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A Study on Homotopy Invariance of Circle and Stereographic Projection

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Abstract

This paper explores fundamental applications of topological spaces and stereographic projection derived from the properties of the circle, employing key concepts such as continuous functions and homotopy theory. By examining the behavior of mappings and deformations within topological spaces, the study demonstrates how the circle serves as a foundational structure for understanding more complex topological constructs. Special attention is given to the use of stereographic projection in visualizing the relationship between the circle and the unit sphere, illustrating how these mathematical tools contribute to a deeper understanding of continuity and homotopy in topological analysis. The discussion offers a concise yet insightful introduction to the interplay between geometric intuition and topological formalism.

Keywords: Homotopy; Continuous Functions; Stereographic Projection; Unit Sphere; Circle

Introduction

In a topology as a field of mathematics, we consider the two circles S^1 and S^2 where, we called S^1 s unit circle and S^2 as sphere. Now we have the varieties of circle as shown below;

In the complex plane, the $z - sphere$ is defined as $S^n = \{(x, y) \in \mathbb{R}^{n+1}: x^2{}_n + \cdots y^2{}_n = 1\}$ (Khatchatourian, 2018), from this the set of points in the Euclidean plane can now be define as ; S^o is two points, S^1 is the circle called unit circle which is written as $\{z \in \mathbb{C}: |z| = 1\}$, S^2 is a sphere and S^3 is the Torus.

In algebraic topology, we have a fundamental notion called homotopy, which is continuous deformation of one function to another function. Note that if we define topology on the space $C(A, B)$, we have that homotopy between two functions $f, g \in C(A, B)$ is a continuous path in the space $C(A, B)$ from function f to g.

Note that circle is the best topological space in determining the stimulus of human heartbeat, the time by which this stimulus occur and beat again is known as Latency and is designated as L.

In this paper, we are going to study homotopy of circle and its application to stereographic projection of the circle.

Some Important Definitions

Definition 1 (Monsuru, *et.al,* 2020): A set X is said to be open, if its neighborhood of each of its points.

Definition 2 (Ahmadu and Monsuru, 2020): Given that (X_1, τ_1) and (X_2, τ_2) are topological spaces. The function $f: X_1 \to X_2$ is said to be continuous, if for every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$.

Definition 3 (Micheal, 2016): Let X_1 and X_2 be two topological spaces and $f, g \in C(X_1, X_2)$. Homotopy from function f to g is designated as $h: f \to g$, is continuous function $h: X_1 \times I \to X_2$, $(x, t) \to h(x, t)$. Such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x), \forall x \in X_1$.

Definition 4 (Adhikari, 2016): Two functions $f, g \in C(X_1, X_2)$ are homotopic, if there is homotopy between them.

Definition 5 (Micheal, 2016): A map $f \in C(X_1, X_2)$ is known as a homotopy equivalence, if there is map $g \in C(X_2, X_1)$ such that the composition map $gof \in C(X_1, X_1)$ is homotopic to id_X and $fog \in C(X_2, X_2)$.

Definition 6 (Adams and Franzosa, 2000): Two topological spaces X_1 and X_2 are said to be topologically equivalent, which is designated as $X_1 \sim X_2$, if there is homotopy equivalence $f: X_1 \to X_2$.

Definition7 (Youness, Jamal, 2021): A topological space X is contractible, if there is homotopy equivalent to one – point space.

Methodology

As defined earlier, homotopy is the continuous deformation of functions. Homotopy from function f to g is designated as $h: f \to g$, is continuous function $h: X_1 \times I \to X_2$, $(x, t) \to h(x, t)$. Such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x), \forall x \in X_1$.

Lemma 1(Micheal, 2016): If $f, g, h \in C(X_1, X_2)$ and $m: f \to g$ and $m': g \to h$ are homotopies, then

$$m_t''(x) = \begin{cases} m_{2k}(x), & if\ k \in [0, \frac{1}{2}] \\ m'_{2k-1}(x), & if\ k \in [\frac{1}{2}, 1] \end{cases}$$, defines a homotopy $m'' = m'om: f \to h$, and for every X_1, X_2, homotopy is an equivalent relation on $C(X_1, X_2)$.

When we want to define topology on the set $C(X_1, X_2)$, we have that a homotopy between $f, g, \in C(X_1, X_2)$ which is a continuous path in the set $C(X_1, X_2)$ from f to g.

For projection, the basic method we used is that we applied ordinary perspective to a sphere embedded in three Dimensions in a particular way. From the perspective projection on to the plane $z = -1$ with the origin as viewing point; $x' = \frac{x}{-z}$, $y' = \frac{y}{-z}$, $z' = -1$. These maps (x, y, z) onto the point on the plane $z = -1$ lying on the line through the point (x, y, z) and $(0,0,0)$. We now restrict this map to the sphere of radius 1, which centered at $(0,0,-1)$. The plane $z = -1$ bisects this sphere into two equal hemispheres. The point $(0,0,0)$ itself is a point of this particular sphere, but its image under this map is not defined. If we consider points on the sphere closed to the origin, then the plane z is close to zero and the image of the point (x, y, z) will be far away. Hence the point $(0,0,0)$ mapped to ∞ so that $\frac{z}{\infty} = 0, z + \infty = \infty, \frac{z}{0} = \infty, z \cdot \infty = \infty\ (z \neq 0)$.

As z represents the complex plane, we can set up a correspondence between the points of another complex plane K and those of sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$ tangent to this plane. If we consider the sphere S of radius $\frac{1}{2}$ and centered at $\left(0, 0, \frac{1}{2}\right)$, we have $S = \left\{ (x, y, z) \in, \mathbb{R}^n : x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4} \right\}$.

Suppose $(0,0,1)$ and $(0,0,0)$ designated the North and South pole of the sphere S above, respectively. The point $z = 0 + i0$ coincides with the point $(0,0,0)$ of the complex plane K above. If $(x, y, 0)$ is an arbitrary point in the complex plane K. Then, corresponding to the point $(x, y, 0)$, there is a unique point on the sphere S. If we draw a straight line through the point $(0,0,1)$ and $(0,0,0)$ which intersect the sphere at (x, y, z). Then, the point (x, y, z) is known as stereographic projection of the sphere S.

For example, let point $(x, y, 0)$ be a point the complex plane that has image on the sphere S as (x, y, z).

Since the points (x, y, z), $(x, y, 0)$ and $(0,0,1)$ are collinear, then

$$X - \frac{0}{x} - 0 = Y - \frac{0}{y} = Z - \frac{0}{z} - 0$$

$\Rightarrow \frac{X}{x} = \frac{Y}{y} = 1 - z$...(1)

$\Rightarrow X = \frac{x}{1-Z}$, and $Y = \frac{y}{1-Z}$ ....(2)

$$\Rightarrow x^2 + y^2 = X^2 + \frac{Y^2}{(1-Z)^2} \Rightarrow \frac{1}{4} - \frac{1-z}{(1-Z)^2}$$

Then, we have $x^2 + y^2 = \frac{z(1-z)}{(1-Z)^2}$...(3)

Sine any point on the sphere is different from point $(0,0,1)$, then $z \neq 1 \Rightarrow 1 - z \neq 0$, so the equation (3) above becomes $x^2 + y^2 = \frac{z}{(1-Z)^2}$.

Hence, $1 - z = \frac{1-x^2+y^2}{1+x^2+y^2} = \frac{1}{1+x^2+y^2}$...(4)

Put equation (4) into (1), we have $\frac{X}{x} = \frac{1}{1+x^2+y^2}$, $\frac{Y}{y} = \frac{1}{1+x^2+y^2}$. Thus, image of point $(x, y, 0)$ on the complex plane is the point (x, y, z) on the sphere S.

Therefore, $X = \frac{x}{1+x^2+y^2}$, $Y = \frac{y}{1+x^2+y^2}$, and $Z = \frac{x^2+y^2}{1+x^2+y^2}$

Results

In this section, we are going to discuss extensively on homotopy and stereographic projection associated with sphere. Here, we are going to make use of Lemma 2 to provide a proof for Theorem 1 below;

Lemma 2 (Micheal, 2016): Let $\varphi \in C(I^n, \mathbb{R}^n)$ satisfy $\varphi\left(I_i^{\pm}\right) \subseteq I_i^{\pm} \forall i$. Then, φ is a contraction map.

Theorem 1: The $n -$ sphere S^n has no a contraction map.

Proof: Let $k \in I^n$, and put $g(x) = \varphi(x) - k$. Now, $x_i = 0 \Rightarrow g_i(x) = 0 - k \leq 0$ and $x_i = 1 \Rightarrow g_i(x) = 1 - k \geq 0$. Thus the function g satisfy the assumption of Poincar$\acute{e}$ – Miranda Theorem, which states that if $g = (g_1, g_2, \dots, g_n) \in C(I^n, \mathbb{R}^n)$ satisfy $g_i(I_i^-) \subseteq (-\infty, 0]$, $g_i(I_i^+) \subseteq (0, \infty]$ for all i, then there is $x \in I^n$ such that $g(x) = 0$. By this, we conclude that, there is $x \in I^n$ with $g(x) = 0$. This implies that $\varphi(x) = k$, so the map φ is a bijective map.

Now, from the fact that topological space is contractible if and only if id_X is homotopic to a constant, then the contraction of boundary sphere, S^{n-1} is equivalent to the existence of a homotopy $h_t: S^{n-1} \rightarrow S^{n-1}$ satisfying $h_o(,) = x_o$ and $h_o = id_{S^{n-1}}$. Therefore, the map $r: D^n \rightarrow \partial D^n$, which is defined by $r(x) =$

$$\begin{cases} x_o, \parallel x \parallel \leq \frac{1}{2} \\ h_{2\|x\|} - \left(\frac{x}{\|x\|}\right), \parallel x \parallel \geq \frac{1}{2} \; is \; a \; retraction, \end{cases}$$

Which is contradicts Lemma 2 above. Therefore, n- sphere S^n has no a contraction map.

Next, we are going to use Poincar$\acute{e}$ – Miranda Theorem to establish the next result as shown in Theorem 2 below;

Theorem 2: Suppose $n \in \mathbb{N}$ and $g \in C(D^n,, \mathbb{R}^n)$ such that $x \cdot f(x) < 0, \forall\, x \in S^{n-1}$. Then, there is $x \in (D^n)^o$ such that $g(x) = 0$.

Proof: In the proof of this Theorem, we have to cases to consider as follows;

Case I: $g(x) \neq 0$, $\forall\, x \in S^{n-1}$ so that $f: x \to \frac{g(x)}{\|g(x)\|}$ defines a continuous map $\varphi: S^{n-1} \to S^{n-1}$. We claim that f is homotopic to $id_{S^{n-1}}$.To prove this, $x \in S^{n-1}$ and $0 \leq p \leq 1$, we have $[(1-p)g(x) + p(x)] \cdot x = (1-p)g(x) \cdot x + p(x) = (1-p)g(x) \cdot x + p > 0$, thus, $(1-p)g(x) + p(x) \neq 0$. Thus, $h(x,p) = \frac{(1-p)g(x)+p(x)}{\|(1-p)g(x)+p(x)\|}$ defines a continuous map $\varphi: S^{n-1} \times [10] \to S^{n-1}$, clearly, $h(\cdot\,,0) = f$ and $h(\cdot\,,1) = id_{S^{n-1}}$. Therefore, h is homotopy from f to $id_{S^{n-1}}$.

Case II: Suppose $g(x) \neq 0$, $\forall\, x \in D^n$, we can define $t: S^{n-1} \times [10] \to S^{n-1}$, so that $t(x,p) \to \frac{g(px)}{\|g(px)\|}$. Then, $t(x,0) = \frac{g(0)}{\|g(0)\|} = k$ and $t(x,1) = \frac{g(x)}{\|g(x)\|} = f(x)$. Thus, there homotopy t from the constant map $x \to k$ to f.

Therefore, there is composite homotopy hot from the constant map $x \to k$ to $id_{S^{n-1}}$, which contradicting Theorem 1 above. Thus, the function g must have a zero which must also lie in the interior of D^n.

In the next result, we are going to provide a proof for the even condition of map $\varphi \in C(\mathrm{S}^1, \mathbb{R}^n)$, as shown below;

Theorem 3: If $\varphi \in C(\mathrm{S}^1, \mathbb{R}^n)$, then, $\exists\, x \in S^1$ for which the map φ is an even function.

Proof: We begin the proof assume that the map φ is not even, that is for every $x \in S^1$, $\varphi(-x) \neq \varphi(x)$. Then, there must be a map $t: S^1 \to S^o = \{\pm 1\}$ and $x \to \frac{t(x)-t(-x)}{|t(x)-t(-x)|}$ is continuous. Since the map satisfies odd conditions, that is $\varphi(-x) = -\varphi(x)$, then, it assumes both values $\pm$, which contradicts the connectedness property of circle. This proved the Theorem.

Next, we shall extend Theorem 3 above to the case of the circle S^2 as follow;

Theorem 4: If $\varphi \in C(\mathrm{S}^2, \mathbb{R}^n)$, then, $\exists\, x \in S^2$ for which the map φ is an even function.

Proof: Suppose the function φ is not even, $\forall\, x \in S^2$. Then $\exists\, t: S^2 \to S^1$ and $x \to \frac{t(x)-t(-x)}{|t(x)-t(-x)|}$ is continuous and odd. Since the sphere $S_t^2 \cong D^2$, then the restriction $K\backslash_{S_t^2}$ is homotopy between $K\backslash_{\partial S_t^2}$, where $\partial S_t^2 = \{x \in S^2 \subseteq \mathbb{R}^3 : x_3 = 0\} \cong S^1$, and the constant map of values $t(\mathbb{N})$, where $\mathbb{N} = (0,0,1)$.

By the inclusion map $i: S^1 \to S^2$, $(x_1, x_2) \to (x_1, x_2, 0)$, by the fact that, for a compact metrizable space X, (i) $f \in C(\mathrm{X}, S^1)$ is homotopic to the constant function e^0 if and only if and only if there is $h \in C(\mathrm{X}, \mathbb{R})$ such that $f(x) = e^{it(x)}$, for every $x \in X$, and (ii) if $f, g \in C(\mathrm{X}, S^2)$, there is homotopy equivalent from f to g if and only if $f/g \sim e^0$, where $(f/g)(x) = f(x)/g(x)$, then, the inclusion map $i: S^1 \to S^1$, $x \to t(i(x))$ is exponential map, that is $i(x) = e^{it(x)}$ for some $t \in C(S^2\, \mathbb{R})$. The map t being odd translates to $e^{it(-x)} = -e^{it(x)}$, which is equivalent to $e^{i(t(x)-t(-x))} = -1, \forall\, x$. Provided that the set $\{p \in \mathbb{R}: e^{ip} = -1\} = i\pi(1 + 2\mathbb{Z})$ is discrete, this implies that $x \to t(x) - t(-x)$ is constant, this implies that $t(x) - t(-x) = k, \forall\, x$, since t is odd, that is $t(-x) = -t(x)$, this shows that $k = 0$. Therefore, $t(x) - t(-x) = 0, \forall\, x$, which contradict $e^{i(t(x)-t(-x))} = -1, \forall\, x$. This end the proof.

Theorem 5.0 : Given that $\mathrm{U} = (0, 0, 1)$ is a point on S^1 that lies on a complex plane , then there is a point $V = (x, y, 0)$on such complex plane which relates V to a unique point V^* on the S^1 .

Proof: We can see that this technique is moving from the plane to the unit sphere.

Now, given that $\mathrm{U} = (0, 0, 1)$ is a point on the north pole of the complex plane and we let $V = V^*$ .Then, the equation of a straight line UV is constructed as

$$\frac{X-x_1}{x_2-x_1} = \frac{Y-y_1}{y_2-y_1} = \frac{Z-z_1}{z_2-z_1} \text{……………………………………………………………….(A)}$$

From the statement of the Theorem, we see that $x_1 = y_1 = 0$, and $z_2 = 0\,, z_1 = 1$.

Substitute these values into the equation (A) above, we have, from the point $U = (0, 0, 1)$

$$\frac{X-0}{x_2-0} = \frac{Y-0}{y_2-0} = \frac{Z-1}{0-1} \quad \text{……………………………………………………………….(B)}$$

From the point $V = (x, y, 0)$, equation (B) above above becomes

$$\frac{X-0}{x_2-0} = \frac{Y-0}{y_2-0} = \frac{Z-1}{0-1} \ \ (x_2 = x\,, y_2 = y\,, Z_2 = 0) \ \Rightarrow \frac{X}{x_2} = \frac{Y}{y_2} = \frac{Z-1}{-1},$$

Now, let $\frac{X}{x} = \frac{Y}{y} = \frac{Z-1}{-1} = \tau$……………………………………………………………..(C)

From (C) above,

$\frac{X}{x} = \tau \Rightarrow \mathrm{X} = \mathrm{x}\tau\,, \frac{Y}{y} = \tau \Rightarrow Y = y\tau \ and \ \frac{Z-1}{-1} = \tau \Rightarrow Z - 1 = -\tau\,, Z = 1 - \tau$

From these illustrations, we see that the coordinate of the straight line

$$\mathrm{UV} = (x\tau\,, y\tau\,, 1-\tau)$$

Next, we shall find the value of τ that satisfies the S^1

$X^2 + Y^2 + Z^2 = 1$……………………………………………………..(D)

Substitute for $\mathrm{X} = x\tau\,, Y = y\tau \; and \; Z = 1-\tau$ in (D), we have

$$(x\tau)^2 + (y\tau)^2 + (1-\tau)^2 = 1 \Longrightarrow x^2\tau^2 + y^2\tau^2 + 1 - 2\tau + \tau^2 = 1,$$

$$\tau^2(x^2+y^2+1) - 2\tau = 1 \Longrightarrow \tau^2(x^2+y^2+1) - 2\tau = 0$$

$\Longrightarrow \tau\{(x^2+y^2+1)-2\} = 0$ ……………………………………………………(E)

Since V is on the complex plane, then, we can write $\mathrm{V} = x + iy$ so that the magnitude of V, is $|V|^2 = x^2 + y^2$……………………………………………………………………(F)

Then from (E) above, either $\tau = 0 \; or \; \{\tau(x^2+y^2+1)-2\} = 0, \;\Longrightarrow \tau(x^2+y^2+1) = 2,$ by solving for τ, we have

$\tau = \frac{2}{x^2+y^2+1}$………………………………………………………………………(G)

Put (F) into (G), we have $\tau = \frac{2}{|v|^2+1}$……………………………………………..(H)

Now, we can substitute τ back into $X = x\tau\,, Y = y\tau \; and \; Z = 1-\tau$ as we have them above , we get the following;

$X = x\tau = x.\frac{2}{|v|^2+1} = \frac{2x}{|v|^2+1}$ ……………………………………………………(I)

$Y = y\tau = y.\frac{2}{|v|^2+1} = \frac{2y}{|v|^2+1}$ ……………………………………………...……………(J)

$Z = 1-\tau = 1 - \frac{2}{|v|^2+1} = \frac{|v|^2+1-2}{|v|^2+1} = \frac{|v|^2-1}{|v|^2+1}$ …………………………………(K)

These shows that V has corresponding point on the sphere. From (J) , (K) and (L) above , we have $V^*(X, Y, Z)$, which is called the stereographic projection.

Example 1: Suppose $V = 1 - i\frac{2}{3}$, where 1 is real part and $-\frac{2}{3}$ is imaginary part, from the fact that complex plane V can be written as $x \pm iy\,, x = 1,\; y = -\frac{2}{3}$,

$$\text{Then, } |v|^2 = 1^2 + \left(-\frac{2}{3}\right)^2 = 1 + \frac{4}{9} = \frac{9+4}{9} = \frac{13}{9}$$

Now, we can find X, Y and Z as follows,

$$X = \frac{2x}{|v|^2+1} = \frac{2(1)}{\frac{13}{9}+1} = \frac{2}{\frac{13}{9}+1} = \frac{2}{\frac{22}{9}} = \frac{2}{1} \times \frac{9}{22} = \frac{9}{11} \quad , X = \frac{9}{11},$$

$$Y = \frac{2y}{|v|^2+1} = \frac{2\left(-\frac{2}{3}\right)}{\frac{13}{9}+1} = \frac{-\frac{4}{3}}{\frac{13}{9}+1} = \frac{-\frac{4}{3}}{\frac{22}{9}} = -\frac{4}{3} \times \frac{9}{22} = -\frac{6}{11} \quad , Y = -\frac{6}{11}$$

$$Z = \frac{|v|^2-1}{|v|^2+1} = \frac{\frac{13}{9}-1}{\frac{13}{9}+1} = \frac{\frac{13-1}{9}}{\frac{13+1}{9}} = \frac{\frac{4}{9}}{\frac{22}{9}} = \frac{4}{9} \times \frac{9}{22} = \frac{2}{11} \quad , Z = \frac{2}{11}$$

Therefore, the stereographic projection $V^*(X, Y, Z) = V^*\left(\frac{9}{11}, -\frac{6}{11}, \frac{2}{11}\right)$

Example 2 : Given that $V = 7 + i\sqrt{5},\ x = 7$ (real part) and $y = \sqrt{5}$ (imaginary part) , from the fact that V can be written in complex form as $V = x + iy \implies \mathrm{x} = 7 \text{ and } \mathrm{y} = \sqrt{5}$ as mentioned above,

Next, we need to find the magnitude of V to be able to compute the corresponding stereographic projection

$$|V|^2 = 7^2 + \left(\sqrt{5}\right)^2 = 49 + 5 = 54\,,$$

Now, we can find X, Y and Z as follows,

$$\mathrm{X} = \frac{2x}{|V|^2+1} = \frac{2(7)}{54+1} = \frac{14}{55},$$

$$\mathrm{Y} = \frac{2y}{|V|^2+1} = \frac{2\left(\sqrt{5}\right)}{54+1} = \frac{2\sqrt{5}}{55},$$

$$\mathrm{Z} = \frac{|V|^2-1}{|V|^2+1} = \frac{54-1}{54+1} = \frac{53}{55}.$$

Therefore, the stereographic projection of $V = 7 + i\sqrt{5} = V^*\left(\frac{14}{55}, \frac{2\sqrt{5}}{55}, \frac{53}{55}\right)$

Example 3 : If $V = \sqrt{2} - i\sqrt{3}$, and we want to find the stereographic projection of V ,say $V^*(X, Y, Z)$, we begin by finding the magnitude of V as $|V|^2 = 2 + 3 = 5$, So, as we have written above,

$X = \frac{2x}{|V|^2+1}$, $Y = \frac{2y}{|V|^2+1}$, $Z = \frac{|V|^2-1}{|V|^2+1}$

From $V = \sqrt{2} - i\sqrt{3}$, if we compare this with $V = x + iy \implies x = \sqrt{2}$ and $y = -\sqrt{3}$

So,

$$X = \frac{2x}{|V|^2+1} = \frac{2\sqrt{2}}{5+1} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3},$$

$$Y = \frac{2y}{|V|^2+1} = \frac{2(-\sqrt{3})}{5+1} = \frac{-2\sqrt{3}}{6} = \frac{-\sqrt{3}}{3},$$

$$Z = \frac{|V|^2-1}{|V|^2+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

$$\therefore \text{ The stereographic projection } V^*(X, Y, Z) = V^*\left(\frac{\sqrt{2}}{3}, \frac{-\sqrt{3}}{3}, \frac{2}{3}\right)$$

Conclusion

This paper studied the homotopy of circles and its stereographic projection.

References

Youness, E.K and Jamal, M. (2021). On Banach Contraction Principle and Sehgal fixed point Theorem in Extended b Metric Space. Adv. Fixed point Theory. 1 -12.

Ahmadu, K., & Monsuru, A. M. (2021). On some topological properties on a certain hyperspace topology. *IOSR Journal of Mathematics*. 17*(1)*, 34-38.

Adam and Franzosa. (2000). "Introduction to topology" Pure and Applied, Upper Saddle River: Pearson Prentice Hall.

Adhikari, M.R. (2016). Basic algebraic topology and its Applications. Springer.

Ahmadu, K., and Monsuru, A.M. (2020). Some topological properties and Stone $\hat{C}$ech compactification of Linear Strongly B – convergent topological space of Maddox. Global Scientific Journal. *8(9). 1657 – 1674.*

Monsuru, A.M., Ahmadu, K., and Balla, M. Y. (2020). On the comparative study of compactness and some of its relative notions in Metric and Topological spaces. *International Journal of scientific and Research Publication. 10(10). 659 – 665.*

Micheal, M. (2016). Topology for the working Mathematician