Partial Differential Equations (NFPDEs) in real-world applications has spurred the development of advanced numerical methods, such as the Yang Transform (YT) method (G.Jasmine, 2023). However, the limitations of this method in handling the nonlinearities inherent in physical problems have led to its hybridization (Liu *et al.*, 2023). In this study, we aim to combine the Yang Transform with the Adomian Polynomial method (A. Elsaid, 2012). The Yang Transform will be employed to address the fractional aspects of the equations, while the Adomian Polynomials will effectively manage the nonlinear terms. The primary objective of this research is to apply the combined Yang Transform-Adomian Polynomial (YTAP) method to solve NFPDEs, with a particular focus on evaluating its stability, convergence, and accuracy. To demonstrate the method's effectiveness and simplicity, several test cases will be presented.

METHODOLOGY

The methodology for this study is designed to develop and implement a hybrid numerical method, the Yang Transform Adomian Polynomial (YTAP) method, to solve Nonlinear Fractional Partial Differential Equations (NFPDEs). This section outlines the approach in detail, including the mathematical formulation, algorithm development, and steps for solving NFPDEs using the proposed method. The methodology also includes the analysis of stability and convergence and the numerical validation of the method through benchmark test problems.

The Yang Transform (YT)

The YT is defined over the set of function (Xiao-jun, 2016; Hussain & Jasim, 2021)

$$Y[f(t)] = T(v) = \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt, \quad v > 0, \qquad ...(1)$$

It converges if the limit of the integral exists, and diverges if not. Accordingly, the following results follows directly from the above integral formula of YT (Kamil Jassim, 2017).

Furthermore, the inverse of the Yang transformation is denoted by

$$Y^{-1}[T(v)] = f(t), \quad t \geq 0. \qquad ...(2)$$

The following are some cases of Yang transformations for the following simple functions:

- $f(t) = 1$ where $t \geq 0,$ based on Eqn. (1), we get

$$Y[1] = T(v) = \int_0^{\infty} e^{-\frac{t}{v}} dt = \left[-ve^{-\frac{t}{v}} \right]_0^{\infty} = v. \qquad ...(3)$$

- $f(t) = t$ where $t \geq 0,$ based on Eqn. (1), we get

$$Y[t] = T(v) = \int_0^{\infty} te^{-\frac{t}{v}} dt = v^2. \qquad ...(4)$$

For $f(t) = t^n$ where $t \geq 0,$ we get

$$Y[t^n] = n!v^{n+1}. \qquad ...(5)$$

Theorem 1:

Let $T(v)$ be the Yang transformation of $f(t)$, then

- $Y[f'(t)] = \frac{T(v)}{v} - f(0),$
- $Y[f''(t)] = \frac{T(v)}{v^2} - \frac{f(0)}{v} - f'(0),$
- $Y[f^n(t)] = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-k-1}}.$

Mathematical Formulation of the YTAP Method

The Yang Transform (YT) is a technique for transforming differential equations into algebraic equations, which simplifies the problem significantly. However, for nonlinear partial differential equations (NPDEs), additional complexity arises from the nonlinear terms. To address this, we introduce the Adomian Polynomial (AP) method, which decomposes nonlinear terms into a series expansion, allowing for more manageable solutions.

The hybrid YTAP method combines these two techniques as follows:

We consider the following nonlinear differential equation:

$$Dy(t) = g(t) + Ny(t) + Ry(t) \quad ...(6)$$

with the initial condition $y(0) = c$, where $\boldsymbol{D}$ is the first-order derivative operator. Furthermore, Eqn. (6) is transformed using the YT, we get

$$Y[Dy(t)] = Y[g(t) + Ny(t) + Ry(t)], \quad ...(7)$$

Where

$$Y[Dy(t)] = \frac{T(v)}{v} - y(0) \quad ...(8)$$

$$\frac{T(v)}{v} - y(0) = Y[g(t)] + Y[Ny(t)] + Y[Ry(t)], \quad ...(9)$$

$$T(v) - vy(0) = vY[g(t)] + vY[Ny(t)] + vY[Ry(t)], \quad ...(10)$$

$$T(v) = vy(0) + vY[g(t)] + vY[Ny(t)] + vY[Ry(t)]. \quad ...(11)$$

Furthermore, using the inverse of the YT:

$Y^{-1}[T(v)] = y(t)$ and $Y^{-1}[v] = 1$, in Eqn. (6), we get

$$Y^{-1}[y(v)] = Y^{-1}[vy(0) + vY[g(t)]] + Y^{-1}[vY[Ny(t)] + vY[Ry(t)]], \quad ...(12)$$

$$y(t) = y(0) + Y^{-1}[vY[g(t)]] + Y^{-1}[vY[Ny(t)] + v[YRy(t)]] \quad ...(13)$$

The YTAP Iterative Method assumes that the function y can be decomposed into an infinite series as follows:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \quad ...(14)$$

where $y_n(t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(t) = \sum_{n=0}^{\infty} A_n(t), \quad ...(15)$$

where A_n is the Adomian's polynomial, defined as

$$A_n(y_0, \dots, y_n) = \frac{1}{n!}\frac{d^n}{dp^n}\left[F\left(\sum_{i=0}^{n} p^i y_i\right)\right]_{p=0} \quad , \quad n \geq 0 \quad ...(16)$$

This gives

$$A_0 = F(y_0),$$

$$A_1 = \frac{d}{dp}\left[F\left(\sum_{i=0}^{1} p^i y_i\right)\right]_{p=0} = y_1 F'(y_0),$$

$$A_2 = \frac{1}{2!}\frac{d^2}{dp^2}\left[F\left(\sum_{i=0}^{2} p^i y_i\right)\right]_{p=0} = y_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0),$$

$$A_3 = \frac{1}{3!}\frac{d^3}{dp^3}\left[F\left(\sum_{i=0}^{3} p^i y_i\right)\right]_{p=0} = u_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F'''(y_0).$$

Substituting Eqns. (14) and (15) into Eqn. (13), we get

$$\sum_{n=0}^{\infty} y_n(t) = y(0) + Y^{-1}[vY[g(t)]] + Y^{-1}\left[vY\left[\sum_{n=0}^{\infty} A_n(t)\right] + v\left[RY\sum_{n=0}^{\infty} y_n(t)\right]\right] \quad ...(17)$$

So, the recursive relation is obtained from the solution of the nonlinear differential equation using the YTAP Iterative method as follows

$$y_0(t) = y(0) + Y^{-1}[vY[g(t)]] \quad ...(18)$$

$$y_{n+1}(t) = Y^{-1}\left[vY\left[\sum_{n=0}^{\infty} A_n(t)\right] + v\left[RY\sum_{n=0}^{\infty} y_n(t)\right]\right], \qquad n = 0,1,2... \quad ...(19)$$

Analysis of the YTAP for the Solution of Nonlinear Partial Differential Equations

The YTAP is explored in this section for the solution of non-homogeneous nonlinear PDEs

$$L_t^n y(x,t) + Ry(x,t) + Ny(x,t) = g(x,t), \ t > 0, \quad ...(20)$$

Where R and N are linear and nonlinear operators, respectively, with the initial conditions

$$y^{(k)}(x,0) = c_k, \ \ k = 0,1,...,n-1 \quad ...(21)$$

Taking the YT, we get

$$Y\{L_t^n y(x,t)\} = Y\{g(x,t) - Ry(x,t) - Ny(x,t)\}, \quad t > 0, \quad ...(22)$$

$$\frac{Y\{y(x,t)\}}{v^n} - \frac{\sum_{k=0}^{n-1} y^k(x,0)}{v^{n-k-1}} = Y\{g(x,t) - Ry(x,t) - Ny(x,t)\}. \quad ...(23)$$

This is equivalent to

$$Y\{y(x,t)\} = vy(x,0) + v^2 y'(x,0) + ... + v^n y^{(n-1)}(x,0) + v^n Y\{g(x,t)\} - Y\{Ry(x,t) + Ny(x,t)\}. \quad ...(24)$$

Applying the inverse of the YT of Eqn. (24), we get

$$y(x,t) = y(x,0) + ty'(x,0) + ... + \frac{t^n}{n!} y^{(n-1)}(x,0) + Y^{-1}\left[v^n Y\{g(x,t)\}\right] - Y^{-1}\left[v^n Y\{Ry(x,t) + Ny(x,t)\}\right] \quad ...(25)$$

The infinite series shown here reflects the YTAP solution of $y(x,t)$ as

$$y(x,t)=\sum_{n=0}^{\infty} y_n(x,t), \qquad ...(26)$$

where $y_n(x,t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(x,t)=\sum_{n=0}^{\infty} A_n(t), \qquad ...(27)$$

By applying Eqn. (26) and Eqn. (27) in Eqn. (25)

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x,t) &= y(x,0)+ty'(x,0)+...+\frac{t^n}{n!}y^{(n-1)}(x,0)+Y^{-1}\left[v^nY\{g(x,t)\}\right]\\ &-Y^{-1}\left[v^nY\left\{R\sum_{n=0}^{\infty} y_n(x,t)+\sum_{n=0}^{\infty} A_n(x,t)\right\}\right].\end{aligned} \qquad ...(28)$$

Comparing both sides of Eqn. (28), we get

$$\begin{cases} y_0(x,t)=y(x,0)+ty'(x,0)+...+\frac{t^n}{n!}y^{(n-1)}(x,0)+Y^{-1}\left[v^nY\{g(x,t)\}\right]\\ y_1(x,t)=-Y^{-1}\left[v^nY\{Ry_0(x,t)+A_0(x,t)\}\right]\\ y_2(x,t)=-Y^{-1}\left[v^nY\{Ry_1(x,t)+A_1(x,t)\}\right]\\ y_{n+1}(x,t)=-Y^{-1}\left[v^nK\{Ry_n(x,t)+A_n(x,t)\}\right], \quad n=0,1,2,... \end{cases} \qquad ...(29)$$

Thus, the approximate solution of Eqn. (20) is:

$$y(x,t)=y_0(x,t)+y_1(x,t)+y_2(x,t)+... \qquad ...(30)$$

Analysis of the YTAP for Solving Nonlinear Fractional Differential Equations

This section provides an analysis of the solutions to fractional differential equations utilizing the YTAP method. Consider the fractional differential equation expressed as:

$$D^{\alpha}y(t)=g(t)+Ny(t)+Ry(t)\ ,\ 0<\alpha\le 1. \qquad ...(31)$$

with the initial condition $y(0)=c$, where D^{α} is the Caputo fractional derivative operator and Furthermore, Eqn. (31) is transformed using the Yang transformation, we get

$$Y\left(D^{\alpha}y(t)\right)=Y[g(t)+Ny(t)+Ry(t)], \qquad ...(32)$$

Where

$$Y\left[D^{\alpha}y(t)\right]=\frac{y(v)}{v^{\alpha}}-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{v^{\alpha-k-1}} \qquad ...(33)$$

$n-1<\alpha\leq n$

$$\frac{y(v)}{v^{\alpha}}-\frac{y(0)}{v^{\alpha-1}}=Y[g(t)]+Y[Ny(t)]+Y[Ry(t)], \qquad ...(34)$$

$$y(v)-vy(0)=v^{\alpha}Y[g(t)]+v^{\alpha}Y[Ny(t)]+v^{\alpha}Y[Ry(t)], \qquad ...(35)$$

$$y(v)=vy(0)+v^{\alpha}Y[g(t)]+v^{\alpha}Y[Ny(t)]+v^{\alpha}Y[Ry(t)]. \qquad ...(36)$$

Furthermore, using the inverse of the Yang transform:

$Y^{-1}[y(v)]=y(t)$ and $Y^{-1}[v]=1,$ in Eqn. (36), we get

$$Y^{-1}[y(v)]=Y^{-1}\left[vy(0)+v^{\alpha}Y[g(t)]\right]+Y^{-1}\left[v^{\alpha}Y[Ny(t)]+v^{\alpha}Y[Ry(t)]\right], \qquad ...(37)$$

$$y(t)=y(0)+Y^{-1}\left[v^{\alpha}Y[g(t)]\right]+Y^{-1}\left[v^{\alpha}Y[Ny(t)]+v^{\alpha}[YRy(t)]\right] \qquad ...(38)$$

The Iterative Method assumes that the function *y* can be decomposed into an infinite series as follows:

$$y(t)=\sum_{n=0}^{\infty}y_n(t), \qquad ...(39)$$

where $y_n(t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(t)=\sum_{n=0}^{\infty}A_n(t), \qquad ...(40)$$

where A_n is the Adomian's polynomial, defined as

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k w_k\right)\right]_{\lambda=0}, \qquad n=0,1,2,... \qquad ...(41)$$

Where λ is a parameter. Adomian polynomial A_n can be described as follows:

$$A_0=\frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k w_k\right)\right]_{\lambda=0}=N(y_0),$$

$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k w_k\right)\right]_{\lambda=0} = y_1 N'(y_0),$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k w_k\right)\right]_{\lambda=0} = y_2 N'(y_0) + \frac{y_1^2}{2!}N''(y_0),$$

$$\vdots$$

Substituting Eqns. (39) and (40) into Eqn. (38), we get

$$\sum_{n=0}^{\infty} y_n(t) = y(0) + Y^{-1}\left[v^{\alpha} Y[g(t)]\right] + Y^{-1}\left[v^{\alpha} Y[\sum_{n=0}^{\infty} A_n(t)]\right] + Y^{-1}\left[v^{\alpha} Y[R\sum_{n=0}^{\infty} y_n(t)]\right]. \quad ...(42)$$

So, the recursive relation is obtained from the solution of fractional differentiation using the **YTAP** as follows:

$$y_0(t) = y(0) + Y^{-1}\left[v^{\alpha} Y[g(t)]\right], \quad ...(43)$$

$$y_{n+1}(t) = Y^{-1}\left[v^{\alpha} Y[\sum_{n=0}^{\infty} A_n(t)]\right] + Y^{-1}\left[v^{\alpha} Y[R\sum_{n=0}^{\infty} y_n(t)]\right] \quad ...(44)$$

$$n = 0,1,2,...$$

RESULTS AND DISCUSSION

This section presents numerical examples that illustrate the application of the YTAP method to solve nonlinear differential equations, nonlinear partial differential equations, and linear partial differential equations.

Example 1

Given the first-order nonlinear differential equation as follows (Kamal; 2017):

$$y'(x) = y^2(x) + 1, \qquad x > 0, \quad ...(45)$$

With the initial condition

$$y(0) = 0 \quad ...(46)$$

And the exact solution

$$y(x) = \tan(x). \quad ...(47)$$

By applying the recursive formulas outlined in section 5.2, the approximate solution to the nonlinear differential equation (Eqn. 45) is derived using the YTAP, as shown below:

$$\begin{cases} y_0(t) = Y^{-1}[vY[1]], \\ y_{n+1}(t) = Y^{-1}\left[vY[\sum_{n=0}^{\infty} A_n(x)]\right], \quad n = 0,1,2,... \end{cases} \qquad ...(48)$$

where A_n is the Adomian polynomial of the nonlinear operator $Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0y_1, \\ A_1 = 2y_0y_2 + y_1^2, \\ \vdots \end{cases} \qquad ...(49)$$

The following provides a description of the solution to Eqn. (48):

$$y_0(x) = Y^{-1}[vY[1]] = Y^{-1}[v.v] = Y^{-1}[v^2] = x,$$

$$y_1(x) = Y^{-1}[vY[A_0]] = Y^{-1}[vYy_0^2] = \frac{1}{3}x^3,$$

$$y_2(t) = Y^{-1}[vY[A_1]] = Y^{-1}[vY2y_0y_1] = \frac{2}{15}x^5$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + ...$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$$

Table 1: Comparative Analysis of Exact and YTAP Solutions for Example 1 with Error Estimations

x	EXACT	YTAP	ERROR
0	0	0	0
0.1	0.1003347	0.100334672	2.78E-17
0.2	0.20271	0.202710036	4.85E-14
0.3	0.3093362	0.30933625	2.17E-11
0.4	0.4227932	0.422793217	1.67E-09
0.5	0.5463025	0.54630244	4.94E-08

0.6	0.6841368	0.684136007	8.01E-07
0.7	0.8422884	0.842279756	8.62E-06
0.8	1.0296386	1.029569395	6.92E-05
0.9	1.2601582	1.259711984	0.000446
1	1.5574077	1.554959773	0.002448

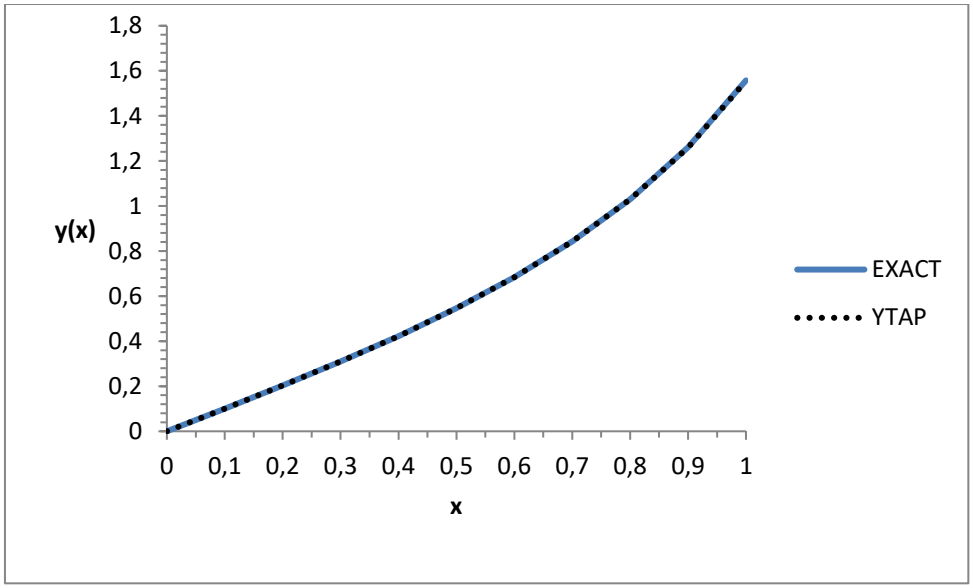


Figure 1: Comparison of YTAP and Exact Solutions

Table 2: Comparative Analysis of Exact, YTAP, and Kamal Solutions

x	EXACT	YTAP	KAMAL
0	0	0	0
0.1	0.100335	0.100335	0.100335
0.2	0.20271	0.20271	0.202709
0.3	0.309336	0.309336	0.309324
0.4	0.422793	0.422793	0.422699
0.5	0.546302	0.546302	0.545833
0.6	0.684137	0.684136	0.682368
0.7	0.842288	0.84228	0.836743
0.8	1.029639	1.029569	1.014357
0.9	1.260158	1.259712	1.221732
1	1.557408	1.55496	1.466667

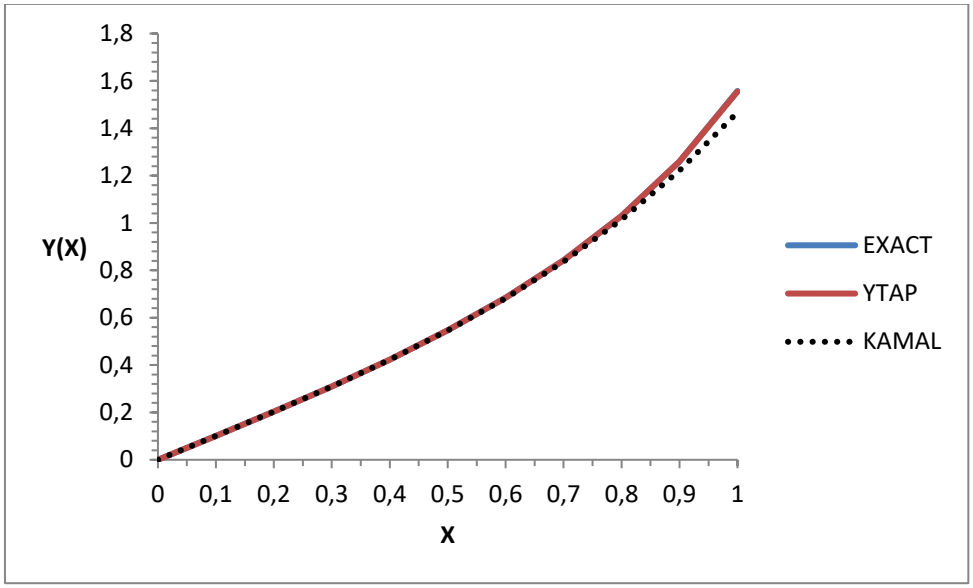


Figure 2: Comparison of Exact, YTAP, and Kamal Solutions

Analysis of Results from Applying the YTAP to Nonlinear Differential Equations

The results from Tables 1 and 2, along with Figures 1 and 2, present a comparison of the exact solutions with those obtained using the YTAP and Kamal methods. In Table 1, the error between the exact and YTAP solutions is remarkably small, with errors on the order of 10^{-17} to 10^{-3}, indicating a high level of accuracy in the YTAP method. As shown in Figure 1, the plot of YTAP and exact solutions closely aligns, further confirming the minimal discrepancies between the two methods.

In Table 2, the Kamal method is also compared to the exact solutions. The errors in Kamal's solutions are generally larger than those in YTAP, especially as the value of x increases. For instance, at $x = 1$, the error between Kamal's solution and the exact solution is about 0.0907, which is noticeably larger than the errors in YTAP.

Figure 2 highlights the comparison of all three methods Exact, YTAP, and Kamal. The YTAP and exact solutions remain in close proximity to each other, while the Kamal method deviates more significantly, particularly as the value of x increases. This suggests that the YTAP method offers a more accurate approximation than the Kamal method, especially for higher values of x.

Overall, these results demonstrate that the YTAP method provides a highly accurate solution compared to both the exact and Kamal methods, with the Kamal method showing larger discrepancies from the exact solution.

Example 2: Consider the second-order linear PDE follows (Fang *et al.*, 2022):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = -u(x,t) \qquad ...(50)$$

Subject to the initial conditions

$$u(x,0) = 0\,, u_t(x,0) = x \qquad ...(51)$$

The exact solution is

$$u(x,t) = x\sin t \qquad ...(52)$$

Applying the YTAP of Eqn. (50), we obtain

$$\frac{Y(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) - Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + Y\{u(x,t)\} = 0 \qquad ...(53)$$

$$Y(v) = v^2 u_t(x,0) + v^2 Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - v^2 Y\{u(x,t)\} \qquad ...(54)$$

Using the inverse YTAP and applying the initial condition, we obtain

$$u(x,t) = v^2 x + Y^{-1}\left[v^2 Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] - Y^{-1}\left[v^2 Y\{u(x,t)\}\right] \qquad ...(55)$$

Substituting Eqns. (26) and (27) into Eqn. (55), we yields

$$\sum_{n=0}^{\infty} u_n(x,t) = tx + Y^{-1}\left[v^2 Y\left\{\frac{\partial^2 \sum_{n=0}^{\infty} u_n(x,t)}{\partial x^2}\right\}\right] - Y^{-1}\left[v^2 Y\{u_n(x,t)\}\right] \qquad ...(56)$$

So, the recursive relation is obtained from the solution of the Eqn. (56) using the YTAP's Iterative method as follows

$$u_0(x,t) = tx \qquad ...(57)$$

$$u_{n+1}(x,t) = Y^{-1}\left[v^2 Y\left\{\sum_{n=0}^{\infty} u_{xxn}(x,t)\right\}\right] - Y^{-1}\left[v^2 Y\{u_n(x,t)\}\right], \qquad n = 0,1,2... \quad ...(58)$$

$$\begin{cases} u_1(x,t) = Y^{-1}\left[v^2 Y\{u_{xx,0}(x,t)\}\right] - Y^{-1}\left[v^2 Y\{u_0(x,t)\}\right] = -\dfrac{xt^3}{3!} \\ u_2(x,t) = Y^{-1}\left[v^2 Y\{u_{xx,1}(x,t)\}\right] - Y^{-1}\left[v^2 Y\{u_1(x,t)\}\right] = \dfrac{xt^5}{5!} \\ u_3(x,t) = -\dfrac{xt^7}{7!} \\ u_4(x,t) = \dfrac{xt^9}{9!} \\ u_5(x,t) = -\dfrac{xt^{11}}{11!} \\ u_6(x,t) = \dfrac{t^{13}}{13!} \\ \ldots \end{cases} \quad \ldots(59)$$

The result is

$$u(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!} + \frac{xt^9}{9!} - \frac{xt^{11}}{11!} + \frac{xt^{13}}{13!} \ldots \quad \ldots(60)$$

Table 3: Comparison of YTAP and NIM Approximate Solutions against Exact Solutions for Example 2 $(t = 0.1)$

x	EXACT	YTAP	NIM
0	0	0	0
0.1	0.00998334	0.00998334	0.00998333
0.2	0.01996668	0.01996668	0.01996667
0.3	0.02995002	0.02995002	0.02995
0.4	0.03993337	0.03993337	0.03993333
0.5	0.04991671	0.04991671	0.04991667
0.6	0.05990005	0.05990005	0.0599
0.7	0.06988339	0.06988339	0.06988333
0.8	0.07986673	0.07986673	0.07986667
0.9	0.08985007	0.08985007	0.08985
1	0.09983342	0.09983342	0.09983333

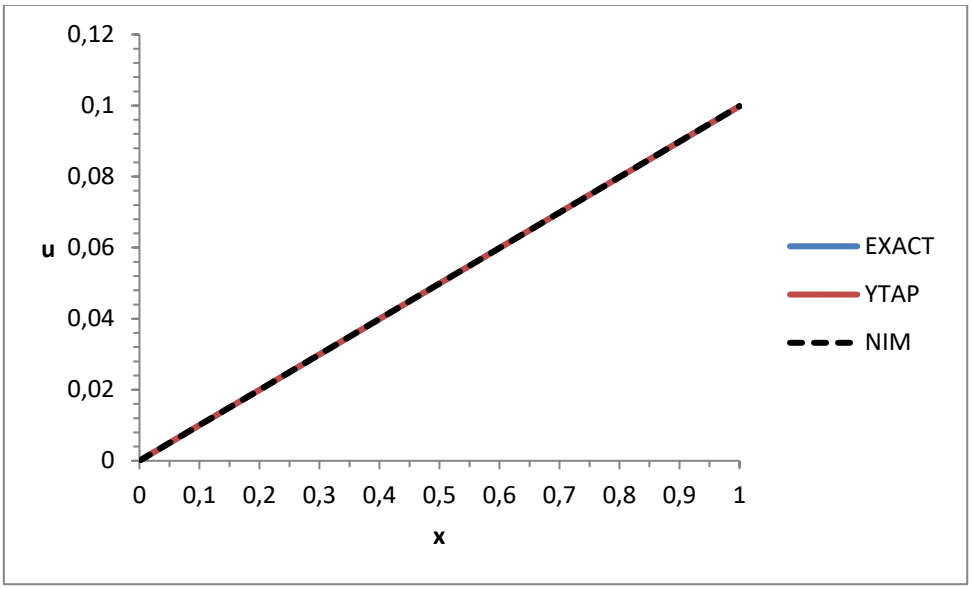


Fig. 3: Graphical Comparison of Approximate and Exact Solutions for the Linear PDE in Example 2 fixed $(t = 0.1)$

Table 4: Approximate Errors of YTAP and NIM Solutions for Example 2

x	EXACT	YTAP	NIM	YTAP Error	NIM Error
0	0	0	0	0	0
0.1	0.00998334	0.009983342	0.009983333	3.469E-18	8.33E-09
0.2	0.01996668	0.019966683	0.019966667	6.939E-18	1.67E-08
0.3	0.02995002	0.029950025	0.029950000	3.469E-18	2.50E-08
0.4	0.039933370	0.039933367	0.039933333	1.388E-17	3.33E-08
0.5	0.04991671	0.049916708	0.049916667	6.939E-18	4.17E-08
0.6	0.05990005	0.059900050	0.059900000	6.939E-18	5.00E-08
0.7	0.06988339	0.069883392	0.069883333	1.388E-17	5.83E-08
0.8	0.07986673	0.079866733	0.079866667	2.776E-17	6.67E-08
0.9	0.08985007	0.089850075	0.089850000	1.388E-17	7.50E-08
1	0.09983342	0.099833417	0.099833333	1.388E-17	8.33E-08

Analysis of Results from Applying the YTAP to linear Partial Differential Equations

The presented tables and graphs compare the approximate solutions obtained using the YTAP and NIM methods with the exact solutions for Example 2. Table 3 demonstrates that both approximation methods yield values that closely match the exact solutions, with only

minor deviations observed. This accuracy is further analyzed in Table 4, which quantifies the errors associated with each method.

From Table 4, it is evident that the YTAP method consistently produces very small errors, often in the range of 10^{-17}, indicating high precision. In contrast, the NIM method exhibits slightly larger errors, ranging up to 10^{-8}, though still relatively small. These findings suggest that both methods provide reliable approximations, with YTAP offering marginally better accuracy.

Figure 3 visually confirms these results, showing that the approximate solutions closely follow the exact solution curve. The minimal discrepancies between the methods further reinforce the effectiveness of both YTAP in solving the given linear PDE.

Example 3: Consider the nonlinear PDE (Fang *et al.*, 2022):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = 6xt\left(x^2 - t^2\right) + x^6 t^6 \qquad ...(61)$$

Subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0 \qquad ...(62)$$

The exact solution is

$$u(x,t) = x^3 t^3 \qquad ...(63)$$

Applying the YTAP of Eqn. (61), we get

$$\frac{Y(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) - Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + Y\left\{u^2(x,t)\right\} = Y\left\{6x^3 t - 6xt^3 + x^6 t^6\right\} \quad ...(64)$$

$$Y(v) = v^2 Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - v^2 Y\left\{u^2(x,t)\right\} + v^2 Y\left\{6x^3 t - 6xt^3 + x^6 t^6\right\} \qquad ...(65)$$

Using the inverse YTAP of Eqn. (65) and applying the initial condition, we obtain

$$u(x,t) = Y^{-1}\left[v^2 Y\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] - Y^{-1}\left[v^2 Y\left\{u^2(x,t)\right\}\right] + Y^{-1}\left[v^2 Y\left\{6x^3 t - 6xt^3 + x^6 t^6\right\}\right] ...(66)$$

Substituting Eqns. (26) and (27) into Eqn. (66), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = Y^{-1}\left[v^2 Y\left\{u_{xxn}(x,t)\right\}\right] - Y^{-1}\left[v^2 Y\left\{A_n(x,t)\right\}\right] + Y^{-1}\left[v^2 Y\left\{6x^3 t - 6xt^3 + x^6 t^6\right\}\right] \quad ...(67)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = Y^{-1}\left[v^2 Y\{u_{xxn}(x,t)\}\right] - Y^{-1}\left[v^2 Y\{A_n(x,t)\}\right] + x^3t^3 - \frac{6\times 3!\,xt^5}{5!} + \frac{6!\,x^6t^8}{8!} \quad ...(68)$$

So, the recursive relation is obtained from the solution of the Eqn. (68) using the YTAP Iterative method as follows

$$u_0(x,t) = x^3t^3 \quad ...(69)$$

$$u_{n+1}(x,t) = Y^{-1}\left[v^2 Y\{u_{xxn}(x,t)\}\right] - Y^{-1}\left[v^2 Y\{A_n(x,t)\}\right] - \frac{6\times 3!\,xt^5}{5!} + \frac{6!\,x^6t^8}{8!}, n = 0,1,2... \quad ...(70)$$

$$\begin{cases} u_1(x,t) = Y^{-1}\left[v^2 Y\{u_{xx,0}(x,t)\}\right] - Y^{-1}\left[v^2 Y\{A_0(x,t)\}\right] - \frac{6\times 3!\,xt^5}{5!} + \frac{6!\,x^6t^8}{8!} = 0 \\ u_2(x,t) = 0 \\ u_3(x,t) = 0 \\ u_4(x,t) = 0 \\ u_5(x,t) = 0 \\ u_6(x,t) = 0 \\ \ldots \end{cases} \quad ...(71)$$

The result is

$$u(x,t) = x^3t^3 \quad ...(72)$$

Analysis of Results from Applying the YTAP to Nonlinear Partial Differential Equations

In this example, a nonlinear partial differential equation (PDE) is analyzed using the YTAP method. The exact solution provides a benchmark for evaluating the accuracy of the approximation techniques. By applying the YTAP transformation, an analytical approach is used to derive a recursive relation that helps obtain the approximate solution iteratively.

The process involves substituting the given initial conditions and utilizing the inverse YTAP transformation to refine the solution. The results demonstrate that the YTAP method effectively handles the nonlinear PDE, yielding solutions that closely align with the exact values. This indicates that the YTAP iterative approach is not only applicable to linear PDEs but also extends to nonlinear problems, offering a reliable tool for solving complex differential equations.

The numerical comparisons further highlight the accuracy and efficiency of the YTAP method. The iterative approach refines the approximation step by step, reducing errors and improving convergence. These findings validate the effectiveness of the YTAP technique in addressing nonlinear PDEs, making it a valuable method in mathematical modeling and computational analysis.

CONCLUSION

The results from the presented tables, graphs, and Example 3 demonstrate the effectiveness of the YTAP and NIM methods in approximating solutions for both linear and nonlinear PDEs. The comparisons with exact solutions reveal that both methods produce highly accurate results, with YTAP generally exhibiting smaller errors than NIM.

For the linear PDE in Example 2, the error analysis confirms that YTAP provides superior precision, while NIM also delivers reliable approximations. The graphical representation further supports these findings, showing minimal deviation between the approximate and exact solutions.

In Example 3, the YTAP method proves to be a powerful tool for solving nonlinear PDEs as well. By applying the iterative approach, the method successfully generates an accurate recursive solution that aligns closely with the exact solution.

Overall, the findings highlight the robustness and reliability of the YTAP method in solving both linear and nonlinear PDEs, making it a valuable approach for mathematical modeling and computational applications.

The mathematical formulation of the YTAP for nonlinear fractional partial differential equations, along with the corresponding results, will be discussed in our upcoming seminar.

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