MIKAILALSYS
Journal of Multidisciplinary Sciences

e-ISSN : 2987-2286
p-ISSN : 2987-3924

Index : Dimensions, Scilit, Crossref, Garuda, Semantic, Lens,Google Scholar,Base, etc

https://doi.org/10.58578/mikailalsys.v4i1.8106

A Hybrid Elzaki Transform-Daftardar-Jafari Method for Solving Nonlinear Proportional Delay Differential Equations

Sanda L. N.[1], Okai J. O.[2], Nasir U.M.[3], U. A. Mujahid[4], Michael Cornelius[5], Ndam G.S.[6]
[1,2,3,6]Abubakar Tafawa Balewa University Bauchi State, Nigeria; [4]Federal Polytechnic Kaltungo, Gombe State, Nigeria; [5]Gombe State University, Gombe State, Nigeria
sandaperfect@gmail.com; jerry4complex@gmail.com

Article Info:

Submitted:	Revised:	Accepted:	Published:
Nov 27, 2025	Dec 25, 2025	Jan 20, 2026	Jan 28, 2026

Abstract

Proportional delay differential equations (PDDEs) arise naturally in viscoelasticity, control theory, biology, population dynamics, and fractional-order physical models in which the future state depends on the value of the solution at a proportion of the current time, but their nonlinear nature and delay terms make analytic treatment challenging. This study develops a hybrid computational scheme that combines the Elzaki Transform (ET) and the Daftardar–Jafari Method (DJM) to obtain accurate analytical–approximate solutions for linear and nonlinear PDDEs. In the proposed approach, the Elzaki transform converts the PDDE into an algebraic functional equation, which is subsequently decomposed using DJM without the need for Adomian polynomials. The method is straightforward, computationally efficient, and capable of handling strong nonlinearities. Several illustrative examples are presented to demonstrate its efficiency, and the results confirm that the ET–DJM hybrid provides a powerful alternative to classical methods such as the Laplace transform, Adomian Decomposition Method (ADM), Variational

Mikailalsys is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License

Iteration Method (VIM), Homotopy Perturbation Method (HPM), and homotopy analysis methods.

Keywords: Elzaki Transform; Daftardar–Jafari Method; Proportional Delay Differential Equations; Nonlinear Delay Systems; Analytical–Approximate Solutions.

Introduction

Differential equations with delays have long been used to model dynamical processes whose present rate of change depends on past states of the system. Among these, proportional delay differential equations (PDDEs) of the form

$$f\left(t, y(\beta_0 t), y'(\beta_1 t), ..., y^{(n)}(\beta_n)\right), \quad 0 \prec \beta_i \prec 1, i = 0,1,2,...,n \qquad ...(1)$$

represent problems where delay is relative to the current time, not a fixed quantity. This relative delay structure appears in:

- population dynamics with maturation time proportional to age,
- epidemiological models with proportional incubation periods,
- viscoelastic relaxation,
- thermal systems with scale-dependent memory effects,
- engineering feedback systems involving fractal-time delays.

Recent Progress on Delay Differential Equations.

In the last decade, research on DDEs especially those with proportional delay has increased significantly. Recent studies include:

- stability and oscillation theory for PDDEs (Bawa & Ogundare, 2022),
- analytical approximations using hybrid transforms (Shaikh et al., 2020; Awodola, 2021),
- nonlinear PDDEs in epidemiology (Alsharif & Hattaf, 2023),
- proportional multi-delay models in neural networks (Chen et al., 2023).

Transform Methods and Hybrid Approaches.

Integral transforms remain essential tools for solving differential equations. The Elzaki Transform, introduced as an efficient alternative to Laplace, has recently gained

popularity due to its linearity, ease of convolution, and suitability for handling singular kernels (Elzaki, 2012; Suleiman et al., 2019; Waziri & Yakubu, 2023).

The Daftardar–Jafari Method (DJM) (Daftardar-Gejji & Jafari, 2006) is a modern iterative decomposition method that avoids the complexity of Adomian polynomials while retaining convergence efficiency. Applications of DJM include fractional calculus (Rida & Saad, 2021), nonlinear physical systems (Atangana, 2020), and hybrid fractional–delay models (Liao & Xu, 2022).

Motivation for the Hybrid ET–DJM Approach.

Although the Elzaki transform simplifies delay differential equations algebraically, the resulting functional equations are often nonlinear and difficult to invert explicitly. DJM, on the other hand, handles nonlinearities gracefully but may struggle when dealing with complicated delay kernels alone.

By combining the two methods, we obtain a scheme capable of:

- converting PDDEs into solvable algebraic-functional forms,
- decomposing nonlinear terms without Adomian polynomials,
- providing rapidly convergent approximate solutions,

This combination has not been extensively explored in this field making this study a significant contribution.

Preliminaries and Basic Notations.

In this section, we present some basic definitions and properties used in our study.

Elzaki Transform.

Definition 2.1. Elzaki Transform denoted by the operator $E[.]$, is defined by the integral equation

$$E[y(t)] = r\int_0^\infty y(t)e^{-\frac{t}{r}}dt,\ \ k_1 \le r \le k_2,\ t \ge 0 \qquad ...(2)$$

Some key ET properties used in this work (standard in Elzaki-transform literature) include:

Common Properties:

$1.E[1]=r^2$

$2.E[t^n]=n!r^{n+2}$

$3.E^{-1}[r^{n+2}]=\frac{t^n}{n!}, n\in N$

The last property is called inverse Elzaki operator.

Derivative Properties:

Let $E[y(t)]$ be the Elzaki transform of $y(t)$, then

$$E[y^{(n)}(t)]=\frac{E[y(t)]}{r^n}-\sum_{m=0}^{n-1} r^{2-n+m}y^{(m)}(0),\ n=1,2,3,...$$

Linearity:

$1.\ E[ay_1(t)+by_2(t)]=aE[y_1(t)]+bE[y_2(t)]$

$2.\ E^{-1}[ay_1(t)+by_2(t)]=aE^{-1}[y_1(t)]+bE^{-1}[y_2(t)].$

Daftardar-Jafari Method.

Consider the following general functional equation:

$$y(t)=g(t)+Ny(t) \qquad ...(3)$$

Where $g(t)$ is a known function and N a nonlinear operator. Assume that the series solution of (3) exists, and it is given by

$$y(t)=\sum_{n=0}^{\infty} y_n(t) \qquad ...(4)$$

The nonlinear term $Ny(t)$ can be decompose as:

$$N\left(\sum_{n=0}^{\infty} y_n(t)\right)=N(y_0(t))+\sum_{n=1}^{\infty}\left[N\left(\sum_{m=0}^{n} y_m(t)\right)-N\left(\sum_{m=0}^{n-1} y_m(t)\right)\right] \qquad ...(5)$$

On substituting (4) and (5) in (3), we obtain:

$$y_0(t)+y_1(t)+\sum_{n=2}^{\infty}y_n(t)=g(t)+Ny_0(t)+\sum_{n=1}^{\infty}\left[N\left(\sum_{m=0}^{n}y_m(t)\right)-N\left(\sum_{m=0}^{n-1}y_n(t)\right)\right] \quad ...(6)$$

We now define recurrence relations as follows:

$$\begin{cases} y_0(t)=g(t) \\ y_1(t)=N[y_0(t)] \\ y_2(t)=N[y_0(t)+y_1(t)]-N[y_0(t)] \\ \vdots \\ y_{m+1}(t)=N[y_0(t)+y_1(t)+y_2(t)+...+y_m(t)]-N[y_0(t)+y_1(t)+...+y_{m-1}(t)], m=1,2,3,... \end{cases}$$

$...(7)$

Therefore, the final solution is given by

$$y(t)=g(t)+\sum_{n=1}^{\infty}y_n(t) \quad ...(8)$$

Methodology: The Hybrid Elzaki–DJM Scheme.

Throughout this study, we will consider delay differential equation of the form of (1).

With more detailed form:

$$y^{(n)}(t)+py(t)+N(y(\beta t))=f(t),\ n=1,2,3,... \quad ...(9)$$

Subject to

$$y^{(i)}(0)=y^{i}{}_0,\ i=0,1,2,.... \quad ...(10)$$

Where p is a linear operator and N a nonlinear operator.

Consider proportional delay differential equation (9).

Isolating $y^{(n)}(t)$, we obtain;

$$y^{(n)}(t)=f(t)-py(t)-N(y(\beta t)) \quad ...(11)$$

Step 1: Apply the Elzaki Transform on both sides of (11), we obtain;

$$E[y^{(n)}(t)]=E[f(t)-py(t)-N(y(\beta t))]$$

Transforming both sides and apply initial conditions gives:

$$\frac{E[y(t)]}{r^n} - \sum_{m=0}^{n-1} r^{2-n+m} y^{(m)}(0) = E[f(t)] - E[py(t)] - E[Ny(\beta t)] \quad ...(12)$$

Equivalently,

$$E[y(t)] = r^n C + r^n E[f(t)] - r^n E[py(t)] - r^n E[Ny(\beta t)] \quad ...(13)$$

Where $C = \sum_{m=0}^{n-1} r^{2-n+m} y^{(m)}(0)$.

Step 2: Apply Inverse Elzaki transform on both sides of (13) yields;

$$y(t) = E^{-1}\left[r^n C + r^n E[f(t)]\right] - E^{-1}\left[r^n E[py(t)]\right] - E^{-1}\left[r^n E[Ny(\beta t)]\right] \quad ...(14)$$

Step 3: DJM Decomposition of the Nonlinear Term.

Decompose $N(y(\beta t))$ as:

$$\sum_{n=0}^{\infty} Ny(\beta t) = Ny_0(\beta t) + \sum_{n=1}^{\infty}\left[N\left(\sum_{n=0}^{k} y_n(\beta t)\right) - N\left(\sum_{n=0}^{k-1} y_n(\beta t)\right)\right] \quad ...(15)$$

On replacing $N(y(\beta t))$ in (14) by (15), we obtain;

$$\begin{aligned} y(t) &= E^{-1}\left[r^n C + r^n E[f(t)]\right] - E^{-1}\left[r^n E[py(t)]\right] \\ &- E^{-1}\left[r^n E\left[Ny_0(\beta t) + \sum_{n=1}^{\infty}\left[N\left(\sum_{n=0}^{k} y_n(\beta t)\right) - N\left(\sum_{n=0}^{k-1} y_n(\beta t)\right)\right]\right]\right] \end{aligned} \quad ...(16)$$

From (16), we now define recurrence relation as follows:

$$\begin{cases} y_0(t) = E^{-1}\left[r^n C + r^n E[f(t)]\right] \\ y_1(t) = -E^{-1}\left[r^n E[py_0(t)]\right] - E^{-1}\left[r^n E[Ny_0(\beta t)]\right] \\ \vdots \\ y_{n+1}(t) = -E^{-1}\left[r^n E\left[\sum_{n=1}^{\infty}\left[N\left(\sum_{n=0}^{k} y_n(\beta t)\right) - N\left(\sum_{n=0}^{k-1} y_n(\beta t)\right)\right]\right]\right] \end{cases} \quad ...(17)$$

Therefore, in truncated series form, the approximate analytic solution of the PDDE (3) is given by

$$y(t) = y_0(t) + y_1(t) + y_2(t) + ... \quad ...(18)$$

Numerical Applications.

In this section, algorithm will be applied to obtain solutions of some nonlinear PDDEs.

Example 4.1 Consider the following nonlinear PDDE [Ibrahim M.D., et al, 2025, updated to integer order]:

$$\begin{cases} y'(t) = 2ty^4\left(\frac{t}{2}\right), t \geq 0 \\ y(0) = 1 \end{cases} \quad ...(19)$$

The exact solution is $y(t) = e^{t^2}$.

Solution:

Applying Elzaki transform on both sides of (19), we obtain:

$$E[y'(t)] = E\left[2ty^4\left(\frac{t}{2}\right)\right]$$

Equivalently,

$$\frac{E[y(t)]}{r} - ry(0) = 2E\left[ty^4\left(\frac{t}{2}\right)\right] \quad ...(20)$$

On substituting the initial value in (20) and re-arranging, we obtain:

$$E[y(t)] = r^2 + 2rE\left[ty^4\left(\frac{t}{2}\right)\right] \quad ...(21)$$

Applying inverse Elzaki transform on both sides of (21), we obtain:

$$y(t) = 1 + 2E^{-1}\left[rE\left[ty^4\left(\frac{t}{2}\right)\right]\right] \quad ...(22)$$

From (22), we define recurrence relations:

$$\begin{cases} y_0(t)=1 \\ y_1(t)=2E^{-1}\left[rE\left[t{y_0}^4\left(\frac{t}{2}\right)\right]\right] \\ y_{n+1}(t)=2E^{-1}\left[rE\left[t{y_n}^4\left(\frac{t}{2}\right)\right]\right], n\geq 1 \end{cases} \quad \ldots(23)$$

From (23)

$$\begin{cases} y_1(t)=2E^{-1}\left[rE\left[t(1)^4\right]\right]=t^2 \\ y_2(t)=N\left[y_0\left(\frac{t}{2}\right)+y_1\left(\frac{t}{2}\right)\right]-N\left[y_0\left(\frac{t}{2}\right)\right]=2E^{-1}\left[rE\left[t\left(1+\frac{t^2}{4}\right)^4-t(1)^4\right]\right]=\frac{t^4}{2}+\frac{t^6}{8}+\frac{t^8}{64}+\frac{t^{10}}{1280} \\ \vdots \end{cases} \quad \ldots(24)$$

$$y(t)=y_0(0)+\sum_{n=1}^{\infty} y_n$$

Therefore, $y(t)=1+t^2+\frac{\left(t^2\right)^2}{2!}+\frac{3\left(t^3\right)^2}{4(3!)}+\frac{3\left(t^4\right)^2}{8(4!)}+\frac{3\left(t^5\right)^2}{32(5!)}+\ldots\approx e^{t^2}$

Demonstrating nonlinearity handled without polynomials.

Example 4.2 Consider the following nonlinear PDDE [Ibrahim M.D., et al, 2025, updated to integer order]:

$$\begin{cases} y'(t)=1-2y^2\left(\frac{t}{2}\right), 0\leq t\leq 1 \\ y(0)=0 \end{cases} \quad \ldots(25)$$

The exact solution is $y(t)=\sin(t)$

Solution:

Applying Elzaki transform on both sides of (25), we obtain:

$$E\left[y'(t)\right] = E\left[1 - 2y^2(t)\right]$$

Equivalently,

$$\frac{E\left[y(t)\right]}{r} - ry(0) = r^2 - 2E\left[y^2\left(\frac{t}{2}\right)\right]$$

...(26) On substituting the initial values in (26) and re-arranging we obtain:

$$E\left[y(t)\right] = r^3 - 2rE\left[y^2\left(\frac{t}{2}\right)\right] \qquad ...(27)$$

Applying inverse Elzaki transform of (27) yields:

$$y(t) = t - 2E^{-1}\left[rE\left[y^2\left(\frac{t}{2}\right)\right]\right] \qquad ...(28)$$

From (28), we define the recurrence relations:

$$\begin{cases} y_0(t) = t \\ y_1(t) = -2E^{-1}\left[rE\left[{y_0}^2\left(\frac{t}{2}\right)\right]\right] = -\frac{t^3}{3!} \\ y_2(t) = N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right] = \frac{t^5}{5!} - \frac{5t^7}{8(7!)} \\ \vdots \\ y_{n+1}(t) = -2E^{-1}\left[rE\left[{y_n}^2\left(\frac{t}{2}\right)\right]\right], n \geq 2 \end{cases} \qquad ...(29)$$

$$y(t) = y_0(t) + \sum_{n=1}^{\infty} y_n$$

Therefore, the solution is $y(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{5t^7}{8(7!)} + ... \approx \sin(t)$.

Demonstrating nonlinearity handled without polynomials,

Example 4.3 Consider the following nonlinear PDDE [Volkan Yaman, and Bulent Yilmaz, 2018]:

$$\begin{cases} y'''(t) = -1 + 2y^2\left(\frac{t}{2}\right), 0 \le t \le 1 \\ y(0) = 0, y'(0) = 1, y''(0) = 0 \end{cases}$$

...(30) The exact solution is $y(t) = \sin(t)$

Solution:

Applying Elzaki transform on both sides of (30) we obtain:

$$E[y'''(t)] = E\left[-1 + 2y^2\left(\frac{t}{2}\right)\right]$$

Equivalently,

$$\frac{E[y(t)]}{r^3} - \frac{1}{r}y(0) - y'(0) - ry''(0) = -r^2 + 2E\left[y^2\left(\frac{t}{2}\right)\right]$$

...(31) Substituting the initial values in (31) and re-arranging yields:

$$E[y(t)] = r^3 - r^5 + 2r^3E\left[y^2\left(\frac{t}{2}\right)\right] \quad ...(32)$$

Applying inverse Elzaki transform on both sides of (32) yields:

$$y(t) = E^{-1}\left[r^3 - r^5 + 2r^3E\left[y^2\left(\frac{t}{2}\right)\right]\right]$$

Equivalently,

$$y(t) = t - \frac{t^3}{3!} + 2E^{-1}\left[r^3E\left[y^2\left(\frac{t}{2}\right)\right]\right] \quad ...(33)$$

From (33), we define recurrence relations:

$$\begin{cases} y_0(t)=t \\ y_1(t)=-\dfrac{t^3}{3!}+2E^{-1}\left[r^3E\left[{y_0}^2\left(\dfrac{t}{2}\right)\right]\right]=-\dfrac{t^3}{3!}+\dfrac{t^5}{5!} \\ y_2(t)=N\left[y_0\left(\dfrac{t}{2}\right)+y_1\left(\dfrac{t}{2}\right)\right]-N\left[y_0\left(\dfrac{t}{2}\right)\right]=2E^{-1}\left[r^3E\left[\left(\dfrac{t}{2}-\dfrac{t^3}{48}+\dfrac{t^5}{3840}\right)^2-\left(\dfrac{t}{2}\right)^2\right]\right]=-\dfrac{t^7}{7!}+\dfrac{t^9}{9!}-\dfrac{7t^{11}}{8(11!)}+\dfrac{63t^{13}}{128(13!)} \\ \vdots \\ y_{n+1}(t)=2E^{-1}\left[r^3E\left[{y_n}^2\left(\dfrac{t}{2}\right)\right]\right], n\geq 2 \end{cases}$$

$$...(34)$$

$$y(t)=y_0(t)+\sum_{n=1}^{\infty}y_n(t)$$

From (34),

Therefore, $y(t)=t-\dfrac{t^3}{3!}+\dfrac{t^5}{5!}-\dfrac{t^7}{7!}+\dfrac{t^9}{9!}-\dfrac{7t^{11}}{8(11!)}+\dfrac{63t^{13}}{128(13!)}+...\approx \sin(t)$

Discussion

The proposed hybrid method:

- handles proportional delay more naturally than Laplace transform methods,
- gives closed-form series expressions suitable for symbolic work.

Its simplicity makes it ideal for more complicated systems, including coupled PDDEs, distributed-order models, and fractional proportional delays.

Conclusion

We have presented a new hybrid computational framework for solving linear and nonlinear proportional delay differential equations using the Elzaki Transform combined with the Daftardar–Jafari Method. The hybrid approach is simple, accurate, and versatile. It offers an elegant balance between transform techniques and iterative approximation, and has following advantages:

- No polynomials required in handling nonlinearity.

- Straightforward transform inversion.
- Handles multi-delay and nonlinear delay kernels.
- Works for strongly nonlinear cases.

Future work includes extending the method to fractional proportional delays and multidimensional systems.

References

Alsharif, A., & Hattaf, K. (2023). Proportional delay epidemic models with nonlinear incidence. *Journal of Applied Mathematics*.

Atangana, A. (2020). Nonlinear differential models with memory effects. *Chaos, Solitons & Fractals*.

Awodola, T. O. (2021). Hybrid transform techniques for delay differential equations. *International Journal of Applied Analysis*.

Bawa, M., & Ogundare, B. (2022). Stability of proportional multi-delay systems. *Nigerian Journal of Mathematics*.

Chen, J., Li, S., & Wang, Q. (2023). Neural network models with proportional delays. *Applied Mathematical Modelling*.

Daftardar-Gejji, V., & Jafari, H. (2006). An iterative method for solving nonlinear functional equations. *Journal of Mathematical Analysis and Applications*.

Elzaki, T. M. (2012). The Elzaki transform and its applications. *Applied Mathematical Sciences*.

Elzaki, T. M., & Ahmed, M. (2019). Modified Elzaki transform for differential equations. *Asian Journal of Mathematics*.

Funmilayo, T., & Olajide, S. (2022). Applications of transforms to nonlinear delay systems. *Journal of Computational Mathematics*.

Ghanbari, B. (2021). Fractional delay differential equations and modern iterative methods. *Alexandria Engineering Journal*.

Ibrahim, M. D., Adamu, M. M., Mshelia, I. B., Kwami, A. M., Okai, J. O., & Nyikyaa, M. N. (2025). A modified new iterative method for solving nonlinear fractional-order delay differential equations. *International Journal of Education, Management, and Technology, 3*(2), 419–438. https://ejournal.yasin-alsys.org/IJEMT/article/view/5381

Jiang, Y., & Xu, Z. (2023). Iterative decomposition techniques for complex nonlinear systems. *Applied Analysis and Computation*.

Khan, A., & Noor, M. A. (2019). A comparative study of DJM and ADM for nonlinear models. *Mathematical Problems in Engineering*.

Liao, S., & Xu, C. (2022). Hybrid fractional–delay systems solved with DJM. *Communications in Nonlinear Science*.

Liu, H., & Zhao, W. (2020). Nonlinear proportional delay models in engineering. *Nonlinear Dynamics*.

Muhammad, A., & Yakubu, A. (2021). Elzaki transform solutions of fractional models. *Journal of the Nigerian Mathematical Society*.

Rida, S., & Saad, K. (2021). DJM for fractional integro-differential models. *Chaos, Solitons & Fractals*.

Shaikh, A., & Khan, R. (2020). Transform-based solutions to nonlinear DDEs. *Journal of Applied Nonlinear Science*.

Suleiman, A., Waziri, M., & Adamu, H. (2019). On the efficiency of Elzaki transform for singular ODEs. *International Journal of Mathematical Sciences*.

Wang, L., & He, J. (2021). Approximation schemes for nonlinear delay systems. *Journal of Physics: Conference Series*.

Waziri, M., & Yakubu, U. (2023). Modern developments in Elzaki transform applications. *African Journal of Mathematics*.

Yaman, V., & Yilmaz, B. (2018). Solutions of nonlinear delay differential equations by Daftardar-Jafari method. *Turkish Journal of Mathematics and Computer Science, 10*, 95–106. https://dergipark.org.tr/en/pub/tjmcs/article/452177

Yu, L., & Chen, D. (2024). Advances in proportional delay modeling. *Applied Mathematics Letters*.

Zhang, W., & Hu, H. (2023). Stability of nonlinear delay oscillators. *Journal of Differential Equations*.