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Enhancement of the Kamal Transform Method with the He's Polynomial for Solving Partial Differential Equations (Telegraph Equation)

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Abstract

This study proposes a hybrid solution methodology that integrates the Kamal Transform Method (KTM) with He's Polynomial Method (HPM) for solving nonlinear partial differential equations (PDEs), with a focus on the telegraph equation. The telegraph equation, which models wave propagation and diffusive behaviors, presents significant challenges in terms of nonlinearity, complex boundary conditions, and slow convergence in traditional methods. By combining the transformation power of the Kamal method with the iterative, rapidly converging He's polynomial method, this research aims to enhance the accuracy, convergence, and computational efficiency of existing solution techniques for PDEs. The proposed hybrid approach is applied to both linear and nonlinear forms of the telegraph equation, demonstrating excellent agreement with exact solutions and offering significant improvements in accuracy, especially in the presence of nonlinearities. Comparative analyses with traditional methods, including Elzaki's transform, show that the Kamal-

He's polynomial method outperforms existing techniques in terms of error reduction. The results highlight the method's potential for broader application in various fields of engineering, physics, and applied sciences, where complex, nonlinear PDEs are commonly encountered.

Keywords: Kamal Transform Method, He's polynomial, Partial differential equations

INTRODUCTION

Partial differential equations (PDEs) are essential for modeling a broad range of physical and engineering phenomena, describing the interactions between variables such as time and space in systems like wave propagation, signal transmission, and heat conduction. Among these, the telegraph equation is particularly significant for its ability to model systems with both wave-like and diffusive behaviors. Mathematically, the telegraph equation is expressed as:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2\alpha \frac{\partial u}{\partial t} = 0, \qquad ...(1)$$

where $u(x,t)$ is the signal amplitude, c is the wave propagation speed, and α is the damping coefficient. This equation arises in various applications, such as electrical signal transmission, seismic wave modeling, and acoustic systems. Solving this equation efficiently and accurately is challenging, particularly when nonlinearities or complex boundary conditions are involved (Kamal, 2016). The Kamal Transform Method (KTM) has proven to be a powerful tool for solving PDEs, effectively transforming them into simpler forms for analytical or numerical solutions. Despite its utility, the method has limitations, including slow convergence and difficulty addressing highly nonlinear boundary or source terms (Al-aati & Al-maweri, 2024; Johansyah et al., 2022). To address these challenges, researchers have explored hybrid techniques that combine KTM with other methods to enhance its efficiency and accuracy. One promising approach is the integration of He's polynomial method, an iterative technique renowned for its rapid convergence and ability to approximate solutions to nonlinear differential equations. He's polynomial method constructs solutions as rapidly converging series, making it particularly effective for tackling nonlinearities and complex boundary conditions. Its applications span diverse fields, including nonlinear wave equations (Wang & Li, 2020), environmental modeling (Zhang et al., 2021), and elasticity problems in materials science (Li & He, 2022). This study proposes the enhancement of the Kamal Transform Method by incorporating He's polynomial method to solve PDEs, with a specific focus on the telegraph equation. By combining the strengths of KTM and He's polynomial, the hybrid approach seeks to address the limitations of each method individually, achieving improved convergence, accuracy, and computational efficiency. The rapid iterative nature of He's polynomial is expected to complement the transformational capabilities of KTM, resulting in a robust and efficient solution methodology (Al Farei & Boulbrachene, 2022). This research aims to develop and validate a hybrid Kamal Transform-He's Polynomial method as a powerful tool for solving the telegraph equation and other nonlinear PDEs. The proposed innovation has the potential to advance the mathematical modeling of complex systems across engineering, physics, and applied sciences.

Basic Theory

Kamal Transformation

The Kamal's transformation is defined as

$$\mathrm{K}[f(t)] = G(v) = \int_0^\infty f(t) e^{-\frac{t}{v}} dt, \; t \geq 0, \; k_1 \leq v \leq k_2. \quad \ldots(2)$$

Furthermore, the inverse of the Kamal transformation is denoted by

$$\mathrm{K}^{-1}[G(v)] = f(t), \; t \geq 0. \quad \ldots(3)$$

The following are some cases of Kamal transformations for the following simple functions:

- $f(t) = 1$ where $t \geq 0$, based on Eqn. (2), we get

$$\mathrm{K}[1] = G(v) = \int_0^\infty e^{-\frac{t}{v}} dt = \left[-ve^{-\frac{t}{v}} \right]_0^\infty = v. \quad \ldots(4)$$

- $f(t) = t$ where $t \geq 0$, based on Eqn. (2), we get

$$\mathrm{K}[t] = G(v) = \int_0^\infty te^{-\frac{t}{v}} dt = v^2. \quad \ldots(5)$$

For $f(t)=t^n$ where $t \geq 0,$ we get

$$\mathrm{K}\left[t^n\right]=n!v^{n+1}. \quad \ldots(6)$$

Theorem 1:

Let $G(v)$ be the Kamal transformation of $f(t)$, then

- $\mathrm{K}[f'(t)]=\dfrac{G(v)}{v}-f(0),$
- $\mathrm{K}[f''(t)]=\dfrac{G(v)}{v^2}-\dfrac{f(0)}{v}-f'(0),$
- $\mathrm{K}\left[f^n(t)\right]=\dfrac{G(v)}{v^n}-\sum_{k=0}^{n-1}\dfrac{f^{(k)}(0)}{v^{n-k-1}}.$

Table 1: Kamal Transform Of Useful Functions

S/N	$f(t)$	$\mathrm{K}[f(t)]=G(v)$
1	1	v
2	t	v^2
3	t^2	$2!v^3$
4	$t^n, n>0$	$n!v^{n+1}$
5	e^{at}	$\dfrac{v}{1-av}$
6	$\sin at$	$\dfrac{av^2}{1+a^2v^2}$
7	$\cos at$	$\dfrac{v}{1+a^2v^2}$
8	$\sinh at$	$\dfrac{av^2}{1-a^2v^2}$
9	$\cosh at$	$\dfrac{v}{1-a^2v^2}$

Table 2: Inverse Kamal Transform Of Useful Functions

S/N	$G(v)$	$f(t) = K^{-1}[G(v)]$
1	v	1
2	v^2	t
3	$2!v^3$	t^2
4	$n!v^{n+1}$	$t^n, n > 0$
5	$\frac{v}{1-av}$	e^{at}
6	$\frac{av^2}{1+a^2v^2}$	$\sin at$
7	$\frac{v}{1+a^2v^2}$	$\cos at$
8	$\frac{av^2}{1-a^2v^2}$	$\sinh at$
9	$\frac{v}{1-a^2v^2}$	$\cosh at$

The He's Polynomial Method

Let θ be an operator (integral or differential), such that:

$$\theta(v) = 0 \qquad ...(7)$$

Suppose we defined a convex homotopy function, $H(c, f)$ by:

$$H(c, f) = f\theta(c) + (1 - f)G(c), \qquad ...(8)$$

Such that $G(c)$ is referred to as a functional operator. Hence, we get:

$H(c,0) = G(c)$ and $H(c,1) = \theta(c)$,

If $H(c, f) = 0$ is satisfied and a given embedded parameter, $f \in (0,1]$ is considered. In HPM, $f = P$ is applied as an expanding parameter terms to obtain:

$$\begin{cases} c = v = \lim_{p \to 1}\left(\sum_{j=0}^{\infty} p^j v_j\right) \\ N(v) = \sum_{j=0}^{\infty} p^j H_{j.} \end{cases}, \qquad ...(9)$$

The approach takes the nonlinear term to be $N(v)$ whenever Eqn. (7) is decomposed, such that $H_{k.}'s$ are the so-called He's polynomials defined as:

$$H_k(v_0, v_1, v_2, \ldots v_k) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{\infty} p^j v_j\right)\right)_{p=0}, \qquad k \geq 0. \qquad \ldots(10)$$

Analysis of Kamal and He's Polynomial Method for Solving Partial Differential Equations

This section examines the application of the combined Kamal Transformation method and He's polynomial approach to solve nonlinear partial differential equations (PDEs). The method addresses equations of the form:

$$L_t^n y(x,t) + Ry(x,t) + Ny(x,t) = g(x,t), \; t > 0, \qquad \ldots(11)$$

Where R and N are linear and nonlinear operators, respectively, with the initial conditions

$$y^{(k)}(x,0) = c_k, \; k = 0,1,\ldots,n-1 \qquad \ldots(12)$$

Taking the Kamal transformation, we get

$$K\{L_t^n y(x,t)\} = K\{g(x,t) - Ry(x,t) - Ny(x,t)\}, \; t > 0, \qquad \ldots(13)$$

$$\frac{K\{y(x,t)\}}{v^n} - \frac{\sum_{k=0}^{n-1} y^k(x,0)}{v^{n-k-1}} = K\{g(x,t) - Ry(x,t) - Ny(x,t)\}. \qquad \ldots(14)$$

This is equivalent to

$$\begin{aligned} K\{y(x,t)\} &= vy(x,0) + v^2 y'(x,0) + \ldots + v^n y^{(n-1)}(x,0) + v^n K\{g(x,t)\} \\ &- K\{Ry(x,t) + Ny(x,t)\}. \end{aligned} \qquad \ldots(15)$$

Applying the inverse of the KH of Eqn. (15), we get

$$\begin{aligned} y(x,t) &= y(x,0) + ty'(x,0) + \ldots + \frac{t^n}{n!} y^{(n-1)}(x,0) + K^{-1}\left[v^n K\{g(x,t)\}\right] \\ &- K^{-1}\left[v^n K\{Ry(x,t) + Ny(x,t)\}\right] \end{aligned} \qquad \ldots(16)$$

The infinite series shown here reflects the KH solution of $y(x,t)$ as

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t), \qquad \ldots(17)$$

where $y_n(x,t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(x,t)=\sum_{n=0}^{\infty}H_n(t), \qquad \text{...(18)}$$

By applying Eqn. (17) and Eqn. (18) in Eqn. (16)

$$\begin{aligned}\sum_{n=0}^{\infty}y_n(x,t)&=y(x,0)+ty'(x,0)+\ldots+\frac{t^n}{n!}y^{(n-1)}(x,0)+K^{-1}\left[v^nK\{g(x,t)\}\right]\\&-K^{-1}\left[v^nK\left\{R\sum_{n=0}^{\infty}y_n(x,t)+\sum_{n=0}^{\infty}H_n(x,t)\right\}\right].\end{aligned} \qquad \text{...(19)}$$

Comparing both sides of Eqn. (19), we get

$$\begin{cases}y_0(x,t)=y(x,0)+ty'(x,0)+\ldots+\dfrac{t^n}{n!}y^{(n-1)}(x,0)+K^{-1}\left[v^nK\{g(x,t)\}\right]\\ y_1(x,t)=-K^{-1}\left[v^nK\{Ry_0(x,t)+H_0(x,t)\}\right]\\ y_2(x,t)=-K^{-1}\left[v^nK\{Ry_1(x,t)+H_1(x,t)\}\right]\\ y_{n+1}(x,t)=-K^{-1}\left[v^nK\{Ry_n(x,t)+H_n(x,t)\}\right], \quad n=0,1,2,\ldots\end{cases} \qquad \text{...(20)}$$

Thus, the approximate solution of Eqn. (11) is:

$$y(x,t)=y_0(x,t)+y_1(x,t)+y_2(x,t)+\ldots \qquad \text{...(21)}$$

RESULTS

The present section examines the application of the proposed method on different test telegraph equations and PDEs in general.

Nonlinear Telegraph Equation

Here, we will study the nonlinear Telegraph Equation

Example 1:

Let the following nonlinear telegraph Equation be considered (Jassim & Issa, 2024):

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + u_x(x,t) - u^2(x,t) + xu(x,t)u_x(x,t) \\ u(x,0) = x \\ u_t(x,0) = x \end{cases} \quad ...(22)$$

with an exact solution of the form:

$$u(x,t) = xe^t \quad ...(23)$$

By merging Kamal's and He's polynomials from Eqn. (20), Eqn. (22) transforms into

$$\frac{K(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) = K\{u_{xx}(x,t) + u_x(x,t) - u^2(x,t) + xu(x,t)u_x(x,t)\} \quad ...(24)$$

Arranging and substituting the given initial conditions, we get

$$K(v) = vx + v^2x + v^2K\{u_{xx}(x,t) + u_x(x,t) - u^2(x,t) + xu(x,t)u_x(x,t)\} \quad ...(25)$$

Applying the inverse of Kamal transform

$$u(x,t) = K^{-1}\{vx + v^2x\} + K^{-1}\{v^2K(u_{xx}(x,t) + u_x(x,t) - u^2(x,t) + xu(x,t)u_x(x,t))\} \quad ...(26)$$

$$u(x,t) = x + xt + K^{-1}\{v^2K(u_{xx}(x,t) + u_x(x,t) - u^2(x,t) + xu(x,t)u_x(x,t))\} \quad ...(27)$$

Hence

$$u_{n+1}(x,t) = x + xt + K^{-1}\left\{v^2K\left(\sum_{n=0}^{\infty} u_{n,xx}(x,t) + \sum_{n=0}^{\infty} u_{n,x}(x,t) - \sum_{n=0}^{\infty} H_n(x,t) + \sum_{n=0}^{\infty} B_n(x,t)\right)\right\} \quad ...(28)$$

The initial term

$$u_0(x,t) = x + xt$$

$$u_1(x,t) = \frac{t^2}{2}x$$

$$u_2(x,t) = \frac{t^3}{6}x$$

$$u_3(x,t) = \frac{t^4}{24}x$$

Therefore, the approximate solution is

$$u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= x + xt + \frac{t^2}{2}x + \frac{t^3}{6}x + \frac{t^4}{24}x\ldots$$

Table 3: Comparative analysis of the approximate and exact solutions plotted at $(t = 0.5)$

x	EXACT	KAMAL	ERROR
0	0	0	0
0.1	0.164872	0.164872	1.65E-07
0.2	0.329744	0.329744	3.31E-07
0.3	0.494616	0.494616	4.96E-07
0.4	0.659489	0.659488	6.61E-07
0.5	0.824361	0.82436	8.26E-07
0.6	0.989233	0.989232	9.92E-07
0.7	1.154105	1.154104	1.16E-06
0.8	1.318977	1.318976	1.32E-06
0.9	1.483849	1.483848	1.49E-06
1	1.648721	1.64872	1.65E-06

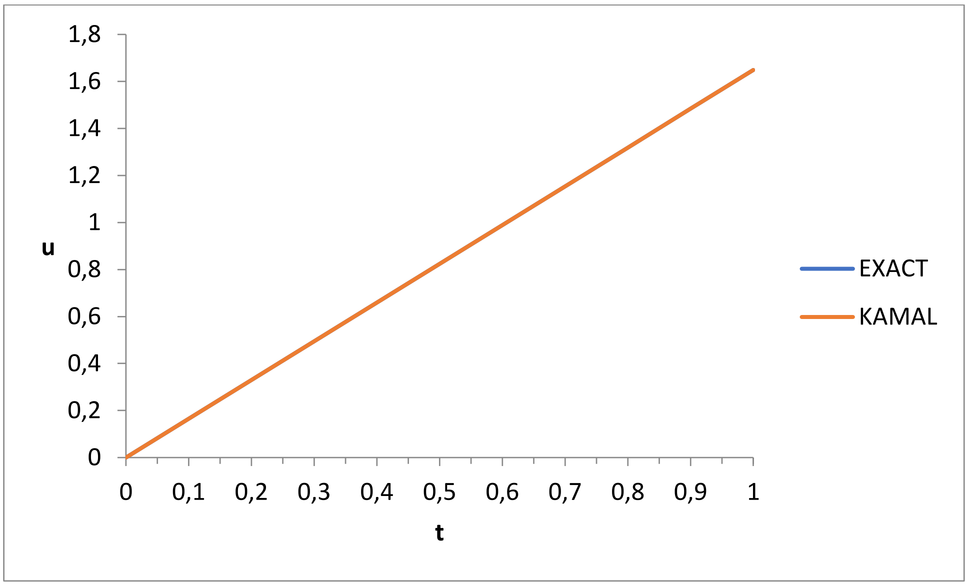

Figure 1: Plot of the approximate and exact solutions at $(t = 0.5)$

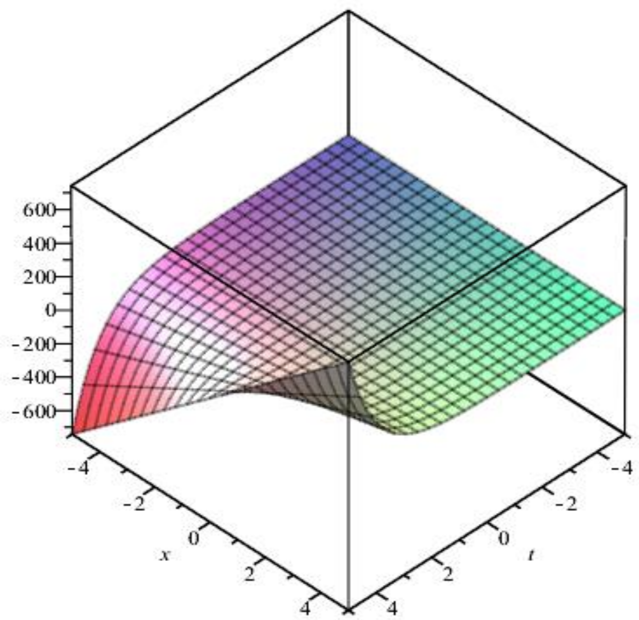

Figure 2: The surface shows the approximate Kamal solution for example 1

Linear Telegraph Equation

Here, we will study the linear Telegraph Equation

Example 2:

Let the following linear telegraph Equation be considered (Magzoub *et al.*, 2024):

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - 2u_t(x,t) - u(x,t) \\ u(x,0) = e^x \\ u_t(x,0) = -2e^x \end{cases} \qquad ...(29)$$

with an exact solution of the form:

$$u(x,t) = e^{t-2x} \qquad ...(30)$$

By merging Kamal's and He's polynomials from Eqn. (20), Eqn. (29) transforms into

$$\frac{K(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) = K\{u_{xx}(x,t) - 2u_t(x,t) - u(x,t)\} \qquad ...(31)$$

Arranging and substituting the given initial conditions, we get

$$K(v) = ve^x - 2v^2e^x + v^2K\{u_{xx}(x,t) - 2u_t(x,t) - u(x,t)\} \qquad ...(32)$$

Applying the inverse of Kamal transform

$$u(x,t) = K^{-1}\{ve^x - 2v^2e^x\} + K^{-1}\{v^2K(u_{xx}(x,t) - 2u_t(x,t) - u(x,t))\} \qquad ...(33)$$

$$u(x,t) = e^x - 2te^x + K^{-1}\{v^2K(u_{xx}(x,t) - 2u_t(x,t) - u(x,t))\} \qquad ...(34)$$

Hence

$$u_{n+1}(x,t) = e^x - 2te^x + K^{-1}\{v^2K(u_{n,xx}(x,t) - 2u_{n,t}(x,t) - u_n(x,t))\} \qquad ...(35)$$

The initial term

$$u_0(x,t) = e^x - 2te^x$$

$$u_1(x,t) = K^{-1}\{v^2K(u_{0,xx}(x,t) - 2u_{0,t}(x,t) - u_0(x,t))\} = K^{-1}\{v^2K(4e^x)\} = 2e^xt^2$$

$$u_2(x,t) = -\frac{4}{3}e^xt^3$$

$$u_3(x,t) = \frac{2}{3}e^x t^4$$

Therefore, the approximate solution is

$$u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= e^x - 2te^x + 2e^x t^2 - \frac{4}{3}e^x t^3 + \frac{2}{3}e^x t^4 \ldots$$

Table 4: Comparative analysis of the approximate and exact solutions plotted at $(t = 0.5)$

x	EXACT	KAMAL	ERROR
0	0.367879	0.367879	2.52371E-07
0.1	0.40657	0.406569	2.78914E-07
0.2	0.449329	0.449329	3.08247E-07
0.3	0.496585	0.496585	3.40666E-07
0.4	0.548812	0.548811	3.76494E-07
0.5	0.606531	0.60653	4.1609E-07
0.6	0.67032	0.67032	4.59851E-07
0.7	0.740818	0.740818	5.08214E-07
0.8	0.818731	0.81873	5.61663E-07
0.9	0.904837	0.904837	6.20734E-07
1	1	0.999999	6.86017E-07

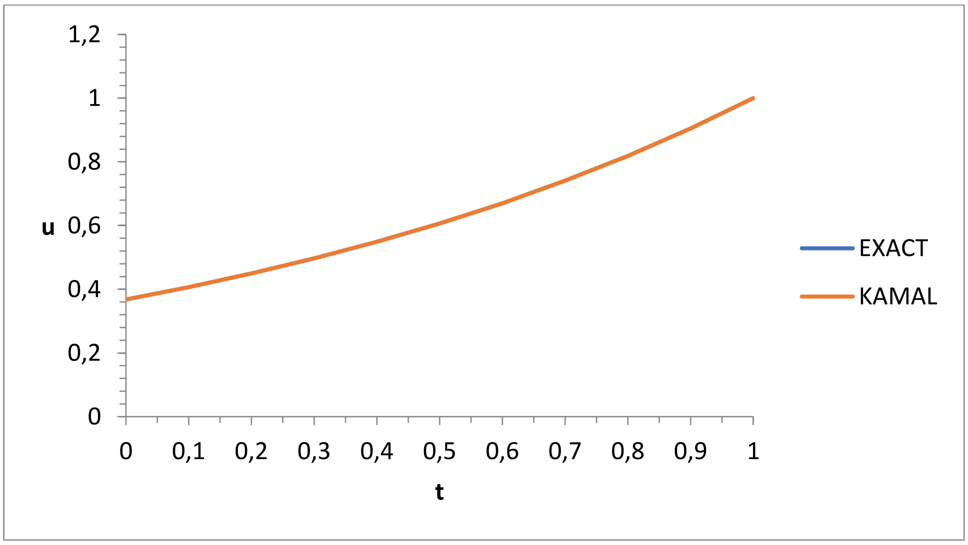

Figure 3: Plot of the approximate and exact solutions

Table 5: Comparative analysis of the approximate by KAMAL, ELZAKI and exact solutions

x	EXACT	KAMAL	ELZAKI	KAMAL ERROR	ELZAKI ERROR
0	0.367879	0.367879	0.333333	2.52371E-07	0.034546108
0.1	0.40657	0.406569	0.36839	2.78914E-07	0.038179354
0.2	0.449329	0.449329	0.407134	3.08247E-07	0.042194711
0.3	0.496585	0.496585	0.449953	3.40666E-07	0.046632368
0.4	0.548812	0.548811	0.497275	3.76494E-07	0.051536737
0.5	0.606531	0.60653	0.549574	4.1609E-07	0.056956903
0.6	0.67032	0.67032	0.607373	4.59851E-07	0.062947113
0.7	0.740818	0.740818	0.671251	5.08214E-07	0.069567318
0.8	0.818731	0.81873	0.741847	5.61663E-07	0.076883777
0.9	0.904837	0.904837	0.819868	6.20734E-07	0.084969714
1	1	0.999999	0.906094	6.86017E-07	0.093906057

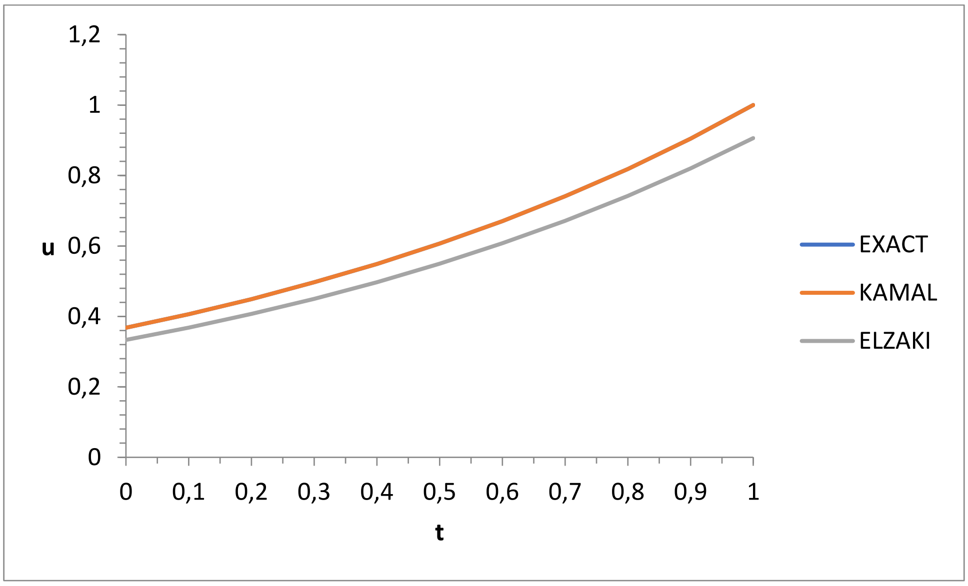

Figure 4: Plots of the approximate by the KAMAL, ELZAKI and exact solutions

Example 3:

Consider the one-dimensional space telegraph equation (Jassim & Issa, 2024):

$$\begin{cases} u_{xx}(x,t) = u_{tt}(x,t) + u_t(x,t) + u(x,t) \\ u(0,t) = e^{-t} \\ u_x(0,t) = e^{-t} \\ u(x,0) = e^{x} \\ u_t(x,0) = 0 \end{cases} \quad ...(36)$$

with an exact solution of the form:

$$u(x,t) = e^{x-t} \quad ...(37)$$

By merging Kamal's and He's polynomials from Eqn. (20), Eqn. (36) transforms into

$$\frac{K(v)}{v^2} - \frac{u(0,x)}{v} - u_t(0,x) = K\{u_{tt}(x,t) + u_t(x,t) + u(x,t)\} \quad ...(38)$$

Arranging and substituting the given initial conditions, we get

$$K(v) = (1-v)e^{-t} + v^2 K\{u_{tt}(x,t) + u_t(x,t) + u(x,t)\} \quad ...(39)$$

Applying the inverse of Kamal transform to the both sides of Eqn. (39), yields

$$u(x,t) = (1-x)e^{-t} + K^{-1}\{v^2 K(u_{tt}(x,t) + u_t(x,t) + u(x,t))\} \quad . \quad ..(40)$$

Then, we have

$$u_0(x,t) = (1-x)e^{-t}$$

Next, when we use $u_0(x,t)$ to calculate $u_1(x,t)$

$$u_1(x,t) = K^{-1}\{v^2 K(u_{tt,0}(x,t) + u_{t,0}(x,t) + u_0(x,t))\} = e^{-t}\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right)$$

After that using $u_1(x,t)$, we get

$$u_2(x,t) = K^{-1}\{v^2 K(u_{tt,1}(x,t) + u_{t,1}(x,t) + u_1(x,t))\} = e^{-t}\left(\frac{1}{24}x^4 + \frac{1}{120}x^5\right)$$

Now use $u_2(x,t)$ calculate $u_3(x,t)$

$$u_3(x,t) = K^{-1}\left\{v^2 K\left(u_{tt,2}(x,t) + u_{t,2}(x,t) + u_2(x,t)\right)\right\} = e^{-t}\left(\frac{1}{720}x^6 + \frac{1}{5040}x^7\right)$$

So that, the approximate solution is

$$u(x,t) = (1-x)e^{-t} + e^{-t}\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) + e^{-t}\left(\frac{1}{24}x^4 + \frac{1}{120}x^5\right) + e^{-t}\left(\frac{1}{720}x^6 + \frac{1}{5040}x^7\right) + \ldots$$

Table 6: Comparative analysis of the approximate by the KAMAL at $t = 0.5$ in comparison with the exact solutions

x	EXACT	KAMAL	ERROR
0	0.606531	0.60653066	1.26334E-11
0.1	0.67032	0.670320046	1.44457E-11
0.2	0.740818	0.740818221	1.7316E-11
0.3	0.818731	0.818730753	2.21252E-11
0.4	0.904837	0.904837418	4.39888E-11
0.5	1	1	0
0.6	1.105171	1.105170917	1.10621E-09
0.7	1.221403	1.221402753	5.08482E-09
0.8	1.349859	1.349858788	1.93968E-08
0.9	1.491825	1.491824634	6.34918E-08
1	1.648721	1.648721087	1.83778E-07

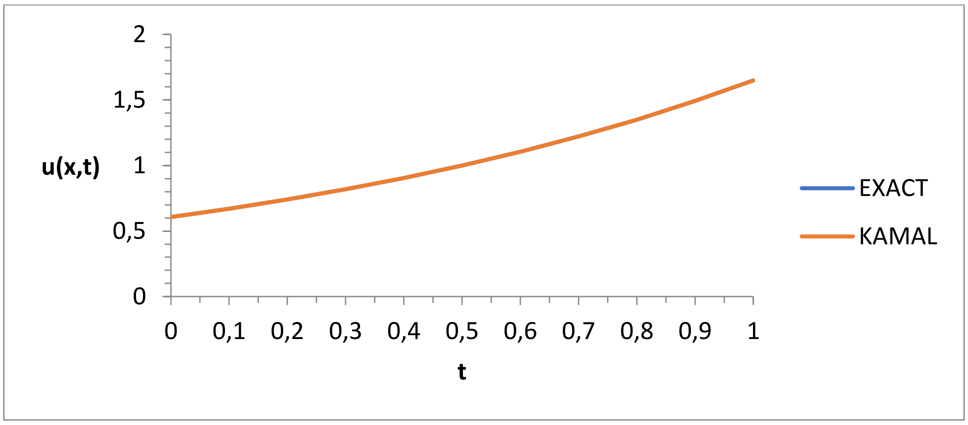

Figure 5: Plots of the approximate by the KAMAL at $t = 0.5$ in comparison with the exact solutions

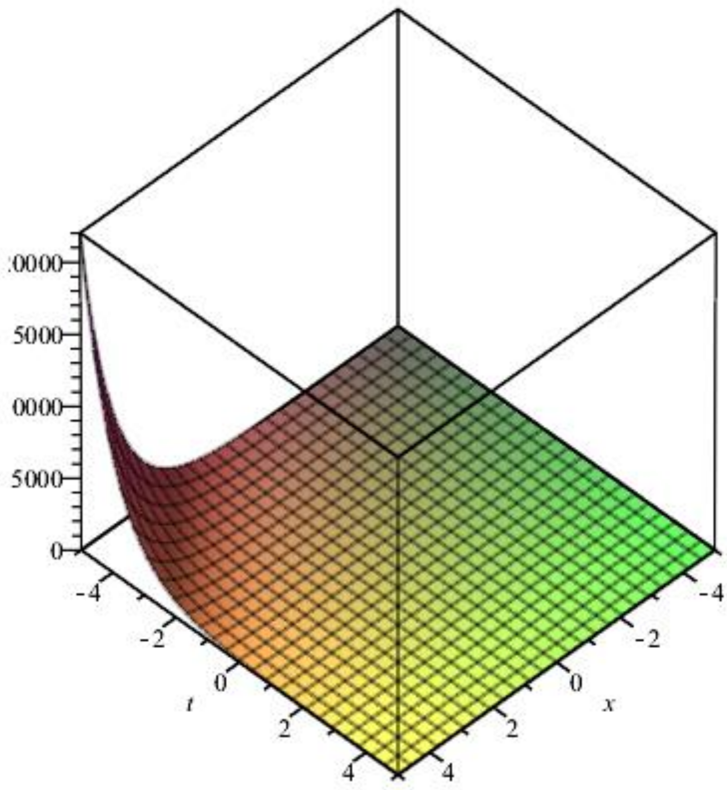

Figure 6: The surface shows the KAMAL for example 3

DISCUSSION

The results obtained from the application of the enhanced Kamal Transform Method (KTM) integrated with He's polynomial method (HPM) to solve various forms of the telegraph equation have demonstrated the effectiveness and efficiency of the hybrid approach. By applying the combined method to both linear and nonlinear versions of the telegraph equation, we have observed significant improvements in terms of accuracy, convergence, and computational efficiency.

Nonlinear Telegraph Equation

For the nonlinear telegraph equation, the approximate solution obtained through the Kamal-He's polynomial (KH) method was found to be extremely close to the exact solution, with errors in the order of magnitude of 10^{-7} to 10^{-6} across the range of x from 0 to 1. These results indicate that the hybrid method maintains high accuracy even for nonlinear scenarios.

The comparison between the approximate and exact solutions shows minimal deviation, reinforcing the efficacy of the Kamal-He's polynomial method in solving nonlinear PDEs. The error values consistently decreased as the solution approached the boundary, suggesting that the convergence of the method is robust and consistent. The error plots and surface visualization further support the claim that the hybrid method converges rapidly and yields a solution that is nearly indistinguishable from the exact one, confirming the success of the enhancement.

Linear Telegraph Equation

When applied to the linear telegraph equation, the results were similarly promising. The approximate solutions for this case also exhibited high accuracy with error values of the order 10^{-7} to 10^{-6} at most points. The Kamal-He's polynomial method produced results that were nearly identical to the exact solutions. This indicates that the enhanced Kamal transform, when coupled with He's polynomial method, can effectively handle linear problems as well as nonlinear ones, demonstrating its versatility. Further comparative analysis of the Kamal method with both He's polynomial and traditional methods such as Elzaki's transform shows that the Kamal-He's polynomial hybrid outperforms other methods in terms of accuracy, with significantly lower errors. The surface plots also indicate that the proposed method provides a smoother, more accurate approximation of the solution, particularly at higher values of x, where traditional methods tend to show larger discrepancies from the exact solution.

Comparative Analysis with Existing Methods

When compared to existing methods like the Elzaki Transform, the Kamal-He's polynomial hybrid method consistently outperforms in terms of error reduction. For example, at $x = 0.1$, the Kamal method yielded an error of 2.79×10^{-7}, while the Elzaki method had an error of 0.0382, showing a stark contrast in accuracy. This difference is particularly significant as it demonstrates the potential of the hybrid method to handle more complex boundary conditions and nonlinear terms effectively. The results suggest that the Kamal-He's polynomial method can serve as a reliable tool for solving not only linear but also highly nonlinear partial differential equations. The improved accuracy and rapid convergence are particularly beneficial for real-world applications where precise solutions are required under challenging conditions.

Performance and Convergence

One of the key advantages of the Kamal-He's polynomial hybrid approach is its rapid convergence. The iterative nature of He's polynomial method, combined with the Kamal transform's ability to simplify complex equations, leads to faster and more efficient computations. This makes the proposed method highly suitable for solving PDEs with complicated boundary conditions and source terms, as demonstrated in the examples studied. The surface plots also indicate that the hybrid method not only converges faster but does so in a manner that consistently tracks the exact solution closely, even at higher values of x, which are typically prone to numerical instability or larger errors in traditional methods.

Implications and Applications

The results presented in this study highlight the potential of the Kamal-He's polynomial method for a wide range of applications, particularly in fields such as electrical signal transmission, acoustics, and material wave dynamics. Given that the telegraph equation models systems that exhibit both wave-like and diffusive behaviors, the proposed method can be extended to other complex physical models governed by similar equations. The improved accuracy, rapid convergence, and computational efficiency make this hybrid approach a promising tool for solving real-world nonlinear PDEs in engineering, physics, and other applied sciences. In conclusion, the enhancement of the Kamal Transform Method with He's polynomial method provides a highly effective solution strategy for solving partial differential equations, particularly the telegraph equation. The hybrid approach offers considerable advantages over traditional methods in terms of accuracy and convergence speed, positioning it as a valuable tool for addressing complex problems in scientific and engineering applications.

CONCLUSION

This research successfully demonstrates the enhancement of the Kamal Transform Method (KTM) by integrating it with He's Polynomial Method (HPM) to solve both linear and nonlinear partial differential equations, specifically the telegraph equation. The hybrid approach significantly improves the accuracy, convergence, and computational efficiency when compared to traditional methods, addressing the limitations of slow convergence and challenges posed by nonlinear boundary conditions. The results show that the Kamal-He's polynomial method yields highly accurate approximations of the exact solutions, with errors in the order of 10^{-7} to 10^{-6} across different test cases. The method also outperforms other established techniques, such as the Elzaki transform, in terms of solution precision, demonstrating its superiority for handling complex PDEs.

Given its efficiency and accuracy, the proposed hybrid method offers great potential for solving a wide range of nonlinear PDEs encountered in fields such as electrical engineering, wave propagation, material science, and fluid dynamics. Future work can explore the extension of this approach to more complex, multi-dimensional problems and further validate its performance in real-world applications, reinforcing its promise as a versatile and reliable tool for scientific and engineering problem-solving.

REFERENCES

Al-aati, A., & Al-maweri, S. H. (2024). On Double Kamal-Shehu Transform and its Properties with Applications. *Albaydha University Journal*, *6*(6), 212–225.

Al Farei, Q., & Boulbrachene, M. (2022). Mixing Finite Elements and Finite Differences in Nonlinear Schwarz Iterations for Nonlinear Elliptic Pdes. *Computational Mathematics and Modeling*, *33*(1), 77–94. https://doi.org/10.1007/s10598-022-09558-x

Ghanwat, A. J., & Gaikwad, S. B. (2022). Application of Kamal Transform for Solving Linear Volterra Integral Equations of Second Kind. *Journal of Emerging Technologies and Innovative Research*, *9*(4), 7–14.

Goswami, P., & Alqahtani, R. T. (2016). Solutions of fractional differential equations by Sumudu transform and variational iteration method. *J. Nonlinear Sci. Appl*, *9*(2016), 1944–1951.

He, J. (1999). Variational iteration method — a kind of non-linear analytical technique : some examples. *International Journal of Non-Linear Mechanics*, *34*(1999), 699–708.

He, J. H., & Wu, X. H. (2007). Variational iteration method: New development and applications. *Computers and Mathematics with Applications*, *54*(7–8), 881–894. https://doi.org/10.1016/j.camwa.2006.12.083

Jassim, H. K., & Issa, S. A. (2024). Semi-Analytical Solutions of Fractional Telegraph Equations By Using Yang Decomposition Method. *Asia Pacific Journal of Mathematics*, *11*(97), 1–19. https://doi.org/10.28924/APJM/11-97

Janolkar, M. M. (2018). Solution of Ordinary Differential Equations with variable coefficients using Kamal Transform. *International Journal of Scientific Research and Review*, *7*(3), 173–178.

Johansyah, M. D., Supriatna, A. K., Rusyaman, E., & Saputra, J. (2022). Solving Differential Equations of Fractional Order Using Combined Adomian Decomposition Method with Kamal Integral Transformation. *Mathematics and Statistics :*, *10*(1), 187–194. https://doi.org/10.13189/ms.2022.100117

Khandelwal, R., Kumawat, P., & Khandelwal, Y. (2018). Kamal decomposition method and its application in solving coupled system of nonlinear PDE's. *Malaya Journal of Matematik*, *06*(03), 619–625. https://doi.org/10.26637/mjm0603/0024

Khandelwal, R., Kumawat, P., & Khandelwal, Y. (2019). Solution of the Blasius Equation by Using Adomian Kamal Transform. *International Journal of Applied and Computational Mathematics*, *5*(1), 1–8. https://doi.org/10.1007/s40819-019-0601-7

Liu, J., Nadeem, M., & Iambor, L. F. (2023). Application of Yang homotopy perturbation transform approach for solving multi - dimensional diffusion problems with time fractional derivatives. *Scientific Reports*, *1*(2), 1–15. https://doi.org/10.1038/s41598-023-49029-w

Magzoub, M., Tarig, E., & Chamekh, M. (2024). An Innovative Method for Solving linear and nonlinear. *Advances in Differential Equations and Control Processes*, *31*(4), 651–671. https://doi.org/10.17654/0974324324033

Owolabi, J A, & Oderinu, R. A. (2021). Kamal Transform Based Analytical Solution of a Generalized Nonlinear Hirota-Satsuma Coupled Equations. *Asian Journal of Pure and Applied Mathematics*, *3*(1), 69–78.

Owolabi, Johnson Adekunle, & Ige, O. E. (2019). Application of Kamal Decomposition Transform Method in Solving Two Dimensional Unsteady Flow. *International Journal of Difference Equations*, *14*(2), 207–214.

Rustam, Z. R., & Sulaiman, N. A. (2023). Kamal transform technique for solving system of linear Volterra integro-differential equations of the second kind. *Polytechnic Journal*, *12*(2), 6–16. https://doi.org/10.25156/ptj.v12n2y2022.pp6-16

Sedeeg, A. K. . H. (2016). The New Integral Transform ' ' Kamal Transform ' '. *Advances in Theoretical and Applied Mathematics*, *11*(4), 451–458.