

A Semi-Analytical Method for Nonlinear System of Delay Differential Equation via Modified He's Polynomial

Yohanna Nehemiah¹, Aminu Barde², A. Madaki³, Michael Cornelius⁴

^{1,2,3}Abubakar Tafawa Balewa University, Bauchi, Nigeria

⁴Gombe State University, Gombe, Nigeria

nehemiahyohanna@gmail.com

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Abstract

In this work, a simple technique based on the combination of sumudu transform and variational iteration method via Modified He's polynomial is introduced to solve systems of non-linear delay differential equations. The introduced technique is simpler and shorter in its computational procedures and time than the other methods. In addition, the modified He's polynomial takes care of the nonlinear terms and hence, making the method less stressful in terms of computations. Also, this technique reduces the complexity of calculating Lagrange's multiplier values which need more computational procedures and time. These advantages make it reliable and its efficiency is demonstrated with numerical examples.

Keywords: Modified He's Polynomial, Sumudu Transform Method, Variational Iteration Method, Delay Differential Equations

Introduction

Delay differential equation (DDE) is a class of functional differential equation arising in numerous applications from different areas of sciences, engineering and many other fields of studies. Over the years, ordinary differential equations (ODEs) are essential in modeling engineering and physical phenomena. In certain time, the mathematical models of real-life problems need to consider both the present and the past process of the system behaviour. However, the standard forms of ODEs are not always effective in modeling the scientific problems with a memory effect. Thus, such phenomena can be better described by means of delay differential equations (DDEs) (Alredi & Al-Jeaid, 2023). Contrary to ordinary differential equations (ODEs) the derivatives of the independent variables in DDEs at a certain time are express in terms of the values of the function at previous time (Barde, 2019). For this purpose, DDEs are used in various applications instead of ODEs. Therefore, researchers in many fields use DDEs for modelling the physical behaviour which contains both the present information and some history of the system.

Delay differential equations (DDEs) are proved to be useful in control systems (Fridman *et al.*, 2000). In many physical phenomena, lasers, traffic models (Davis, 2002), metal cutting, epidemiology, neuroscience, population dynamics (Kuang, 1993), chemical kinetics (Epstein & Luo, 1991). Different kinds of vigorous techniques have been developed in recent time to find an approximate solution for these such delay differential equations, as the Sumudu variational iteration method (SVIM) (Vilu *et al.*, 2019), the Laplace variational iteration method (LVIM) (Biala *et al.*, 2014), the Optimal homotopy asymptotic method (OHAM) (Anakira *et al.*, 2013), the Differential transform method (DTM) (Karakoc & Bereketoglu, 2009; Rashidi *et al.*, 2011), the Homotopy perturbation method (HPM) (Shakeri & Dehghan, 2008), the Adomian decomposition method (ADM) (Evans & Raslan, 2007) and the Homotopy analysis method (HAM) (Alomari *et al.* 2009; Hassan & Rashidi 2011). The method of variational iteration method (VIM) was developed for solving nonlinear problems by He (1999, 2000, 2004, 2007) He and Wang (2007) and He and Wu (2007). Subsequent Batiha *et al.* (2007), Mahdy *et al.* (2015), Noor and Mohy-Din (2008), Wazwaz (2009) and Wu (2013) reflect the VIM procedure's versatility, reliability and performance. A new modified variational iteration method was found, inspired and driven by Wu's thoughts and combining with the Sumudu transform (Liu *et al.*, 2016). The new approach is based on variational iteration theory and Sumudu transform. Subashini *et al.*

(2019) in their research, used the basic motivation and extend this new reliable approach for the solution of linear and nonlinear DDEs which are normally challenging to analyze due to their multifaceted nature and boundless dimensionality. In this paper, the modified He's polynomial is added to the new approach by Subashini *et al.* (2019), thereby making it easier to extend the application of the modified method to solving nonlinear systems of delay differential equations.

The Sumudu Variational Iteration Method Via Modified He's Polynomial (SVIMH)

To establish the procedure of the proposed method, consider the following form of nonlinear RDDEs.

$$v_i^{(n)}(t) = f_i \left(t, v(t), v'(t), \dots, v^{(n-1)}(t), v(\alpha_j(t)), v'(\alpha_j(t)), \dots, v^{(n-1)}(\alpha_j(t)) \right), \\ t \in [0, \infty), \quad \dots (1)$$

Where, $i = 1, 2, \dots, m$, $j = 0, 1, 2, \dots, r$, $f_i: [0, \infty) \times \mathbb{R}^{nm} \times \mathbb{R}^{nmr} \rightarrow \mathbb{R}$ are continuous

functions $v^{(k)}(t) = [v_1^{(k)}(t), v_2^{(k)}(t), \dots, v_m^{(k)}(t)]$ and $v^{(k)}(\alpha_j(t)) = [v_1^{(k)}(\alpha_j(t)), v_2^{(k)}(\alpha_j(t)), \dots, v_m^{(k)}(\alpha_j(t))]$, for $k = 0, 1, 2, \dots, n-1$

and $\alpha_j(t)$ are continuous delay functions. This research considered two forms of real time delay (proportional and constant delays) therefore, the delay function $\alpha_j(t)$ is defined as follows:

- (i) Proportional delay: $\alpha_j(t) = b_j t$, for $b_j \in (0, 1)$
- (ii) Constant delay: $\alpha_j(t) = t - \tau_j$, where $\tau_j > 0$ is real constant.

The initial condition of Equation (28) with respect to proportional delay is defined as follows:

$$v_i(0) = \mu_{i,0}, v_i'(0) = \mu_{i,1}, \dots, v_i^{(n-1)}(0) = \mu_{i,n-1}, \quad \dots (2)$$

where $\mu_{i,0}, \mu_{i,1}, \dots, \mu_{i,n-1}$ are constants real numbers. While the initial condition with constant delay is given as:

$$y_i(t) = p_i(t), t \in [-D, 0], i = 1, 2, \dots, m, \quad \dots (3)$$

where $p_i(t), p'_i(t), \dots, p_i^{(n-1)}(t)$ are continuous functions on $[-D, 0]$ and $D = \max_{j=1,2,\dots,r} [\tau_j]$

In this work, we extend the idea in finding the unknown Lagrange multiplier. Then the equation (1) is converted into an algebraic equation below:

$$\frac{d^n[v_i(t)]}{dt^n} + R_i[\mathbf{v}(t)] + F_i[\mathbf{v}(t)] = g_i(t), \quad \dots (4)$$

with Equations (2) and (3) as initial condition, $\mathbf{v}(t) = (v_1(t), v_2(t), \dots, v_m(t))$, R_i are linear operators and N_i are nonlinear operators represent the nonlinear terms. Apply Sumudu transform with its fundamental properties to (4) to obtain the following equation.

$$u^{-m}v_i(u) - u^{-m}v_i(0) - \dots - u^{-1}v^{m-1}_i(0) + S(R_i[V] + F_i[V]) = S(g_i(t)) \quad \dots (5)$$

Then by using the concept Variational Iteration Method gives the correction functional as

$$\begin{aligned} \tilde{v}_{i(nt1)}(u) = & \tilde{v}_{i(n)}(u) + \lambda(u) [u^{-m}\tilde{v}_i(u) - u^{-m}v_i(0) - \dots - u^{-1}v^{m-1}_i(0) + \\ & S(R_i[V] + F_i[V]) - \\ & S(g_i(t))] \end{aligned} \quad \dots (6)$$

Considering the terms $S(R_i[\tilde{v}_n] + F_i[\tilde{v}_n]) - S(g_i(t))$ as restricted variations, we let (6) be stationary with respect to $\tilde{v}_n$

$$\delta \tilde{v}_{i(nt1)}(u) = \delta \tilde{v}_n(u) + \lambda(u) \left(\frac{1}{u^m} \delta \tilde{v}_n(u) \right) \quad \dots (7)$$

From (3), we define Lagrange multiplier as

$$\lambda(u) = -u^m \quad \dots (8)$$

Succeeding approximations can then be attained with the application of inverse Sumudu transform S^{-1} into (5) and equation (6) becomes

$$\begin{aligned} v_{i(nt1)}(u) = & v_{i(n)}(u) - S^{-1} \left[u^m \left\{ u^{-m}v_i(u) - u^{-m}v_i(0) - \dots - u^{-1}v^{m-1}_i(0) + \right. \right. \\ & S(R_i[V] + H_{i,n}(\mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_m(t))) - \\ & \left. \left. (g_i(t)) \right\} \right] \end{aligned} \quad \dots (9)$$

Where the nonlinear operator $F_i[\mathbf{v}]$ are computed using the modified He's polynomial $H_{i,n}(\mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_m(t))$ and the initial approximation is given as:

$$v_0(t) = S^{-1}\left(v(0) + \dots + u^{m-1}v^{m-1}(0)\right) = v(0) + v'(0) + \dots + \frac{t^{m-1}v^{m-1}(0)}{(m-1)!} \quad (10)$$

Now, as $n \rightarrow \infty$, $v_n(t)$ converges to the exact solutions v_i .

Numerical Applications

In this section the proposed method shall be applied to solve some problems of nonlinear system of DDEs

Example 1: Consider the first order nonlinear system of delay differential equation [Subashini *et al.*, 2019]

$$y'(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad y(0) = 0 \quad \dots (11)$$

The exact solution is known as , $y(t) = \sin(t)$

Taking the sumudu Transform we obtain;

$$\frac{y(u)}{u} - \frac{y(0)}{u} = S\left[1 - 2y^2\left(\frac{t}{2}\right)\right] \quad \dots (12)$$

The iteration formula thus is;

$$y_{i(n+1)}(u) = y_{i(n)}(u) + \lambda(u) \left[\frac{y(u)}{u} - \frac{y(0)}{u} + S\left(-1 + H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right) \right] \quad \dots (13)$$

And its Lagrange multiplier

$$\lambda(u) = -u^m = -u \quad \dots (14)$$

Applying inverse Sumudu Transform gives

$$\begin{aligned} y_{i(n+1)}(u) &= y_{i(n)}(u) + S^{-1} \left\{ -u \left[\frac{y(u)}{u} - \frac{y(0)}{u} + S\left(-1 + H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right) \right] \right\} \quad \dots (15) \\ &= y_{i(n)}(u) + S^{-1} \left\{ u \left[S\left(1 - H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right) \right] \right\} \end{aligned}$$

Therefore

$$y_{i(n+1)}(u) = y_{i(n)}(u) + S^{-1} \left\{ u \left[S \left(1 - H_{i,n} \left(2y^2 \left(\frac{t}{2} \right) \right) \right) \right] \right\} \quad \dots (16)$$

With initial approximation $y_0(t) = 0$ and applying the iteration formula (16) above, we obtain

$$\begin{aligned} y_1(t) &= t \\ y_2(t) &= t - \frac{t^3}{6} \\ y_3(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{8064} \end{aligned} \quad \dots (17)$$

From these approximations, it can be seen that the solution tends to form the Taylor series expansion of $\sin(t)$. So, as $n \rightarrow \infty$, $y_n(t)$ converges to the exact solutions $\sin(t)$. In order to attest numerically whether or not the suggested approach maintains the accurateness, numerical solutions of the approximation up to y_3 were evaluated. The table and graph of the approximate solutions were plotted and compared with that of the exact solution. Table 1 shows the comparison between exact solution, SVIMH and SVIM and Figure 1 displays behavior of the graphs of Example 1. The outcome is in good agreement with each other.

Table 1: The comparison between exact solution $y(t)$, SVIMH and SVIM

T	EXACT	SVIMH(n=3)	SVIM(n=4)
0	0	0	0
0.1	0.099833	0.099833417	0.0998334
0.2	0.198669	0.198669331	0.1986693
0.3	0.29552	0.295520207	0.2955202
0.4	0.389418	0.389418342	0.3894183
0.5	0.479426	0.479425533	0.4794255
0.6	0.564642	0.564642446	0.5646424
0.7	0.644218	0.644217577	0.6442176
0.8	0.717356	0.717355723	0.7173557
0.9	0.783327	0.78332585	0.7833258
1	0.841471	0.841468254	0.8414683

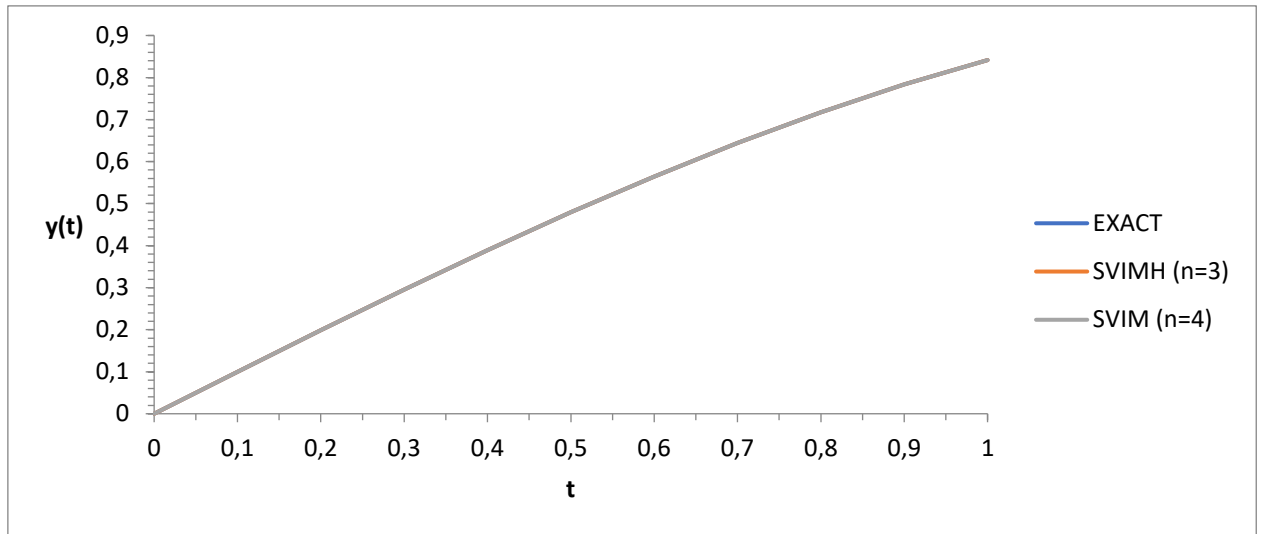


Figure 1: Graphs of exact solution $y(t)$, SVIMH and SVIM.

Example 2: Consider the first order nonlinear system of delay differential equation [Mohammed, 2021, Barde, 2019 and Rostam *et. al*, 2010]

$$y_1'(t) = y_2^2(t)$$

$$y_2'(t)$$

$$= \frac{1}{2} y_1\left(\frac{t}{2}\right) \quad \dots (18)$$

$$y_3'(t) = e^{\frac{5t}{2}} y_2(t) + 9e^{2t} y_3\left(\frac{t}{3}\right)$$

With initial condition

$$\begin{aligned} y_1(0) = 1, y_2(0) = 1, y_3(0) \\ = 0 \end{aligned} \quad \dots (19)$$

Equation (18), together with the initial conditions equation (19), has the exact solution

$$\begin{aligned} y_1(t) = e^t, y_2(t) = e^{\frac{t}{2}}, y_3(t) \\ = te^{3t} \end{aligned} \quad \dots (20)$$

Taking Sumudu Transform

$$\frac{y_1(u)}{u} - \frac{y_1(0)}{u} = S[y_2^2(t)]$$

$$\begin{aligned} & \frac{y_2(u)}{u} - \frac{y_2(0)}{u} \\ &= S \left[\frac{1}{2} y_1 \left(\frac{t}{2} \right) \right] \end{aligned} \quad \dots (21)$$

$$\frac{y_3(u)}{u} - \frac{y_3(0)}{u} = S \left[e^{\frac{5t}{2}} y_2(t) + 9e^{2t} y_3 \left(\frac{t}{3} \right) \right]$$

The iteration formulae thus is;

$$\begin{aligned} y_{i(nt1)}(u) &= y_{i(n)}(u) + \lambda(u) \left[\frac{y_1(u)}{u} - \frac{y_1(0)}{u} - S \left(H_{i,n}(y_2^2(t)) \right) \right] \\ y_{i(nt1)}(u) &= y_{i(n)}(u) + \lambda(u) \left[\frac{y_2(u)}{u} - \frac{y_2(0)}{u} - S \left(H_{i,n} \left(\frac{1}{2} y_1 \left(\frac{t}{2} \right) \right) \right) \right] \quad \dots (22) \\ y_{i(nt1)}(u) &= y_{i(n)}(u) + \lambda(u) \left[\frac{y_3(u)}{u} - \frac{y_3(0)}{u} - S \left(H_{i,n} \left(e^{\frac{5t}{2}} y_2(t) + 9e^{2t} y_3 \left(\frac{t}{3} \right) \right) \right) \right] \end{aligned}$$

And its Lagrange multiplier

$$\lambda(u) = -u^m = -u \quad \dots (23)$$

Applying inverse Sumudu Transform to (22) gives

$$\begin{aligned} y_{i(nt1)}(u) &= y_{i(n)}(u) + S^{-1} \left\{ -u \left[\frac{y_1(u)}{u} - \frac{y_1(0)}{u} - S \left(H_{i,n}(y_2^2(t)) \right) \right] \right\} \\ y_{i(nt1)}(u) &= y_{i(n)}(u) \\ &+ S^{-1} \left\{ -u \left[\frac{y_2(u)}{u} - \frac{y_2(0)}{u} - S \left(H_{i,n} \left(\frac{1}{2} y_1 \left(\frac{t}{2} \right) \right) \right) \right] \right\} \quad \dots (24) \end{aligned}$$

$$\begin{aligned} y_{i(nt1)}(u) &= y_{i(n)}(u) \\ &+ S^{-1} \left\{ -u \left[\frac{y_3(u)}{u} - \frac{y_3(0)}{u} \right. \right. \\ &\quad \left. \left. - S \left(H_{i,n} \left(e^{\frac{5t}{2}} y_2(t) + 9e^{2t} y_3 \left(\frac{t}{3} \right) \right) \right) \right] \right\} \end{aligned}$$

Therefore

$$y_{i(nt1)}(u) = y_{i(n)}(u) + S^{-1} \left\{ u \left[S \left(H_{i,n}(y_2^2(t)) \right) \right] \right\}$$

$$y_{i(n+1)}(u) = y_{i(n)}(u) + S^{-1} \left\{ u \left[S \left(H_{i,n} \left(\frac{1}{2} y_1 \left(\frac{t}{2} \right) \right) \right) \right] \right\} \quad \dots (25)$$

$$y_{i(n+1)}(u) = y_{i(n)}(u) + S^{-1} \left\{ u \left[S \left(H_{i,n} \left(e^{\frac{5t}{2}} y_2(t) + 9e^{2t} y_3 \left(\frac{t}{3} \right) \right) \right) \right] \right\}$$

With initial approximation $y_1(0) = 1, y_2(0) = 1, y_3(0) = 0$ and applying the iteration formula (25) above, we obtain

$$\begin{aligned} y_{1,0}(t) &= 1, & y_{2,0}(t) &= 1, & y_{3,0}(t) &= 0 \\ y_{1,1}(t) &= 1 + t, & y_{2,1}(t) &= 1 + \frac{t}{2}, & y_{3,1}(t) &= t \left[1 + \frac{5t}{4} + \frac{25t^2}{24} + \frac{125t^3}{192} + \dots \right] \\ y_{1,2}(t) &= 1 + t + \frac{t^2}{2}, & y_{2,2}(t) &= 1 + \frac{t}{2} + \frac{t^2}{8}, & y_{3,2}(t) &= t \left[1 + 3t + \frac{495t^2}{216} + \frac{7875t^3}{5184} + \dots \right] \\ y_{1,3}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6}, & y_{2,3}(t) &= 1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48}, \end{aligned}$$

From these approximations above, it can be seen that the solutions to equation (18) with initial conditions in equation (19) are as follows:

$$\begin{aligned} y_{1,3}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots \\ y_{2,3}(t) &= 1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \dots \\ y_{3,2}(t) &= t \left[1 + 3t + \frac{495t^2}{216} + \frac{7875t^3}{5184} + \dots \right] \end{aligned}$$

Thus as $n \rightarrow \infty$ the approximate solutions tend to converge as follows:

$$y_1(t) = e^t, y_2(t) = e^{\frac{t}{2}}, y_3(t) = te^{3t}$$

Table 2: The comparison between exact solution $y_1(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

T	EXACT	SVIMH(n=3)	ADM (n=6)	HAM(n=5)
0	1	1	1	1
0.1	1.105170918	1.105170918	1.105170917	1.105167
0.2	1.221402758	1.221402756	1.221402667	1.221333
0.3	1.349858808	1.349858763	1.34985775	1.3495
0.4	1.491824698	1.491824356	1.491818667	1.490667
0.5	1.648721271	1.648719618	1.648697917	1.645833
0.6	1.8221188	1.8221128	1.822048	1.816
0.7	2.013752707	2.013734818	2.013571417	2.002167
0.8	2.225540928	2.225494756	2.225130667	2.205333
0.9	2.459603111	2.459496363	2.45875825	2.4265
1	2.718281828	2.718055556	2.716666667	2.666667

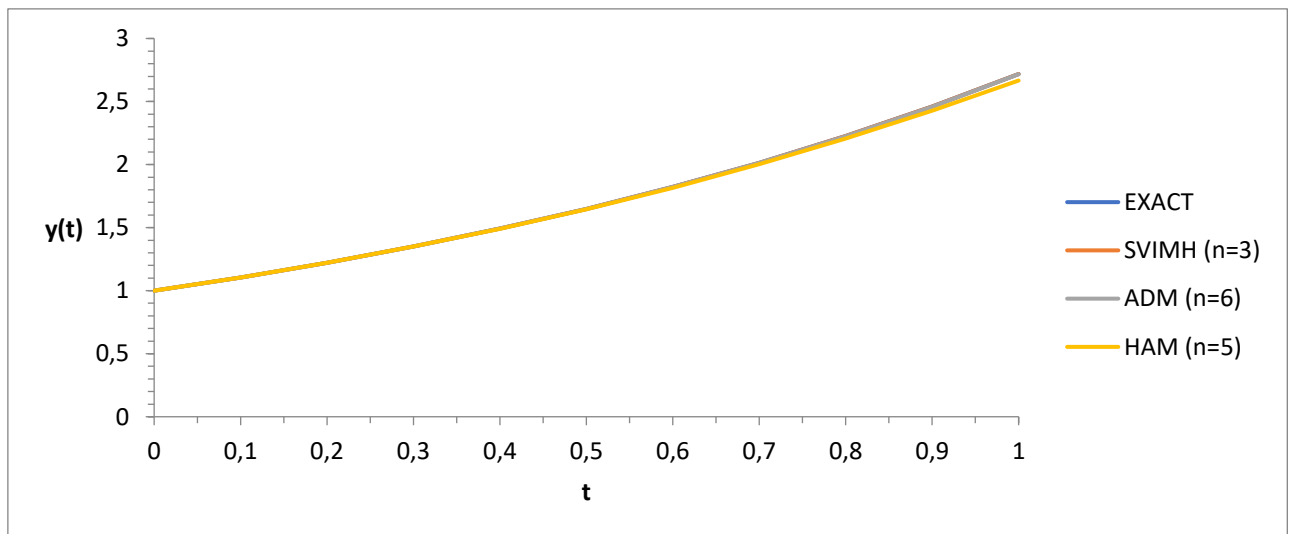


Figure 2: Graphs of exact solution $y_1(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 2 and figure 2 above, the results obtained from our proposed method were in good agreement with the results from ADM, HAM and the exact solutions. This shows that the proposed method is effective and efficient.

Table 3: The comparison between exact solution $y_2(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

t	EXACT	SVIMH(n=3)	ADM(n=7)	HAM(n=6)
0	1	1	1	1
0.1	1.051271	1.05127109	1.0512711	1.05127083
0.2	1.105171	1.10517083	1.1051708	1.10516667
0.3	1.161834	1.16183359	1.1618336	1.1618125
0.4	1.221403	1.2214	1.2214	1.22133333
0.5	1.284025	1.28401693	1.2840169	1.28385417
0.6	1.349859	1.3498375	1.3498375	1.3495
0.7	1.419068	1.41902109	1.4190211	1.41839583
0.8	1.491825	1.49173333	1.4917333	1.49066667
0.9	1.568312	1.56814609	1.5681461	1.5664375
1	1.648721	1.6484375	1.6484375	1.64583333

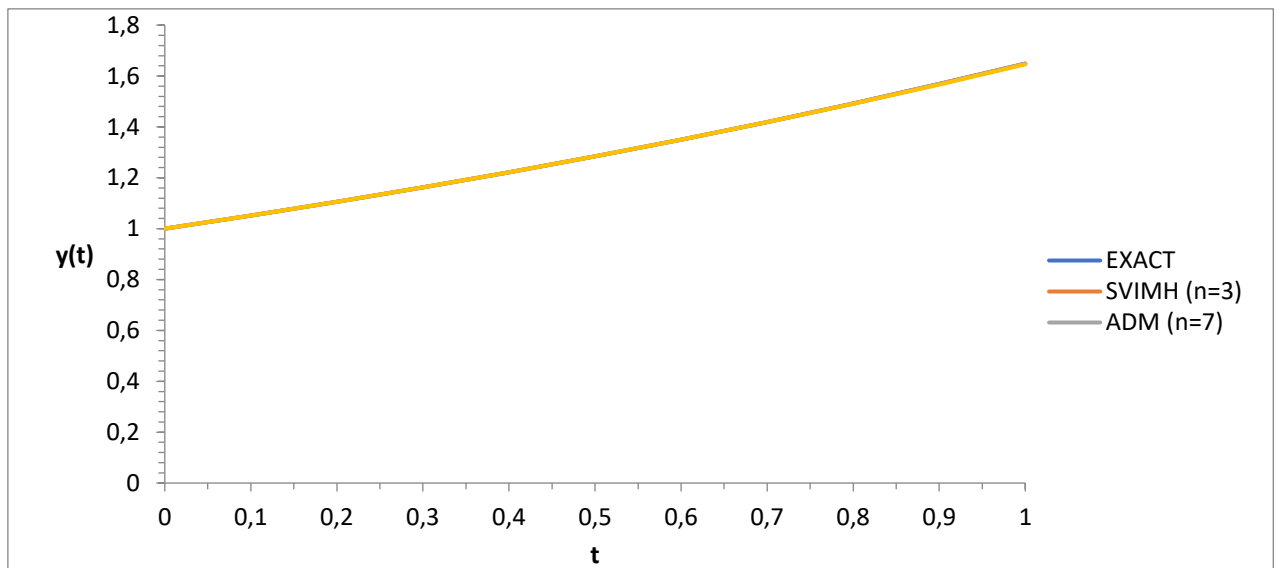


Figure 3: Graphs of exact solution $y_2(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 3 and figure 3 above, the values demonstrate that the outcomes are in excellent agreement with those of the other approaches. This shows that the proposed method is effective and efficient.

Table 4: The comparison between exact solution $y_3(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

t	EXACT	SVIMH(n=3)	ADM(n=5)	HAM(n=4)
0	0	0	0	0
0.1	0.134986	0.134985881	0.1345	0.13495
0.2	0.364424	0.364423671	0.356	0.3632
0.3	0.737881	0.737877379	0.6915	0.72795
0.4	1.328047	1.327997659	1.168	1.2832
0.5	2.240845	2.240464565	1.8125	2.09375
0.6	3.629788	3.627750254	2.652	3.2352
0.7	5.716319	5.707824322	3.7135	4.79395
0.8	8.818541	8.789099374	5.024	6.8672
0.9	13.39176	13.30308941	6.6105	9.56295
1	20.08554	19.84642857	8.5	13

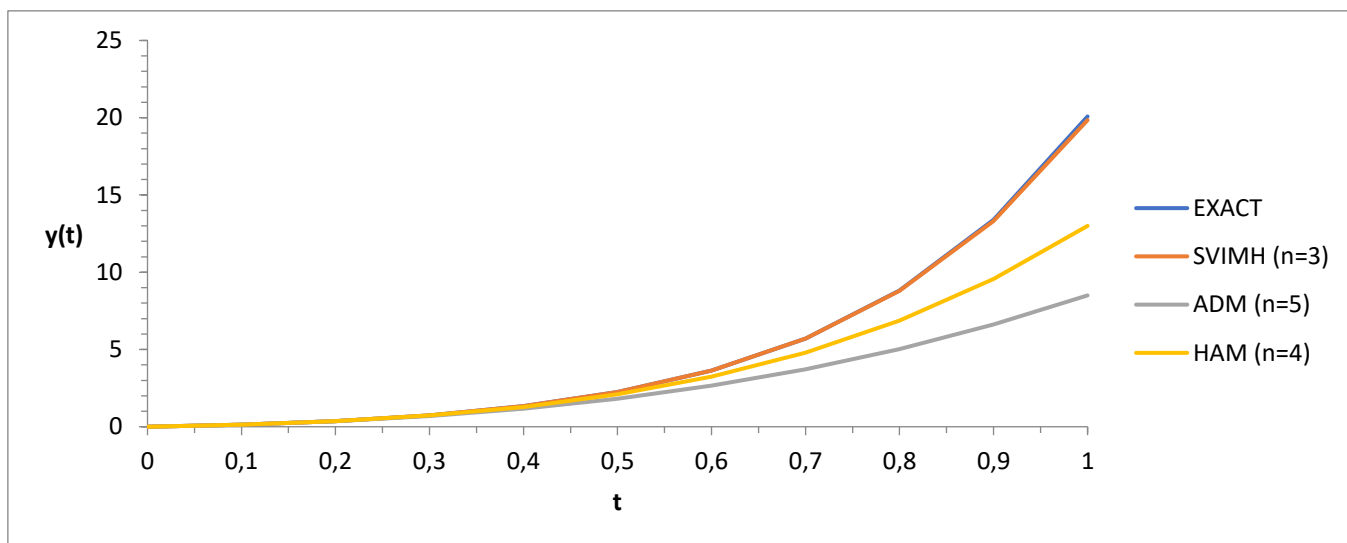


Figure 4: Graphs of exact solution $y_3(t)$, the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 4 and figure 4 above, the values demonstrate that the approximate solutions obtained through the proposed method is closer to exact than other approaches. This shows that the proposed method is effective and efficient.

Example 3: Consider the second order nonlinear system of delay differential equation [Barde, 2019 and Rebenda *et. al*, 2015]

$$y_1''(t) - y_1'(t) - 2y_1\left(\frac{t}{2}\right)y_2(t-1) - 4 = -t^4 - 2t^3x - t^2 - 4t$$

$$y_2''(t) - y_2'(t-2) - y_2\left(\frac{t}{3}\right)y_1\left(\frac{t}{2}\right) - 6 = -\frac{t^4}{18} - 2t \quad \dots (53)$$

With initial condition

$$y_1(0) = y_1'(0) = 0, \quad y_2(0) = y_2'(0) = 0 \quad \dots (54)$$

From the given initial conditions, the initial approximations can be chosen as

$$y_{1,0}(t) = 2t^2, \quad y_{2,0}(t) = t^2 \quad \dots (55)$$

And using our method, the next iterations gives the following results;

$$y_{1,1}(t) = 2t^2 + \frac{2t^3}{3} - \frac{t^4}{12}$$

$$y_{2,1}(t) = t^2 + \frac{t^3}{6}$$

$$y_{1,2}(t) = 2t^2 + \frac{2t^3}{3} - \frac{5t^4}{72} - \frac{17t^5}{720} - \frac{37t^6}{17280} - \frac{155t^7}{12096} - \frac{5t^8}{4032} - \frac{7t^9}{72576}$$

$$y_{2,2}(t) = t^2 + \frac{t^3}{2} - \frac{t^6}{270} - \frac{t^7}{3402} - \frac{t^8}{870912} - \frac{t^9}{2239488}$$

Table 5: The comparison between exact solution $y_1(t)$ and the approximate solution using SVIMH for Example3.

t	EXACT	SVIMH (n=3)
0	0	0
0.1	0.02	0.020659463
0.2	0.08	0.085213123
0.3	0.18	0.197361557
0.4	0.32	0.360537167
0.5	0.5	0.577813946
0.6	0.72	0.851779839
0.7	0.98	1.184363234
0.8	1.28	1.576604537
0.9	1.62	2.028363064
1	2	2.5379488

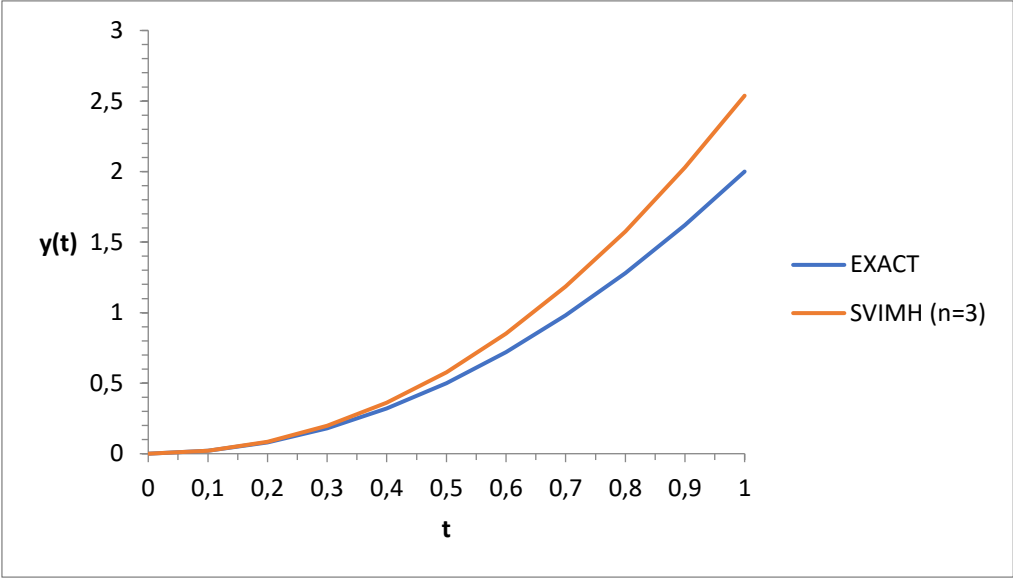


Figure 5: Graphs of exact solution $y_1(t)$ and the approximate solution using SVIMH for Example3.

Table 6: The comparison between exact solution $y_2(t)$ and the approximate solution using SVIMH for Example3.

t	EXACT	SVIMH(n=3)
0	0	0
0.1	0.01	0.0105
0.2	0.04	0.04399976
0.3	0.09	0.10349723
0.4	0.16	0.19198434
0.5	0.25	0.31243978
0.6	0.36	0.46781873
0.7	0.49	0.66103921
0.8	0.64	0.89496493
0.9	0.81	1.17238443
1	1	1.4959864

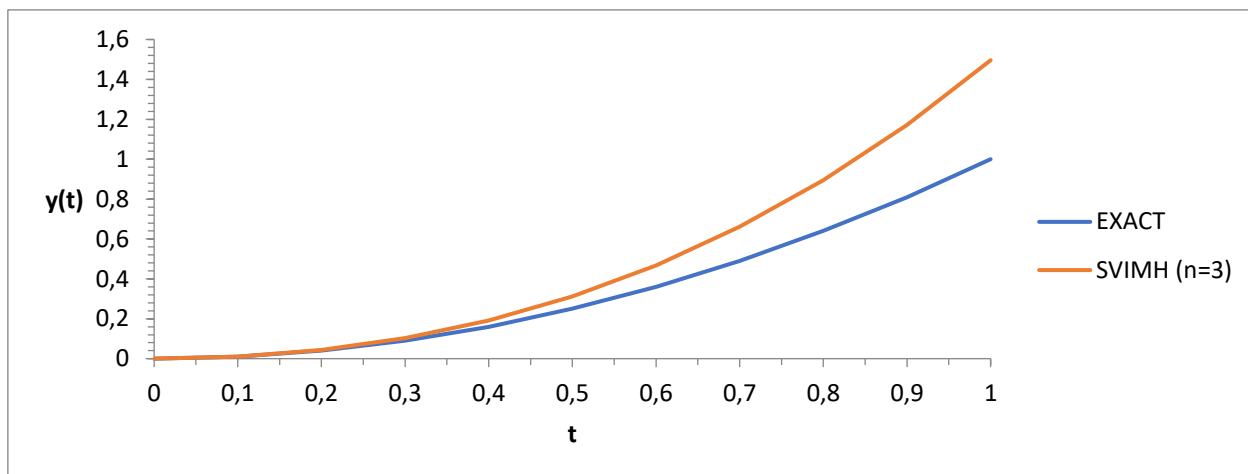


Figure 6 Graphs of exact solution $y_2(t)$ and the approximate solution using SVIMH for Example 3.

From table 5 and 6, figure 5 and 6 above, the proposed method was able to solve a problem of system of delay differential equation with constant delay which is one of the objective of this research. The approximate solution has a good agreement with the exact solution.

Conclusion

The Sumudu transform is a simple variant of the Laplace transform and is essentially identical with the Laplace. It has many interesting properties that make it easy to visualize and hence making the process of the solution simpler. Sumudu Lagrange multiplier is derived from Sumudu transform plus integrating with procedures of VIM to obtain the approximate solutions to DDEs. Lagrange multipliers, $\lambda(u)$ could be known optimally with this different approach of variational theory. A new modification of the VIM was attained. This recommended procedure was obtained by not having applied any linearization, discretization, or impractical rules. This new technique provides more convincing or accurate sequence of results that converges quickly in physical problems. In this article, the SVIMH was successfully applied in solving nonlinear system of DDEs. In order to attest numerically whether or not the suggested approach maintains the accurateness, numerical solutions of the approximation up to y_3 were evaluated. The results were compared in the tables 1,2,3,4,5 and 6 while Figures 1, 2, 3,4,5 and 6 display graphs of the compared values between our methods and others in Example 1 to Example 3. The outcomes are in good agreement with each other. It is worth stating that this method is efficacious in minimizing

number of computations contrasting with traditional methods in spite of sustaining great accurateness of the numerical outcome.

References

- Alomari AK, Noorani MS, Nazar R (2009). Solution of delay differential equation by means of homotopy analysis method. *AcaApplicandaeMathematicae* 108(2):395-412.
- Alrebdi, R.& Al-Jeaid, H.K. (2023). Accurate Solution for the Pantograph Delay Differential Equation via Laplace Transform. *Mathematics*, 11, 2031. <https://doi.org/10.3390/math11092031>.
- Aminu Barde & Normah Maan (2019).Efficient Analytical Approach for Nonlinear System of Delay Differential Equations.*International Journal of Mathematics and Computer Science*, 14(2019), no. 3, 693–712.
- AminuBarde, A.B. Jafar and A.G. Madaki. (2023). Enhanced Efficient Analytical Approach for Nonlinear System of DDEs . *Bima Journal of Science and Technology*, 7(1): 123-135.
- Anakira N.R., Alomari A.K., & Hashim I. (2013). “Optimal homotopy asymptotic method for solving delay differential equations”. *Mathematical Problems in Engineering*, ArticleID498902,11pages.
- Batiha B, Noorani MSM, Hashim I (2007). Application of variational iteration method to a general Riccati equation. *International Mathematical Forum* 2(56):2759-2770.
- Belgacem F.M &Karaballi A.A., (2006).Sumudu Transform Fundamental Properties Investigations And Application.*Journal of Applied Mathematics and Stochastic Analysis*. Pages 1–23 DOI 10.1155/JAMSA/2006/91083
- Biala T.A., Asim O.O.,& Afolabi Y.O.,(2014) “A combination of the laplace transform and the variational iteration method for the analytical treatment of delay differential equations,” *International Journal of Differential Equations and Applications*, vol.13,no.3,.
- Davis C.L., (2002). Modification of the optimal velocity traffic model to include delay due to driver reaction time. *Physica A* 319:557-567.
- Demiray S.T.,Bulut H. & Belgacem F.M. (2015).Sumudu Transform Method for Analytical Solutions of Fractional Type Ordinary Differential Equations.*Mathematical Problems in Engineering* Volume 2015, Article ID 131690, 6 pages <http://dx.doi.org/10.1155/2015/131690>.
- Epstein I.R.& Luo Y. (1991). Differential delay equations in chemical kinetics. Nonlinear models: The cross-shaped phase diagram and the Oregonator. *The Journal of Chemical Physics* 95(1):244-254.
- Evans D.J.&Raslan K.R. (2007). The Adomian decomposition method for solving delay differential equation. *International Journal computer Mathematical*. Doi.org / 10.1080/00207160412331286815.
- Fridman E., Fridman L., Shustin E (2000). Steady models in really control systems with a time delay and periodic disturbances. *Journal of Dynamical Systems Measurement and Control* 122:732-737.

- Goswami, P., & Alqahtani, R. T. (2016). Solutions of fractional differential equations by Sumudu transform and variational iteration method. *J. Nonlinear Sci. Appl*, 9(2016), 1944–1951.
- Ghorbani, A. and Saberi-Nadjafifi, J. (2007). He's Homotopy Perturbation Method for Calculating Adomian Polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*, 8(2): 229–232.
- Hassan H, Rashidi MM (2011). An analytical solution of micropolar flow in aporous channel with mass injection using homotopy analysis method. *International Journal of Numerical Methods for Heat and Fluid Flow* 24(2):419-437. Doi.org/10.1108/HHF-08-2011-0158
- He J.H. (1999). Variational iteration method, a kind of nonlinear analytical technique: Some examples. *International Journal of Nonlinear Mechanics* 34(4):699-708.
- He J.H (2000). Variational iteration method for autonomous ordinary differential. *Applied Mathematics and Computation* 114:115-123.
- He J.H (2004). Variational principle for some nonlinear partial differential equations with variable coefficients. *Chaos, Solitons and Fractals* 19:847-851.
- He J.H (2007). Variational iteration method - Some recent results and new interpretations. *Journal of Computational and Applied Mathematics* 207(1):3-17.
- He J.H, Wang SQ (2007). Variational iteration method for solving integro-differential equations. *Physics Letters A* 207:3-17.
- He, J. H., & Wu, X. H. (2007). Variational iteration method: New development and applications. *Computers and Mathematics with Applications*, 54(7–8), 881–894. <https://doi.org/10.1016/j.camwa.2006.12.083>
- Karakoc F. Bereketoglu (2009). Solutions of delay differential equations by using differential transform method. *International Journal Computer Mathematical*. Doi.org/10.1080/ 00207160701750575.
- Kuang Y., (1993). Delay differential equations with applications in population biology. *Academic Press, Boston, San Diego, New York*.
- Mahdy AMS, Mohamed AS, Mtawal AAH (2015). Variational homotopy perturbation method for solving the generalized time-space fractional Schrodinger equation. *International Journal of Physical Sciences* 10(11):342-350.
- Mtawal A.A.M., Saif A. E. M., & Ayiman A. A., (2020). Application of the alternative variational iteration method to solve delay differential equations. *International Journal of Physical Sciences*. Vol. 15(3), pp. 112-119, DOI: 10.5897/IJPS2020.4879.
- Mohammed S.M. Bahgat and A.M. Sebaq (2021). An Analytical Computational Algorithm for Solving a System of Multipantograph DDEs Using Laplace Variational Iteration Algorithm. *Hindawi Advances in Astronomy*, Article ID 7741166, 16 pages <https://doi.org/10.1155/2021/7741166>.
- Mohyud-Din S.T., & Yildirim A., (2010). Variational Iteration Method for Delay Differential Equations Using He's Polynomials.
- Noor MA, Mohyud-Din ST (2008). Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. *International Journal of Nonlinear Science and Numerical Simulation* 9(2):141-156.

- Rashidi MM, Iaragi N, Basiriparsa A (2011). Analytical modeling of heat convection in magnetized micropolar fluid by using modified differential transform method. *Heat Transfer* 40(3): 187-204. Doi. Org/ 10.1002/ htj.20337.
- Rostam K S. & Botan M. R., (2010). Adomian decomposition method for solving system of delay differential equation. *Australian Journal of Basic and Applied Sciences*, 4, no. 8:3613–3621.
- Sara Barati and Karim Ivaz(2012).Variational Iteration Method for Solving Systems of Linear Delay Differential Equations.*International Journal of Computational and Mathematical Sciences*.
- Shakeri F.&Dehghan M. (2008). Solution of delay equation via homotopy perturbation method. *Mathematical and Computer Modeling* 48(3-4): 486-498.
- SubashiniVilu et al. (2019).Variational Iteration Method and Sumudu Transform for Solving Delay Differential Equation.*International Journal of Differential Equations* Volume 2019, Article ID 6306120, 6 pages <https://doi.org/10.1155/2019/6306120>.
- Torabi A. G.,Ranjbar M., Shafiee M., &Roomi V., (2021).VIM-Pad'e technique for solving nonlinear and delay initial value problems.*Computational Methods for Differential Equations* <http://cmde.tabrizu.ac.ir> Vol. 9, No. 3 pp. 749-761 DOI:10.22034/cmde.2020.35417.1606.
- Watugala G.K., (1993). “Sumudu transform: a new integral transform to solve differential equations and control engineering problems”. *International Journal of Mathematical Education in Science and Technology*, vol.24 no.1, pp.35–43.
- Watugala G.K., (1998). “Sumudutransform—anewintegraltransform to solve differential equations and control engineering problems”. *Mathematical Engineering in Industry*, vol. 6, no. 4, pp. 319–329,1998.
- Watugala G.K., (2002). “The Sumudu transform for functions of two variables”. *Mathematical Engineering in Industry*, vol. 8, no. 4, pp.293–302,2002.
- Wazwaz AM (2009). The variational iteration method for analytic treatment of linear and nonlinear ODEs. *Applied Mathematics and Computation* 212(1):120-134.
- Wazwaz A.M., RAJA M.Z., &Syam M.I., (2017). Reliable Treatment For Solving Boundary Value Problems Of Pantograph Delay Differential Equation.*Romanian Reports in Physics* 69, 102.
- Wu G (2013). Challenge in the Variational iteration method - A new approach to identification of the Lagrange multipliers. *Journal of King Saud University - Science* 25(2):175-178.