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Enhancing the Daftardar Jafari Method for Solving the Bagley–Torvik Equation through Numerical Approaches

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Abstract

A robust algorithm is introduced in the development of the Enhanced Daftardar Jafari Method (DJM) to effectively address both linear and nonlinear Bagley–Torvik equations (BTE) and other fractional order differential equations. The method's efficacy is demonstrated through numerical examples, showcasing its ability to solve these equations without resorting to linearization or small perturbations. The results affirm the method's strength, accuracy, and simplicity in comparison to alternative approaches.

Keywords: Daftardar Jafari Method; Bagley–Torvik Equation

INTRODUCTION

Fractional calculus, a branch of mathematics dealing with derivatives and integrals of non-integer order, has gained significant attention and utility across various scientific and engineering disciplines in recent years. Its ability to describe complex phenomena with more precision than classical calculus makes it invaluable for modeling processes in physics, engineering, biology, and finance. In particular, fractional-order differential equations have emerged as essential tools for understanding behaviors that exhibit memory effects and anomalous diffusion, such as those described by the Bagley–Torvik equation [Manish K. & Rashmi J. 2016, Bagley &Torvik, 1983].

The generic form of Bagley-Torvik equation can be written as

$$Ay^{n}(t)+BD^{\alpha}y(t)+Cy(t)=f(t) \qquad \ldots(1)$$

$$y^{(i)}(0)=\delta_i,\ i=0,1,2,\ldots,r-1,\ r\in \mathrm{N} \qquad \ldots(2)$$

The equation under consideration, with certain constraints $(1\le n, n\in N, r-1\le \alpha \le r,$ constants $B\ne A, C\in \Re, \delta_i$ determined by initial conditions, and f being a continuous function defined on the interval [0, 1] × ℝ), has its origins in the modeling of the motion of a rigid plate submerged in a Newtonian fluid. This mathematical framework was initially introduced in 1984 and (Raja et al., 2011) extensively discussed the BTE. Later, (Mahdy, A. M. & Mukhtar, 2017) explored the inhomogeneous (BTE), presenting an analytical solution.

Since then, numerous research efforts have been dedicated to solving this BTE. Initially, numerical techniques were employed to reformulate the BTE as a system of functional differential equation of order 1.5. Subsequently, a numerical approach was adopted, which encompassed a generalized version of Taylor's and Bessel's collocation method, as proposed by (Daftardar-gejji & Jafari, 2006) and (Ramadan & Al-luhaibi, 2014). In order to obtain a unique solution for BTE, homogeneous initial conditions are assumed. Here in particular D^{α} denotes the fractional differential operator of order $\alpha \notin N$ in the caputo sense defined by:

$$D^{\alpha}=J^{m-\alpha}y^{m}(t), \qquad \ldots(3)$$

where m is the integer defined by the relation $m-1<q<m$ and J^{α} is a fractional integral operator.

The Bagley–Torvik equation, a fractional-order differential equation, has been employed to represent the mechanical behavior of viscoelastic materials in engineering contexts. Due to its inherent complexity, analytical solutions are often elusive. Consequently, researchers have relied on numerical techniques to obtain solutions [Podlubny, 1999]. One such numerical method is the DaftardarJafari Method, which has demonstrated effectiveness in solving fractional-order differential equations [Daftardar-Gejji&Jafari, 2011]. Nevertheless, there is room for improvement in terms of modification, accuracy and computational efficiency when this method is applied to the Bagley–Torvik equation.

METHODOLOGY

Introduction

To elucidate the concept behind the initial strategy in the DaftardarJafari Method (DJM), one can refer to the subsequent general functional equation, drawing inspiration from works by Podlubny (1999), Raja *et al.* (2011), Ramadan& Al-luhaibi(2015), as well as Ashitha & Ranjini(2020).

Let us consider the nonlinear functional equation [Daftardar-gejji & Bhalekar (2010)]

$$y(x) = g(x) + N[y(x)] \text{ ...(4)}$$

where$N$ is the nonlinear operator and g is a known function. We are looking for y which has the series solution inthe form

$$y = \sum_{i=0}^{\infty} y_i \,. \qquad \text{...(5)}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad \text{...(6)}$$

From Eqns. (5) and (6), Eqn. (4) is equivalent to

$$\sum_{i=0}^{\infty} y_i = g + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad \text{...(7)}$$

We define the recurrence relation:

$$\begin{cases} y_0 = g, \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1}), m = 1,2,\ldots \end{cases} \qquad \ldots(8)$$

Then

$$(y_1 + \ldots + y_{m+1}) = N(y_0 + \ldots + y_m), m = 1,2,\ldots \qquad \ldots(9)$$

and

$$y = g + \sum_{i=0}^{\infty} y_i \ . \qquad \ldots(10)$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eqn. (4).

The proposed Enhanced DaftardarJafari Method (DJM)

In a prior investigation, the utilization of the DJM method was applied to obtain an approximate solution for fractional differential equations. We are now introducing novel algorithms aimed at simplifying the resolution of the Bagley-Torvic Equation (BTE). To elucidate the concept behind these newly developed generalized DJM algorithms, we will examine a general form of the Bagley-Torvic Equation (BTE).

$$Ay^{(n)}(x) + BD^{\alpha} y(x) + Cy(x) = f(x) \qquad \ldots(11)$$

$$y^{(i)} = \delta_i, \qquad i = 0,1,2,\ldots \qquad \ldots(12)$$

Next, by isolating the term associated with the fractional derivative...

$$D^{\alpha} y(x) = F(x) - Gy^{(n)}(x) - Hy(x) \qquad \ldots(13)$$

Where

$$F(x) = \frac{f(x)}{B}, \quad G = \frac{A}{B}, \text{ and } H = \frac{C}{B}, \text{ and then}$$

$$y(x) = J^{\alpha}[F(x) - Gy^{(n)}(x) - Hy(x)] + \sum_{i=0}^{r} y^{(i)}(0) \frac{x^i}{i!} \qquad \ldots(14)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x) = N(y(x)) + g(x) \qquad \ldots(15)$$

where

$$N(y(x)) = J^{\alpha}[F(x) - Gy^{(n)}(x) - Hy(x)] \quad ...(16)$$

In typical cases, N serves as the nonlinear operator; however, when dealing with the BTE, it is employed with linear functions. Additionally, "g" represents a known function, defined as:

$$g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \quad ...(17)$$

In our quest for a solution to Eqn. (15), we seek a representation in the form of a series:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \ ...(18)$$

The operator N can be decomposed into the following

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \quad ...(19)$$

From Eqns. (15), (17) and Eqn. (18)

$$\sum_{i=0}^{\infty} y_i = g(x) + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \quad ...(20)$$

We define the recurrence relation:

$$\begin{cases} y_0 = g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \\ y_1 = J^{\alpha}[F(x) - Gy_0^{(n)}(x) - Hy_0(x)] \\ y_2 = N(y_0 + .y_1) - N(y_0) = \\ J^{\alpha}[F(x) - Gy_0^{(n)}(x) - Hy_0(x) - Gy_1^{(n)}(x) - Hy_1(x)] \\ -J^{\alpha}[F(x) - Gy_0^{(n)}(x) - Hy_0(x)] = J^{\alpha}[-Gy_1^{(n)}(x) - Hy_1(x)] \\ y_3 = N(y_0 + .y_1 + .y_2) - N(y_0 + .y_1) = \\ J^{\alpha}[F(x) - Gy_0^{(n)}(x) - Hy_0(x) - Gy_1^{(n)}(x) - Hy_1(x) - Gy_2^{(n)}(x) - Hy_2(x)] \\ -J^{\alpha}[F(x) - Gy_0^{(n)}(x) - Hy_0(x) - Gy_1^{(n)}(x) - Hy_1(x)] = J^{\alpha}[-Gy_2^{(n)}(x) - Hy_2(x)] \end{cases}$$

...(21)

Then k-term series solution will be in the form

$$y(x) = y_0 + y_1 + y_2 + y_3 + \dots \qquad ...(22)$$

From Eqns. (21), we can deduce the following:

$$\begin{aligned}
y_0 &= g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \\
y_1 &= J^{\alpha}[F(x) - Gy_0{}^{(n)}(x) - Hy_0(x)] \\
y_2 &= J^{\alpha}[-Gy_1{}^{(n)}(x) - Hy_1(x)] \\
y_3 &= J^{\alpha}[-Gy_2{}^{(n)}(x) - Hy_2(x)] \\
&\vdots \\
y_m &= J^{\alpha}[-Gy_{m-1}{}^{(n)}(x) - Hy_{m-1}(x)]
\end{aligned} \qquad ...(23)$$

Or

$$\begin{aligned}
y_0 &= g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \\
y_1 &= J^{\alpha}[y_0(x)] \\
y_2 &= J^{\alpha}[y_1(x)] \\
y_3 &= J^{\alpha}[y_2(x)] \\
&\vdots \\
y_m &= J^{\alpha}[y_{m-1}(x)]
\end{aligned} \qquad ...(24)$$

Suitable Algorithm for Nonlinear BTE

In this section, we introduce a suitable algorithm for solving nonlinear BTE using the enhanced DaftardarJafari Method (DJM). Consider the following nonlinear fractional differential equation of arbitrary order:

$$\begin{cases} D^{\alpha} y(x) + Ly(x) + Ny(x) = g(x), & x > 0, \\ y^{(i)} = \delta_i, & i = 0,1,2,\dots \end{cases} \qquad ...(25)$$

where L is a linear operator, N, represent a nonlinear operator, $g(x)$ is the source term, and D^{α} is the Caputo fractional derivative of order with $m-1<\alpha-m$. To solve Eqn. (25) bymeans of the enhanced DJM, we apply the operator J^{α}, the inverse of the operator D^{α}, to both sides of Eqn. (25) as follows:

$$y(x)=\sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!}+J^{\alpha}[-Ly(x)-Ny(x)+g(x)]. \qquad ...(26)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x)=N(y(x))+\sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} \qquad ...(27)$$

where

$$N(y(x))=J^{\alpha}[g(x)-Ly(x)-Ny(x)] \qquad ...(28)$$

In our quest for a solution to Eqn. (27), we seek a representation in the form of a series:

$$y(x)=\sum_{i=0}^{\infty} y_i(x). \qquad ...(29)$$

The operator N can be decomposed into the following

$$N\left(\sum_{i=0}^{\infty} y_i\right)=N(y_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right)-N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad ...(30)$$

From Eqns. (27), (29) and Eqn. (30)

$$\sum_{i=0}^{\infty} y_i=\sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!}+N(y_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right)-N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad ...(31)$$

We define the recurrence relation:

$$\begin{cases}
y_0 = \sum_{i=0}^{r} y^{(i)}(0)\dfrac{x^i}{i!}, \\
y_1 = J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x)] \\
y_2 = N(y_0 + y_1) - N(y_0) \\
= J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x)] \\
- J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x)] = J^{\alpha}[-Ly_1(x) - Ny_1(x)] \\
y_3 = N(y_0 + y_1 + y_2) - N(y_0 + y_1) = \\
J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x) - Ly_2(x) - Ny_2(x)] \\
- J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x)] = \\
J^{\alpha}[-Ly_2(x) - Ny_2(x)] \\
y_4 = N(y_0 + y_1 + y_2 + y_3) - N(y_0 + y_1 + y_2) = \\
J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x) - Ly_2(x) - Ny_2(x) - Ly_3(x) - Ny_3(x)] \\
- J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x) - Ly_2(x) - Ny_2(x)] = \\
J^{\alpha}[-Ly_3(x) - Ny_3(x)] \\
y_5 = N(y_0 + y_1 + y_2 + y_3 + y_4) - N(y_0 + y_1 + y_2 + y_3) = \\
J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x) - Ly_2(x) - Ny_2(x) - Ly_3(x) - Ny_3(x) - Ly_4(x) - Ny_4(\\
- J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - Ly_1(x) - Ny_1(x) - Ly_2(x) - Ny_2(x) - Ly_3(x) - Ny_3(x)] = \\
J^{\alpha}[-Ly_4(x) - Ny_4(x)]
\end{cases} \quad \text{...(32)}$$

$$\vdots$$

Then k-term series solution will be in the form

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + \dots \quad \text{...(33)}$$

From Eqns. (32), we can deduce the following:

$$y_0 = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!},$$

$$y_1 = J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x)]$$

$$y_2 = J^{\alpha}[-Ly_1(x) - Ny_1(x)] \quad \text{...(34)}$$

$$y_3 = J^{\alpha}[-Ly_2(x) - Ny_2(x)]$$

$$\vdots$$

$$y_m = J^{\alpha}[-Ly_{m-1}(x) - Ny_{m-1}(x)]$$

RESULTS AND DISCUSSION

Introduction

This chapter presents the results obtained through the application of the enhanced DaftardarJafari Method to solve the Bagley–Torvik equation. The numerical simulations and experiments conducted to evaluate the effectiveness of the modified method are outlined in this section. The discussion will focus on interpreting the results and their significance in the context of the research objectives.We will apply the enhanced DaftardarJafari Method as follows:

Bagley–Torvik Equation

The effectiveness of the enhanced DaftardarJafari Methodfor solving Bagley–Torvik equation equations can be illustrated as follows:

Example 1:

Consider the following linear BTE (Abu Arqub & Maayah, 2018)

$$y''(x) + D^{\frac{3}{2}} y(x) + y(x) = 1 + x \quad ...(35)$$ Subject to

$$y(0) = y'(0) = 1, \quad ...(36)$$

where the exact solution is

$$y(x) = 1 + x \quad ...(37)$$

Now, by applying the Eqn. (23), we get:

$$y_0(x) = 1 + x \quad ...(38)$$

$$y_1(x) = J^{\frac{3}{2}}[1 + x - 1 - x] = 0 \quad ...(39)$$

$$y_2(x) = J^{\frac{3}{2}}[-0 - 0] = 0 \quad ...(40)$$

Next, it becomes readily apparent that the precise solution will assume the following format. $y(x) = y_0 + y_1 + y_2 + ... = 1 + x$ ...(41)

It is noteworthy to mention that (Hemeda, 2015) addressed the same problem but introduced a minor modification. He utilized distinct initial values, resulting in an approximate solution. The 6-term approximate solution assumed the following structure.

$$y(x)=\sum_{i=0}^{5}y_i=1+x-\frac{x^6}{144}-\frac{5x^7}{5040}-\frac{x^9}{36288}-\frac{x^{10}}{36880}-\frac{x^{12}}{479001600}-\frac{x^{13}}{6227020800}$$
$$-\frac{32x^{4.5}}{945\sqrt{\pi}}-\frac{64x^{5.5}}{10395\sqrt{\pi}}-\frac{512x^{7.5}}{405405\sqrt{\pi}}-\frac{1024x^{8.5}}{681885\sqrt{\pi}}-\frac{512x^{10.5}}{687465529\sqrt{\pi}}-\frac{128x^{5.5}}{1976463395\sqrt{\pi}} \quad ...(42)$$

However, here we got the exact solution after our modification

Example 2:

Consider the following nonlinear BTE (Torvik & Bagley, 1984; Wiley, 2012)

$$y'''(x)+D^{\frac{5}{2}}y(x)+y^2(x)=x^4,\ y(0)=y'(0)=0,\ y''(0)=2 \quad ...(43)$$

where the exact solution is

$$y(x)=x^2 \quad ...(44)$$

Isolating the unknown function and using Eqn. (20)

$$y(x)=J^{\frac{5}{2}}[-D^3y-y^2+x^4]+x^2 \quad ...(45)$$

Now, by applying the Eqn. (23), we get:

$$y_0(x)=x^2 \quad ...(46)$$

and

$$Ny(x)=J^{\frac{5}{2}}[-D^3y-y^2+x^4] \quad ...(47)$$

$$y_1(x)=N(y_0)=J^{\frac{5}{2}}[-D^3y_0-y_0{}^2+x^4]=0 \quad ...(48)$$

$$y_2(x)=N(y_0+y_1)-N(y_0)=J^{\frac{5}{2}}[-D^3(y_0+y_1)-(y_0+y_1)^2+x^4]=0 \quad ...(49)$$

$$\vdots$$

When we consider the higher-order terms, it becomes evident that they all give zero when the enhanced DJM is employed. It is straightforward to determine that the exact solution will adopt the following form.

$$y(x)=y_0+y_1+y_2+...=x^2 \quad ...(50)$$

It's worth mentioning that we attempted the conventional DJM approach, but we only obtained an approximate solution to the same problem, which assumes the following form.

$$y(x) = x^2 - \frac{x^6}{30} - \frac{17x^{\frac{39}{2}}}{221760} + \dots \qquad \dots(51)$$

Table 1: Numerical solution of DJM and Exact solution

x	EXACT	DJM (n=5)	ABSOLUTE ERROR
0	0	0	0
0.1	0.01	0.01	0
0.2	0.04	0.039998	2.13333E-06
0.3	0.09	0.089976	2.43E-05
0.4	0.16	0.159863	0.000136533
0.5	0.25	0.249479	0.000520833
0.6	0.36	0.358445	0.001555204
0.7	0.49	0.486078	0.003921706
0.8	0.64	0.631261	0.008739121
0.9	0.81	0.792275	0.017724524
1	1	0.96659	0.033409993

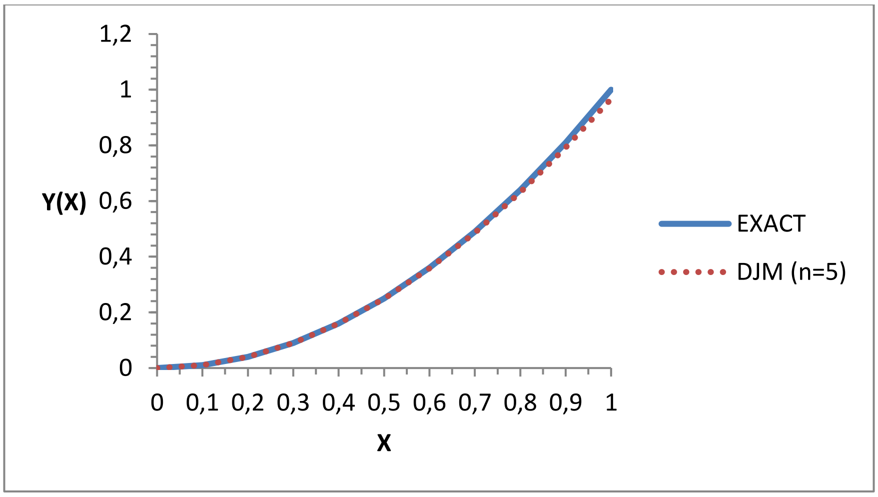

Figure 1: the graph of for exact solution and DJM

Example 3:

Consider the fractional Bagley-Torvik equation (Askari, 2023)

$$y''(x)+y^{\frac{3}{2}}(x)-6x-\frac{8x^{\frac{3}{2}}}{\sqrt{\pi}}=0 \qquad ...(52)$$

with the initial conditions

$$y(0)=y'(0)=1. \qquad ...(53)$$

The exact solution is as follows:

$$y(x)=x^{3}+x+1 \qquad ...(54)$$

Separating the unknown function and employing Eqn. (24), we get

$$y(x)=J^{\frac{3}{2}}\left[6x+\frac{8x^{\frac{3}{2}}}{\sqrt{\pi}}-y''(x)\right]+1+x \qquad ...(55)$$

Now, by applying the Eqn. (27), we get:

$$y_0(x)=1+x \qquad ...(56)$$

and

$$Ny(x)=J^{\frac{3}{2}}\left[6x+\frac{8x^{\frac{3}{2}}}{\sqrt{\pi}}-y''(x)\right] \qquad ...(57)$$

$$y_1(x)=N(y_0)=J^{\frac{3}{2}}\left[6x+\frac{8x^{\frac{3}{2}}}{\sqrt{\pi}}-y_0''(x)\right]=\frac{16}{5}\frac{x^{5/2}}{\sqrt{\pi}}+x^{3} \qquad ...(58)$$

$$y_2(x)=J^{\frac{3}{2}}\left[-y_1''(x)\right]=-3x^{2}-\frac{16}{5}\frac{x^{5/2}}{\sqrt{\pi}} \qquad ...(59)$$

$$y_3(x)=J^{\frac{3}{2}}\left[-y_2''\right]=\frac{8x^{3/2}}{\sqrt{\pi}}+3x^{2} \qquad ...(60)$$

$$y_4(x)=J^{\frac{3}{2}}\left[-y_3''\right]=-6x-\frac{8x^{3/2}}{\sqrt{\pi}} \qquad ...(61)$$

$$y_5(x) = J^{\frac{3}{2}}\left[-y_4''\right] = 6x \qquad ...(62)$$

$$y_6(x) = J^{\frac{3}{2}}\left[-y_5''\right] = 0 \qquad ...(63)$$

$$\vdots$$

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + ... = x^3 + x + 1 \qquad ...(64)$$

It's noteworthy to highlight that we achieved the precise solution using only fewer number of terms from the enhanced DJM approach, as illustrated in the aforementioned example 3.

CONCLUSION

The aim of this study is to enhance the DJM for obtaining an exact solution to the Bagley–Torvik equation. The proposed modification, referred to as the enhanced DJM, forms the basis for understanding the fundamental concept of obtaining an accurate solution. Consequently, we have introduced a successful algorithm for solving the BTE. The algorithms presented in this work yielded precise solutions for all examples examined, demonstrating the efficiency, accuracy, and reliability of the method in solving both linear and nonlinear Bagley–Torvik equations.

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