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A Revised Kamal Integral Transformation Method for Solving First-Order Nonlinear Fractional Differential Equations

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Abstract

This study integrates the Kamal transform method with Adomian polynomials to provide approximate analytical solutions for non-linear fractional differential equations. Since the Kamal transform alone cannot handle non-linear terms, Adomian polynomials are used to decompose these terms. The resulting solutions are compared with those from other methods applied to the same problems. The results show that the solution obtained with the Kamal transform and Adomian polynomial closely aligns with those from other methods, demonstrating the effectiveness of this combined approach. This work emphasizes the strength of the Kamal transform method and Adomian polynomials as integral transform techniques for solving non-linear equations.

Keywords: Kamal transform method; Adomian polynomials; Fractional differential equations

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Introduction

Solving fractional-order differential equations has emerged as a compelling area of research due to their ability to more accurately model complex phenomena across various scientific and engineering disciplines. Unlike traditional integer-order equations, fractional-order models capture memory effects and hereditary properties, making them particularly suited for real-world systems (Podlubny, 1999). However, the inherent complexity of these equations especially in nonlinear cases poses significant analytical and computational challenges.

Recent advancements have shown promise in tackling such equations through integral transformations. The Kamal Integral Transformation (KIT), in particular, has been applied successfully to linear differential equations, offering a pathway to simplification and solution [(Mushtt, 2023), (Sedeeg, 2016), (Kamal & Hosseini, 2017)]. Despite these successes, the application of KIT to nonlinear fractional-order equations remains underexplored.

This research aims to bridge that gap by proposing a novel method that combines the strengths of the Adomian Polynomial (AP) technique with the Kamal Integral Transformation. The integrated approach is designed to offer a systematic, accurate, and efficient framework for solving nonlinear fractional-order differential equations. By developing and validating this methodology, the study seeks to provide a valuable tool for both theoretical analysis and practical applications across fields such as physics, biology, and engineering.

Basic Theory

Kamal Transformation

Definition 1:

The Kamal's transformation is defined as

$$\mathrm{K}[f(t)] = G(v) = \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2. \qquad ...(1)$$

Furthermore, the inverse of the Kamal transformation is denoted by

$$\mathrm{K}^{-1}[G(v)] = f(t), \quad t \geq 0. \qquad ...(2)$$

The following are some cases of Kamal transformations for the following simple functions:

- $f(t)=1$ where $t\geq 0,$ based on Eqn. (1), we get

$$\mathrm{K}[1]=G(v)=\int_0^{\infty} e^{-\frac{t}{v}}dt=\left[-ve^{-\frac{t}{v}}\right]_0^{\infty}=v. \qquad ...(3)$$

- $f(t)=t$ where $t\geq 0,$ based on Eqn. (1), we get

$$\mathrm{K}[t]=G(v)=\int_0^{\infty} te^{-\frac{t}{v}}dt=v^2. \qquad ...(4)$$

For $f(t)=t^n$ where $t\geq 0,$ we get

$$\mathrm{K}[t^n]=n!v^{n+1}. \qquad ...(5)$$

Based on that, $n=\alpha$, where α is a fractional number, then applies

$$\mathrm{K}[t^{\alpha}]=\Gamma(\alpha+1)v^{\alpha+1} \qquad ...(6)$$

Definition 2:

The Caputo fractional derivative of the function f with respect to t of order α, where $\alpha>0,$ is defined as

$$D^{\alpha}[f(t)]=\frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^{t}(t-u)^{n-\alpha-1}f^{(n)}(u)du, \quad n-1<\alpha\leq n. \qquad ...(7)$$

Theorem 1:

Let $G(v)$ be the Kamal transformation of $f(t)$, then

- $\mathrm{K}[f'(t)]=\frac{G(v)}{v}-f(0),$
- $\mathrm{K}[f''(t)]=\frac{G(v)}{v^2}-\frac{f(0)}{v}-f'(0),$
- $\mathrm{K}[f^{n}(t)]=\frac{G(v)}{v^n}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{n-k-1}}.$

Theorem 2:

The Kamal transformation of Caputo fractional derivative for $\alpha = 0$ is defined as

- $\mathrm{K}\left[D^{\alpha} f(t)\right] = \frac{G(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{\alpha-k-1}}, \qquad n-1 < \alpha \le n.$
- $\mathrm{K}\left[f''(t)\right] = \frac{G(v)}{v^2} - \frac{f(0)}{v} - f'(0),$

Table 1: Kamal Transform Of Useful Functions

S/N	$f(t)$	$\mathrm{K}\left[f(t)\right] = G(v)$
1	1	v
2	t	v^2
3	t^2	$2!v^3$
4	$t^n, n > 0$	$n!v^{n+1}$
5	e^{at}	$\frac{v}{1-av}$
6	$\sin at$	$\frac{av^2}{1+a^2v^2}$
7	$\cos at$	$\frac{v}{1+a^2v^2}$
8	$\sinh at$	$\frac{av^2}{1-a^2v^2}$
9	$\cosh at$	$\frac{v}{1-a^2v^2}$

Table 2: Inverse Kamal Transform Of Useful Functions

S/N	$G(v)$	$f(t) = K^{-1}\left[G(v)\right]$
1	v	1
2	v^2	t
3	$2!v^3$	t^2
4	$n!v^{n+1}$	$t^n, n > 0$

5	$\frac{v}{1-av}$	e^{at}
6	$\frac{av^2}{1+a^2v^2}$	$\sin at$
7	$\frac{v}{1+a^2v^2}$	$\cos at$
8	$\frac{av^2}{1-a^2v^2}$	$\sinh at$
9	$\frac{v}{1-a^2v^2}$	$\cosh at$

Basic Properties

Hence, we have the following properties:

i. $J^{\alpha}J^{v}f = J^{\alpha+v}f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

ii. $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}\ ,\ \alpha > 0\ , \gamma > -1\ ,\ x > 0$

iii. $J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0)\frac{x^{k}}{k!}\ \ ,\ x > 0\ ,\ m-1 < \alpha \leq m$

iv. $D^{\alpha}J^{\alpha}\ f(x) = f(x),\ x > 0\ ,\ m-1 < \alpha \leq m$

v. $D^{\alpha}C = 0,\ where\ C\ is\ a\ constant.$

vi. $D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}\ ,\ \beta \epsilon N_0\ \ ,\ \ \beta \geq \alpha$

where α is an integer and N_0 are natural numbers

vii. The n-fold integral formula

$$\int_a^{x_1}\int_a^{x_2}\dots\int_a^{x_n} u(x_n)dx_n \dots dx_2 dx_1$$

$$= \frac{1}{(n-1)!}\int_a^{x}(x-t)^{n-1}u(t)\,dt \qquad \dots (8)$$

The Combined Kamal and Adomian Polynomial for the Solution of Fractional Differential Equation

The combined method is the method obtained from the combination of the Adomian's polynomial method with the "Kamal" integral transformation (APKIT). This section presents solutions to fractional differential equations using the combined method.

Given the fractional differential equation as follows:

$$D^{\alpha} y(t) = g(t) + Ny(t) + Ry(t) \quad ...(9)$$

with the initial condition $y(0) = c$, where D^{α} is the Caputo fractional derivative operator and $0 < \alpha \leq 1$. Furthermore, Eqn. (9) is transformed using the Kamal transformation, we get

$$K\left[D^{\alpha} y(t)\right] = K\left[g(t) + Ny(t) + Ry(t)\right], \quad ...(10)$$

Where

$$K\left[D^{\alpha} y(t)\right] = \frac{y(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{v^{\alpha-k-1}} \quad ...(11)$$

$n-1 < \alpha \leq n$

$$\frac{y(v)}{v^{\alpha}} - \frac{y(0)}{v^{\alpha-1}} - = K\left[g(t)\right] + K\left[Ny(t)\right] + K\left[Ry(t)\right], \quad ...(12)$$

$$y(v) - vy(0) = v^{\alpha} K\left[g(t)\right] + v^{\alpha} K\left[Ny(t)\right] + v^{\alpha} K\left[Ry(t)\right], \quad ...(13)$$

$$y(v) = vy(0) + v^{\alpha} K\left[g(t)\right] + v^{\alpha} K\left[Ny(t)\right] + v^{\alpha} K\left[Ry(t)\right]. \quad ...(14)$$

Furthermore, using the inverse of the Kamal transform:

$K^{-1}\left[y(v)\right] = y(t)$ and $K^{-1}\left[v\right] = 1$, in Eqn. (14), we get

$$K^{-1}\left[y(v)\right] = K^{-1}\left[vy(0) + v^{\alpha} K\left[g(t)\right]\right] + K^{-1}\left[v^{\alpha} K\left[Ny(t)\right] + v^{\alpha} K\left[Ry(t)\right]\right], \quad ...(15)$$

$$y(t) = y(0) + K^{-1}\left[v^{\alpha} K\left[g(t)\right]\right] + K^{-1}\left[v^{\alpha} K\left[Ny(t)\right] + v^{\alpha}\left[KRy(t)\right]\right] \quad ...(16)$$

The Iterative Method assumes that the function *y* can be decomposed into an infinite series as follows:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \quad ...(17)$$

where $y_n(t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(t) = \sum_{n=0}^{\infty} A_n(t), \quad ...(18)$$

where A_n is the Adomian's polynomial, defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k w_k\right)\right]_{\lambda=0}, \qquad n = 0,1,2,... \qquad ...(19)$$

Where λ is a parameter. Adomian polynomial A_n can be described as follows:

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k w_k\right)\right]_{\lambda=0} = N(y_0),$$

$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k w_k\right)\right]_{\lambda=0} = y_1 N'(y_0),$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k w_k\right)\right]_{\lambda=0} = y_2 N'(y_0) + \frac{y_1^2}{2!}N''(y_0),$$

$$\vdots$$

Substituting Eqns. (17) and (18) into Eqn. (16), we get

$$\sum_{n=0}^{\infty} y_n(t) = y(0) + K^{-1}\left[v^\alpha K[g(t)]\right] + K^{-1}\left[v^\alpha K[\sum_{n=0}^{\infty} A_n(t)]\right] + K^{-1}\left[v^\alpha K[R\sum_{n=0}^{\infty} y_n(t)]\right]. \quad ...(20)$$

So, the recursive relation is obtained from the solution of fractional differentiation using the Kamal decomposition as follows:

$$y_0(t) = y(0) + K^{-1}\left[v^\alpha K[g(t)]\right], \qquad ...(21)$$

$$y_{n+1}(t) = K^{-1}\left[v^\alpha K[\sum_{n=0}^{\infty} A_n(t)]\right] + K^{-1}\left[v^\alpha K[R\sum_{n=0}^{\infty} y_n(t)]\right] \qquad ...(22)$$

$n = 0,1,2,...$

Results and Discussion

This section presents some numerical examples of solving fractional differential equations using the combine Kamal transform and the Adomian polynomials method discussed above

Example 1

Given the fractional differential equation as follows:

$$D^\alpha y(t) = y^2(t) + 1, \qquad t > 0, \quad 0 < \alpha \le 1, \qquad ...(23)$$

With the initial condition

$$y(0) = 0 \qquad ...(24)$$

And the exact solution

$$y(t) = \tan(t). \qquad ...(25)$$

Using the recursive formulas in Eqn. (21) and (22), the approximate solution to the fractional differential Eqn. (23) is derived through the combined method of the Adomian polynomial and the Kamal integral transformation, as follows:

$$\begin{cases} y_0(t) = K^{-1}\left[v^{\alpha} K[1]\right], \\ y_{n+1}(t) = K^{-1}\left[v^{\alpha} K[\sum_{n=0}^{\infty} A_n(t)]\right], \; n = 0,1,2,... \end{cases} \qquad ...(26)$$

where A_n is the Adomian polynomial of the nonlinear operator $Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0 y_1, \\ A_2 = 2y_0 y_2 + y_1^2, \\ \vdots \end{cases} \qquad ...(27)$$

The following is a description of the solution to the Eqn. (26):

$$y_0(t) = K^{-1}\left[v^{\alpha} K[1]\right] = K^{-1}\left[v^{\alpha} v\right] = K^{-1}\left[v^{\alpha+1}\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1(t) = K^{-1}\left[v^{\alpha} K[A_0]\right] = K^{-1}\left[v^{\alpha} K y_0^2\right] = K^{-1}\left[v^{\alpha} K\left[\frac{x^{2\alpha}}{\Gamma^2(\alpha+1)}\right]\right] = K^{-1}\left[v^{\alpha}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} v^{2\alpha+1}\right]\right]$$

$$= K^{-1}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} v^{3\alpha+1}\right] = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha},$$

$$y_2(t) = K^{-1}\left[v^{\alpha} K[A_1]\right] = K^{-1}\left[v^{\alpha} K 2y_0 y_1\right] = K^{-1}\left[v^{\alpha} K\left[2\frac{t^{\alpha}}{\Gamma(\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}\right]\right]$$

$$= K^{-1}\left[v^{\alpha} K\left[\frac{2\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{4\alpha}\right]\right] = K^{-1}\left[v^{\alpha}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{4\alpha+1}\right]\right]$$

$$= K^{-1}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{5\alpha+1}\right] = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha}$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} + \ldots$$

Figure 1 presents the approximate solution to Eqn. (23) for the second iteration of the fractional differential equation using the Combined Method, with varying α values: α = 0.9, α = 0.8,

α = 0.7, and α = 1, generated with the help of Excel and MAPLE software.

Table: 3. The numerical solution of the fractional differential equation (Eqn. 23) using the Combination of Adomian polynomial and the Kamal Integral Transform for α=0.9, α=0.8, α=0.7 and α=1

T	APKIT α=1	APKIT α=0.9	APKIT α=0.8	APKIT α=0.7
0	0	0	0	0
0.1	0.1003347	0.13177167	0.13177167	0.22521847
0.2	0.2027093	0.25006258	0.25006258	0.38221117
0.3	0.309324	0.36969838	0.36969838	0.53741796
0.4	0.4226987	0.4961631	0.4961631	0.70405938
0.5	0.5458333	0.63427755	0.63427755	0.89091018
0.6	0.682368	0.78915309	0.78915309	1.10562379
0.7	0.8367427	0.96655629	0.96655629	1.35561347
0.8	1.0143573	1.17311494	1.17311494	1.64837174
0.9	1.221732	1.41646192	1.41646192	1.99160513
1	1.4666667	1.70534864	1.70534864	2.39329533

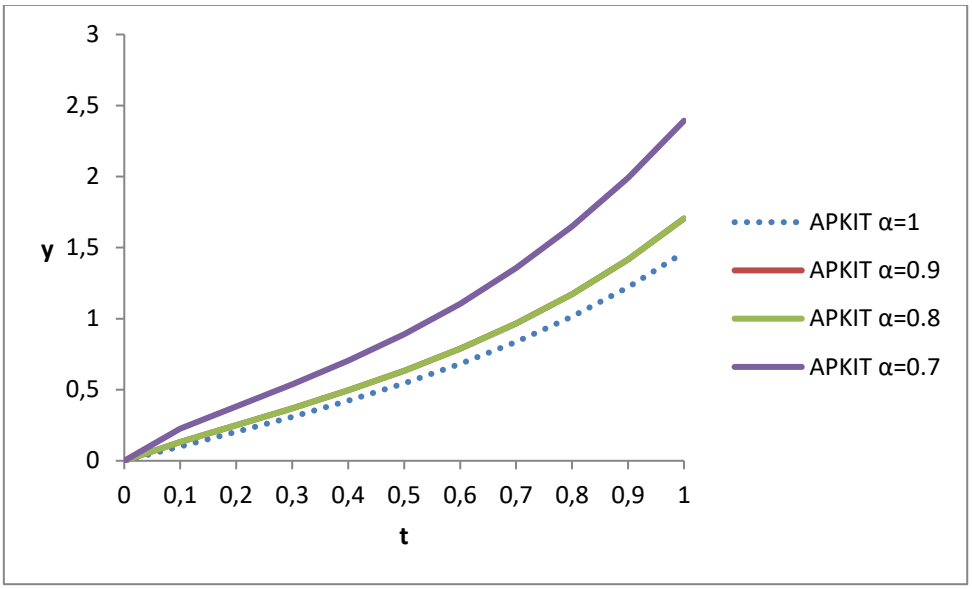


Figure 1. Graph of the solution to the fractional differential equation using the combination of the Adomian polynomial and the Kamal Integral Transform for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$.

Table: 4.The numerical value of the exact solution and the approximation solution using the Combined Method for α=1

t	EXACT	APKIT α=1	ERROR
0	0	0	0
0.1	0.100335	0.1003347	5.42E-09
0.2	0.20271	0.2027093	7.02E-07
0.3	0.309336	0.309324	1.22E-05
0.4	0.422793	0.4226987	9.46E-05
0.5	0.546302	0.5458333	0.000469
0.6	0.684137	0.682368	0.001769
0.7	0.842288	0.8367427	0.005546
0.8	1.029639	1.0143573	0.015281
0.9	1.260158	1.221732	0.038426
1	1.557408	1.4666667	0.090741

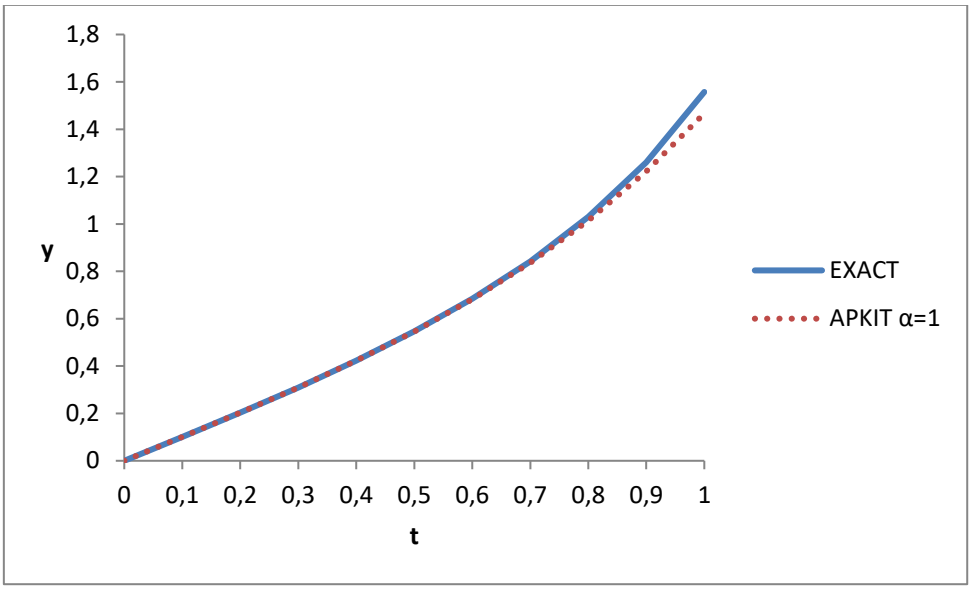


Figure 2. Graph of the exact solution, which coincides with the approximate solution obtained using the Combined Method for $\alpha = 1$.

Furthermore, Figure 2 presents a graph where the exact solution of Eqn. (23) coincides with the approximate solution obtained using the Combined Method for $\alpha = 1$, generated with the help of MS Excel and MAPLE software.

From Figure 2, it is clear that the graph of the exact solution aligns perfectly with the graph of the approximate solution for Eqn. (23) at $\alpha = 1$. This indicates that the combination of the Adomian polynomial and the Kamal Integral Transformation is highly accurate in solving fractional-order differential equations.

Example 2

Given the fractional differential equation as follows [(Opanuga *et al.*, 2015)]:

$$D^{\alpha} y(t) = 1 - y^2(t), \qquad t > 0, \quad 0 < \alpha \leq 1, \qquad \ldots(28)$$

With the initial condition

$$y(0) = 0 \qquad \ldots(29)$$

And the exact solution

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \qquad \ldots(30)$$

Based on the recursive formula in Eqn. (21) and (22), the approximate solution of the fractional differential Eqn. (28) uses the combined method of the Adomian polynomial and the Kamal integral transformation as follows:

$$\begin{cases} y_0(t) = K^{-1}\left[v^{\alpha} K[1]\right], \\ y_{n+1}(t) = -K^{-1}\left[v^{\alpha} K[\sum_{n=0}^{\infty} A_n(t)] \right], \qquad n = 0,1,2,... \end{cases} \qquad ...(31)$$

where A_n is the Adomian polynomial of the nonlinear operator $Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0 y_1, \\ A_2 = 2y_0 y_2 + y_1^2, \\ \vdots \end{cases} \qquad ...(32)$$

The following is a description of the solution to the Eqn. (28):

$$y_0(t) = K^{-1}\left[v^{\alpha} K[1]\right] = K^{-1}\left[v^{\alpha} v\right] = K^{-1}\left[v^{\alpha+1}\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1(t) = -K^{-1}\left[v^{\alpha} K[A_0]\right] = -K^{-1}\left[v^{\alpha} K y_0^2\right] = -K^{-1}\left[v^{\alpha} K\left[\frac{x^{2\alpha}}{\Gamma^2(\alpha+1)} \right] \right]$$

$$= -K^{-1}\left[v^{\alpha}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} v^{2\alpha+1} \right] \right] = -K^{-1}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} v^{3\alpha+1} \right] = -\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha},$$

$$y_2(t) = -K^{-1}\left[v^{\alpha} K[A_1]\right] = -K^{-1}\left[v^{\alpha} K 2y_0 y_1\right] = -K^{-1}\left[v^{\alpha} K\left[2\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha} \right) \right] \right]$$

$$= K^{-1}\left[v^{\alpha} K\left[\frac{2\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{4\alpha} \right] \right] = K^{-1}\left[v^{\alpha}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{4\alpha+1} \right] \right]$$

$$= K^{-1}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} t^{5\alpha+1} \right] = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha}$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + ...$$

$$= -\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} + ...$$

Figure 3 presents a graph of the approximate solution to Eqn. (28) for the second iteration of the fractional differential equation using the Combined Method, with varying values of α: $\alpha = 0.7$,

$\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$, generated using MS Excel and Maple software.

Additionally, Figure 4 displays a graph of the exact solution to Eqn. (28), where the green curve coincides with the approximate solution (represented by black dots) using the Combined Method for $\alpha = 1$, also created with MS Excel and Maple software.

From Figure 4, it is evident that the green curve, representing the exact solution, aligns perfectly with the black-dot graph of the approximate solution for $\alpha = 1$. This demonstrates that the Combined Method of the Adomian polynomial and the Kamal Integral Transformation is highly accurate in solving fractional-order differential equations.

Table: 5. The numerical solution of the fractional differential equation (Eqn. 28) using the Combination of Adomian polynomial and the Kamal Integral Transform for α=0.9, α=0.8, α=0.7 and α=1

t	APKIT α=1	APKIT α=0.9	APKIT α=0.8	APKIT α=0.7
0	0	0	0	0
0.1	0.099668	0.130037517	0.168003035	0.214342304
0.2	0.197376	0.238793994	0.285273387	0.335584046
0.3	0.291324	0.336022738	0.382167028	0.42816573
0.4	0.380032	0.42293959	0.464358727	0.504164513
0.5	0.4625	0.500523314	0.535903434	0.571526543
0.6	0.538368	0.570328099	0.600987125	0.637249244
0.7	0.608076	0.63477396	0.664397476	0.708200283
0.8	0.673024	0.697307193	0.731697823	0.791403688
0.9	0.735732	0.762513756	0.809303466	0.894154663
1	0.8	0.83621102	0.904518895	1.024067693

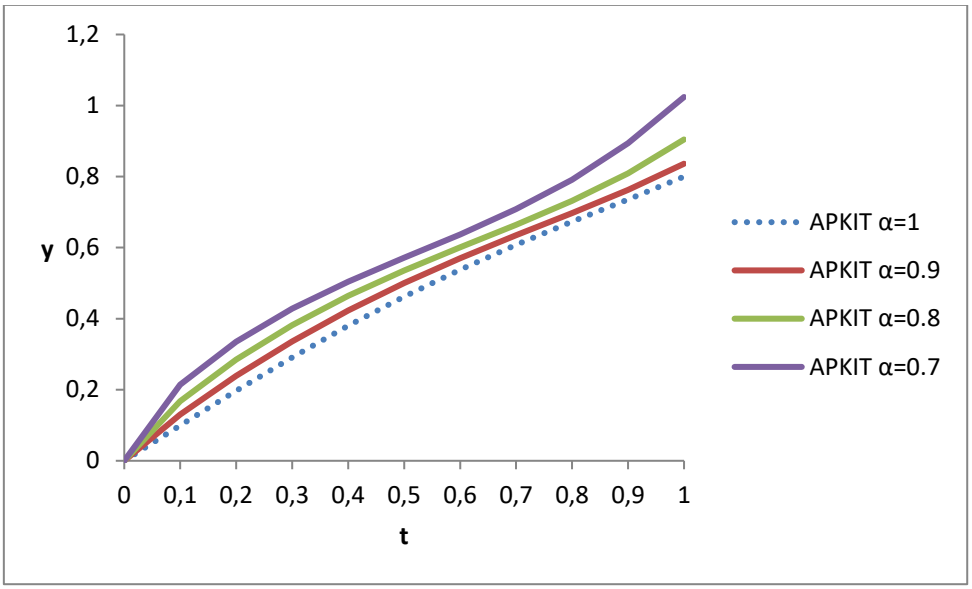


Figure 3. Graph of the solution to the fractional differential equation using the Combined Method of the Adomian polynomial and the Kamal Integral Transform for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$.

Table: 6.The numerical value of the exact solution and the approximation solution using the Combined Method for α=1

t	EXACT	APKIT α=1	ERROR
0	0	0	0
0.1	0.099668	0.099668	5.38E-09
0.2	0.197375	0.197376	6.8E-07
0.3	0.291313	0.291324	1.14E-05
0.4	0.379949	0.380032	8.3E-05
0.5	0.462117	0.4625	0.000383
0.6	0.53705	0.538368	0.001318
0.7	0.604368	0.608076	0.003708
0.8	0.664037	0.673024	0.008987
0.9	0.716298	0.735732	0.019434
1	0.761594	0.8	0.038406

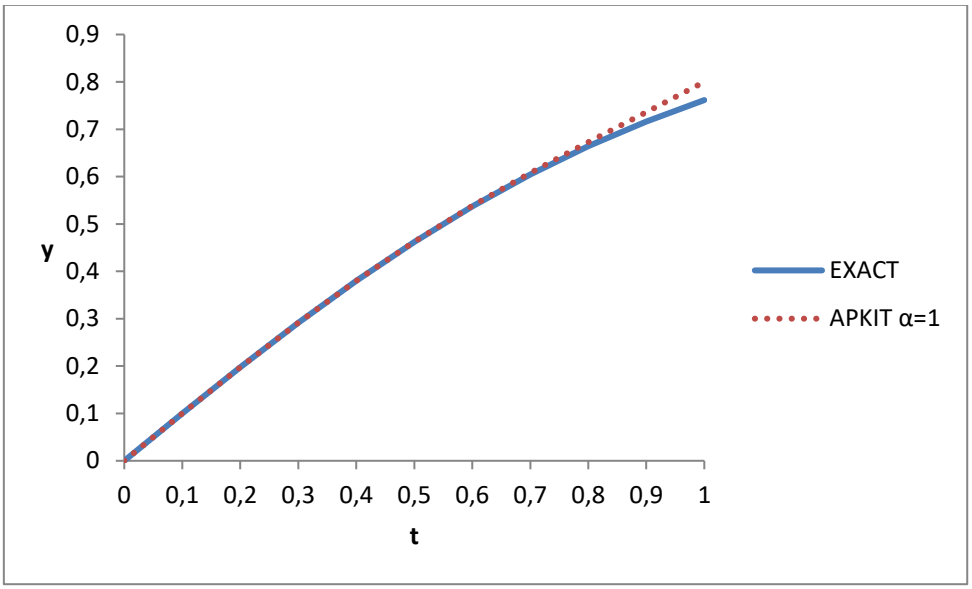


Figure 4. Graph showing the exact solution, which coincides with the approximate solution obtained using the Combined Method for $\alpha = 1$.

Example 3

Given the fractional differential equation as follows [(Odetunde & Taiwo, 2014)]:

$$D^{\alpha} y(t) = -y(t) + y^2(t), \qquad t > 0, \quad 0 < \alpha \leq 1, \qquad \ldots(33)$$

With the initial condition

$$y(0) = \frac{1}{2} \qquad \ldots(34)$$

And the exact solution

$$y(t) = \frac{e^{-t}}{1+e^{-t}}. \qquad \ldots(35)$$

Using the recursive formulas in Eqn. (21) and (22), the approximate solution to the fractional differential Eqn. (33) is obtained through the combined method of the Adomian polynomial and the Kamal integral transformation, as follows:

$$\begin{cases} y_0(t) = \frac{1}{2}, \\ y_{n+1}(t) = -K^{-1}\left[v^{\alpha} K[R_n]\right] + K^{-1}\left[v^{\alpha} K[\sum_{n=0}^{\infty} A_n(t)]\right], \qquad n = 0,1,2,\ldots \end{cases} \qquad \ldots(36)$$

where A_n is the Adomian polynomial of the nonlinear operator $Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0y_1, \\ A_2 = 2y_0y_2 + y_1^2, \\ \vdots \end{cases} \quad \ldots(37)$$

The following is a description of the solution to the Eqn. (33):

$$y_0(t) = \frac{1}{2},$$

$$y_1(t) = -K^{-1}\left[v^\alpha K[R_0]\right] + K^{-1}\left[v^\alpha K[A_0]\right],$$

$$y_1(t) = -K^{-1}\left[v^\alpha K[y_0]\right] + K^{-1}\left[v^\alpha K[y_0^2]\right]$$

$$y_1(t) = -K^{-1}\left[v^\alpha K\left[\frac{1}{2}\right]\right] + K^{-1}\left[v^\alpha K\left[\frac{1}{4}\right]\right]$$

$$= -K^{-1}\left[v^\alpha K\left[\frac{1}{4}\right]\right] = -K^{-1}\left[v^\alpha\left[\frac{1}{4}v^\alpha v\right]\right]$$

$$= -K^{-1}\left[v^\alpha\left[\frac{1}{4}v^{\alpha+1}\right]\right] = -\frac{1}{4\Gamma(\alpha+1)}t^\alpha,$$

$$y_2(t) = -K^{-1}\left[v^\alpha K[R_1]\right] + K^{-1}\left[v^\alpha K[A_1]\right],$$

$$y_2(t) = -K^{-1}\left[v^\alpha K[y_1]\right] + K^{-1}\left[v^\alpha K[2y_0y_1]\right]$$

$$= -K^{-1}\left[v^\alpha K\left[-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right]\right],$$

$$= K^{-1}\left[v^\alpha K\left[2\frac{1}{2}\left(-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right)\right]\right]$$

$$= K^{-1}\left[v^\alpha K\left[-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right]\right]$$

$$= -K^{-1}\left[v^\alpha K\left[-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right]\right] = 0$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$= \frac{1}{2} - \frac{1}{4\Gamma(\alpha+1)} t^{\alpha} + 0 + \ldots$$

Figure 5 displays the graph of the approximate solution to Eqn. (33) for the second iteration of the fractional differential equation using the Combined Method, with different values of α: $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$. The graph was generated using MS Excel and Maple software. Additionally, Figure 6 presents the graph of the exact solution of Eqn. (33) alongside the approximate solution obtained using the Combined Method for $\alpha = 1$, also created with MS Excel and Maple software.

From Figure 6, it is evident that the graph of the exact solution of Eqn. (33) matches closely with the approximate solution for $\alpha = 1$. This demonstrates the high accuracy of the Combined Method in solving fractional-order differential equations.

Table: 7. The numerical solution of the fractional differential equation (Eqn. 33) using the Combination of Adomian polynomial and the Kamal Integral Transform for α=0.9, α=0.8, α=0.7 and α=1

t	APKIT α=1	APKIT α=0.9	APKIT α=0.8	APKIT α=0.7
0	0.5	0.5	0.5	0.5
0.1	0.475	0.467275677	0.45745864	0.44510298
0.2	0.45	0.438934255	0.42593119	0.41081953
0.3	0.425	0.412041099	0.39755085	0.38155039
0.4	0.4	0.38604729	0.37103872	0.3551259
0.5	0.375	0.360702389	0.34583445	0.3306334
0.6	0.35	0.335862887	0.32162567	0.30757804
0.7	0.325	0.311435949	0.29821458	0.28565301
0.8	0.3	0.287356725	0.27546537	0.26465133
0.9	0.275	0.263577436	0.25327944	0.24442491
1	0.25	0.240061467	0.23158218	0.22486315

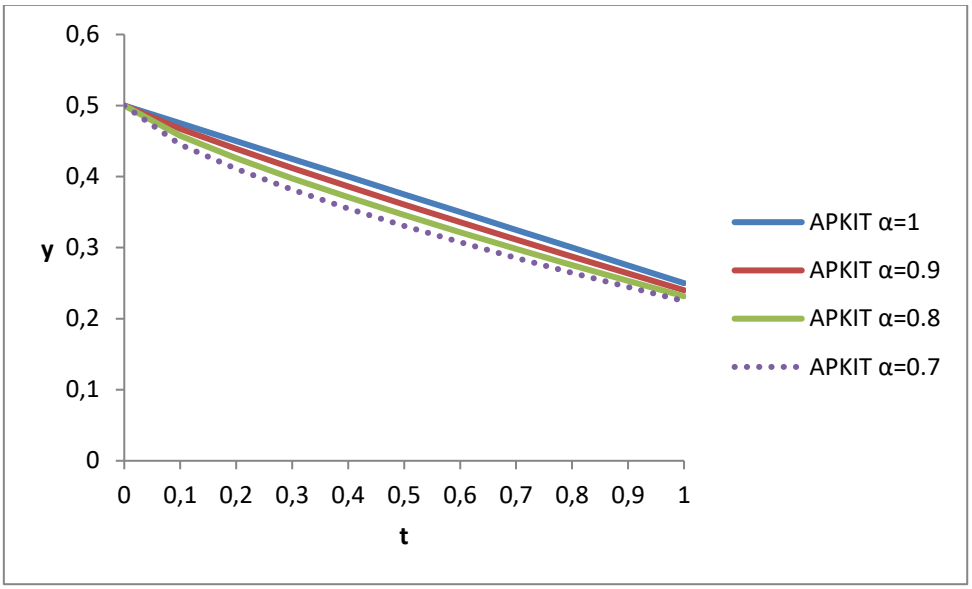


Figure 5. Graph of the solution to the fractional differential equation using the Combined Method for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$.

Table: 8.The numerical value of the exact solution and the approximation solution using the Combined Method for α=1

t	EXACT	APKIT α=1	ERROR
0	0.5	0.5	0
0.1	0.475021	0.475	2.08E-05
0.2	0.450166	0.45	0.000166
0.3	0.425557	0.425	0.000557
0.4	0.401312	0.4	0.001312
0.5	0.377541	0.375	0.002541
0.6	0.354344	0.35	0.004344
0.7	0.331812	0.325	0.006812
0.8	0.310026	0.3	0.010026
0.9	0.28905	0.275	0.01405
1	0.268941	0.25	0.018941

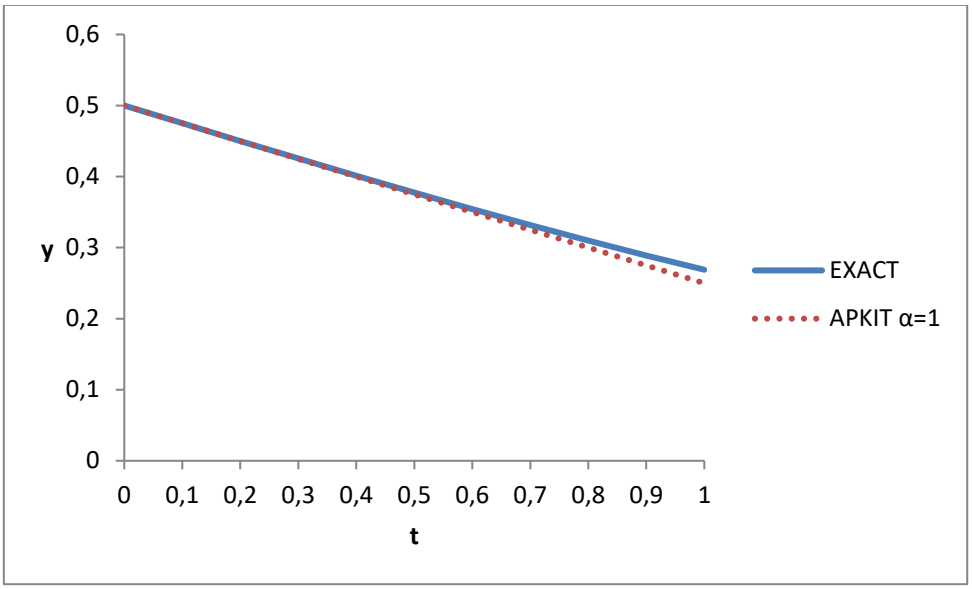


Figure 6. Graph showing the exact solution alongside the approximate solution obtained using the Combined Method for $\alpha = 1$.

Conclusion

This study presents a method for solving fractional-order differential equations by combining the Adomian polynomial with the Kamal integral transformation. We also explored the solution of fractional differential equations using this combined approach. The main conclusion from our work is that this combined method is highly accurate, straightforward, and effective for solving fractional-order differential equations. It simplifies the process of determining the terms in the approximation solution of such equations.

References

Adomian, G. (1994). Solving Frontier Problems of Physics: The Decomposition Method. *Kluwer Academic Publishers*.

Diethelm, K., Ford, N. J., & Freed, A. D. (2002). A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29(1-4), 3-22.

Elzaki, T. M. (2011). The new integral transform "Elzaki transform." *Global Journal of Pure and Applied Mathematics*, *7*(1), 57–64.

Ghanwat, A. J., & Gaikwad, S. B. (2022). Application of Kamal Transform for Solving Linear Volterra Integral Equations of Second Kind. *Journal of Emerging Technologies and Innovative Research*, *9*(4), 7–14.

Hussain, E. A., & Jasim, A. S. (2021). Z-transform solution for nonlinear difference equations. *Al-Mustansiriyah Journal of Science*, *32*(4), 51–56. https://doi.org/10.23851/mjs.v32i4.1019

Janolkar, M. M. (2018). Solution of Ordinary Differential Equations with variable coefficients using Kamal Transform. *International Journal of Scientific Research and Review*, *7*(3), 173–178.

Kamal, A. H. M., &Hosseini, K. (2017). An analytical approach to fractional-order differential equations using Kamal integral transform. *Computers & Mathematics with Applications*, 73(10), 2031-2044.

Khandelwal, R., Kumawat, P., & Khandelwal, Y. (2018). Kamal decomposition method and its application in solving coupled system of nonlinear PDE's. *Malaya Journal of Matematik*, *06*(03), 619–625. https://doi.org/10.26637/mjm0603/0024

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). *Elsevier*.

Mushtt, I. Z. (2023). A Novel Integral Transform : INEM - Transform. *Journal of Kufa for Mathematics and Computer*, *10*(2), 109–113.

Odetunde, O. S., & Taiwo, O. A. (2014). A Decomposition Algorithm for the Solution of Fractional Quadratic Riccati Differential Equations with Caputo Derivatives. *American Journal of Computational and Applied Mathematics*, *4*(3), 83–91. https://doi.org/10.5923/j.ajcam.20140403.03

Opanuga, A. A., Edeki, S. O., Okagbue, H. I., & Akinlabi, G. O. (2015). A novel approach for solving quadratic Riccati differential equations. *International Journal of Applied Engineering Research*, *10*(11), 29121–29126.

Owolabi, J. A., & Oderinu, R. A. (2021). Kamal Transform Based Analytical Solution of a Generalized Nonlinear Hirota-Satsuma Coupled Equations 2 Properties of Kamal Transform. *Asian Journal of Pure and Applied Mathematics*, *3*(1), 69–78.

Rustam, Z. R., & Sulaiman, N. A. (2023). Kamal transform technique for solving system of linear Volterra integro-differential equations of the second kind. *Polytechnic Journal*, *12*(2), 6–16. https://doi.org/10.25156/ptj.v12n2y2022.pp6-16

Sedeeg, A. K. . H. (2016). The New Integral Transform' Kamal Transform'. *Advances in Theoretical and Applied Mathematics*, *11*(4), 451–458.

Shah, R., Khan, H., Kumam, P., Arif, M., & Baleanu, D. (2019). Natural transform decomposition method for solving fractional-order partial differential equations with proportional delay. *MDPI*, *7*(6), 1–14. https://doi.org/10.3390/MATH7060532