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Application of the Kamal-He's Iterative Method to Klein-Gordons Equations

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Abstract

This study demonstrates the effectiveness and accuracy of the KHM for solving both linear and nonlinear Klein-Gordon equations. Through graphical comparisons with other methods such as VIM, TAM, and NIM, and error analysis, the results confirm the high precision and reliability of KHM. The approach is shown to be straightforward, easy to implement, and highly efficient for solving linear PDEs. Additionally, KHM provides the exact solution for nonlinear Klein-Gordon equations in a single iteration, highlighting its computational efficiency. Overall, the KHM is proven to be a powerful and reliable tool for solving a wide range of equations in mathematical physics.

Keywords: Application, Kamal-He's Iterative Method, Klein-Gordons Equations

INTRODUCTION

The Klein-Gordon equation (KGE) is a cornerstone of relativistic quantum mechanics and field theory, essential for describing scalar fields, such as those encountered in particle physics and cosmology (Peskin& Schroeder, 1995; Ryder, 1996). As a second-order partial differential equation, it encapsulates the behaviour of quantum particles with mass and provides insights into phenomena such as particle creation, decay, and interactions in various potential fields (Birrell& Davies, 1984).

While analytical solutions to the KGE are obtainable in certain idealized scenarios, such as free fields or simple potentials, real-world applications often demand numerical or semi-analytical approaches due to the complexity introduced by nonlinearities and time-dependent potentials (Triki H. *et al*.; de Oliveira *et al*., 2020). Recent advances in computational techniques have highlighted the necessity for robust numerical methods that can handle the intricacies of these equations, particularly in dynamic regimes where standard perturbative methods fail (Chowdhury & Hashim, 2009).

The nonlinear KG equation is given as follows [Raza *et. al* (2016)]:

$$\phi_{tt} + \alpha\phi_{xx} + \beta\phi^3 + \gamma\phi = 0 \qquad \text{...(1)}$$

α, β and γ are constants, whereas (x, t) denotes the wave profile.

Therefore, this equation plays a crucial role in the explanation of several fascinating phenomena in various areas in applied science [Ablowitz M., & Clarkson P (1991)]. The nonlinear KG equation is used to describe dispersive wave phenomena in field theory and relativistic quantum mechanics. This equation, for example, is used to investigate multi-solitons in splintring resonator (SRR) meta-materials in Klein–Gordon latices [Marek D.,& Lucjan D., (2017)], to investigate the quaternionic field theory in perfect crystals [Bandyopadhyay A., Sarkar B., & Das S., (2017)].

The integral transform generally used and its applications such as the Laplace, Foureir, Mellin, Hankel and Sumudu, to name but a few. Presently, Abdelilah Kamal found a new integral transform, called the Kamal transform, and then applied to the solution of ordinary and partial differential equations. Integral transforms are mathematical tools for solving differential and integral equations for centuries. However, this old area has recently got a center stage among many researchers by introducing many integral transforms among which are [(Hussain& Jasim, 2021); (Shah et al., 2019); (Elzaki, 2011); (Funmilayo et al., 2022); (Kilbas *et al*., 2006); (Sedeeg, 2016); (Kamal &Hosseini, 2017)].

In this seminar, the recently introduced integral transforms by Abdelilah Kamal (Rustam & Sulaiman, 2023) closely studied in relation to the some existing famous polynomials..

In this work we will modify the Kamal transform and apply them in solving some types of Klein-Gordon equation, enhancing our understanding of scalar field dynamics in various physical contexts.

Brief Literature Review

The Klein-Gordon equation (KGE) is one of the fundamental equations in relativistic quantum mechanics, providing a relativistic extension of the Schrödinger equation to describe spin-zero particles. Originally formulated in the 1920s by Oskar Klein and Walter Gordon, the equation is used to model scalar fields, mesons, and other relativistic particles. Despite its importance, finding exact solutions to the KGE, particularly for complex potential models, remains a challenging problem in theoretical physics. This challenge has driven the development of various analytical, numerical, and approximation methods over the years.

In the pursuit of solving complex differential equations that often arise in various scientific and engineering disciplines, researchers have explored a multitude of mathematical techniques. Among these, two notable methodologies have emerged as valuable tools for tackling challenging problems: the Kamal Integral Transformation (KIT) and the Adomian Decomposition Method (ADM).

Sedeeg, (2016) provide several examples of how this transform can be used to solve initial value problems described by ordinary differential equations. The authors also present a theorem that provides a mathematical proof of the properties of the Kamal transform. Overall, the Kamal transform appears to be a promising new tool for solving differential equations. While more research is needed to fully explore its potential applications, the simplicity and versatility of this transform make it a valuable addition to the field of mathematics.

Owolabi & Oderinu, (2021)presents a new scheme for solving the generalized Hirota-Satsuma coupled Kdv equations using the Kamal transform and Adomian polynomial. The paper is structured into four main sections: Introduction, Mathematical Preliminaries, Solution of the Generalized Nonlinear Hirota-Satsuma Coupled Equations, and Application. The introduction provides an overview of the problem and the motivation for the study. The mathematical preliminaries section presents the necessary mathematical concepts and definitions required for the study. The solution section presents the new scheme for solving the generalized Hirota-Satsuma coupled Kdv equations using the Kamal transform and Adomian polynomial. Finally, the application section provides an example of the application of the new scheme to a specific problem.

Overall, the paper is well-structured and provides a clear and concise presentation of the new scheme for solving the generalized Hirota-Satsuma coupled Kdv equations. The authors provide a detailed explanation of the mathematical concepts and definitions used in the study, making it accessible to readers with a background in mathematics. The application section provides a practical example of the new scheme, demonstrating its effectiveness in solving real-world problems.

Rustam & Sulaiman, (2023) demonstrate the effectiveness of the Kamal transform in obtaining the exact solution for such equations. The paper includes some numerical problems that have been solved using the Kamal transform to illustrate its applicability. The results of the numerical problems show that the Kamal transform is a very effective and beneficial integral transform to determine the exact solution of a linear system of Volterra integro-ordinary differential equations of the second kind. The proposed scheme can be applied to a nonlinear system of Volterra-integral equations. Overall, the paper provides a valuable contribution to the field of integral transforms and their applications in solving differential equations.

Janolkar, (2018) discusses the Kamal transform, a recently introduced integral transform that has gained attention among researchers for its effectiveness in solving differential and integral equations. The paper provides a detailed explanation of the Kamal transform and its properties, including its application to integral equations, partial differential equations, and ordinary differential equations with variable coefficients. The Kamal transform is defined for functions of exponential order, and the paper provides examples of its application to solve various types of differential equations. The paper also includes a list of references to other integral transforms, such as the Sumudu transform, Natural transform, Elzaki transform, and Aboodh transform, and compares the Kamal transform to these other transforms. Overall, the paper provides a comprehensive introduction to the Kamal transform and its potential applications in solving differential and integral equations.

Ghanwat & Gaikwad, (2022) offers a comprehensive exploration of the Kamal Transform, delving into its fundamental properties and highlighting its practical utility in the context of solving Second Kind linear Volterra Integral Equations. The authors elucidate the intricacies of the Kamal Transform with clarity, providing a valuable resource for both novices and experts in the field. The authors commence by meticulously expounding the essential characteristics of the Kamal Transform, elucidating its mathematical underpinnings and elucidating its significance in solving integral equations. Through lucid explanations and illustrative examples, they ensure that even those unfamiliar with the topic can grasp its essence. One notable strength of this paper is its pragmatic approach, replete with real-world applications of the Kamal Transform. These practical instances underscore the transform's versatility and its efficacy in tackling complex problems. By offering a variety of applications, the authors not only demonstrate its utility but also inspire further exploration of its potential applications. Furthermore, the paper exhibits a commendable commitment to accessibility. Technical jargon and mathematical equations, often daunting in similar academic works, are presented in a manner that facilitates comprehension. This approach ensures that a wider readership can engage with the material, fostering a more inclusive learning experience. To bolster the credibility of their research, the authors judiciously incorporate references to other seminal works in the field. This intertextuality underscores the depth of their investigation and situates their findings within the broader academic discourse.

In summation, this paper stands as an invaluable resource for individuals interested in harnessing the Kamal Transform to solve integral equations. It not only furnishes a comprehensive overview of the topic but also supplements theoretical discussions with practical examples, all while maintaining an accessible and well-structured narrative. The authors' adept incorporation of references further solidifies the paper's standing as a reliable reference in the field.

Khandelwal *et al.*, (2018) introduces the Kamal decomposition method, which is a new method for solving coupled systems of nonlinear partial differential equations. The authors combine the Kamal transform and the Adomian decomposition method to find exact solutions for both linear and nonlinear equations. They compare their results to those obtained using the NDM method and conclude that the Kamal decomposition method is a powerful and easy-to-use analytic tool for PDE's. The paper also provides a list of references for further reading on the topic.

METHODOLOGY

Basics of the Kamal Transformation Method (KTM)

The Kamal transform denoted by the operator (.), defined by the integral equation:

$$K[f(t)] = G(v) = \int_0^\infty f(t)e^{\frac{-t}{v}}dt. \quad t \geq 0,\ k_1 \leq v \leq k_2$$

...(2)

The variable v in this transform is used to factor the variable t in the argument of the function f. This transform has deeper connection with the Laplace, Elzaki, Aboodh, Mahgoub transforms.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

The Combined Kamal-He's Iterative Method for the Solution of Nonlinear Ordinary Differential Equation

The combined method is the method obtained from the combination of the He's polynomial method with the "Kamal" integral transformation. This section presents solutions to nonlinear differential equations using the combined method.

Given the nonlinear differential equation as follows:

$$Dy(t) = g(t) + Ny(t) + Ry(t) \qquad ...(3)$$

with the initial condition $y(0) = c$, where $\boldsymbol{D}$ is the first-order derivative operator. Furthermore, Eqn. (3) is transformed using the Kamal transformation, we get $K[Dy(t)] = K[g(t) + Ny(t) + Ry(t)],$ ...(4)

Where

$$K[Dy(t)] = \frac{y(v)}{v} - y(0) \qquad \text{...(5)}$$

$$\frac{y(v)}{v} - y(0) = K[g(t)] + K[Ny(t)] + K[Ry(t)],$$
...(5)

$$y(v) - vy(0) = vK[g(t)] + vK[Ny(t)] + vK[Ry(t)],$$
...(6)

$$y(v) = vy(0) + vK[g(t)] + vK[Ny(t)] + vK[Ry(t)].$$
...(7)

Furthermore, using the inverse of the Kamal transform:

$K^{-1}[y(v)] = y(t)$ and $K^{-1}[v] = 1,$ in Eqn. (7), we get

$$K^{-1}[y(v)] = K^{-1}[vy(0) + vK[g(t)]] + K^{-1}[vK[Ny(t)] + vK[Ry(t)]], \qquad \text{...(8)}$$

$$y(t) = y(0) + K^{-1}[vK[g(t)]] + K^{-1}[vK[Ny(t)] + v[KRy(t)]]$$

The Kamal He's Iterative Method assumes that the function y can be decomposed into an infinite series as follows:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \qquad \text{...(9)}$$

where $y_n(t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(t) = \sum_{n=0}^{\infty} H_n(t), \qquad \text{...(10)}$$

where H_n is the He's polynomial, defined as $H_n(y_0, \dots, y_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[F\left(\sum_{i=0}^{n} p^i y_i\right)\right]_{p=0}$, $n \geq 0$ ... (11)

This gives

$H_0= F(y\_0),$

$$H_1 = \frac{\partial}{\partial p}\left[F\left(\sum_{i=0}^{1} p^i y_i\right)\right]_{p=0} = y_1 F'(y_0),$$

$$H_2 = \frac{1}{2!}\frac{\partial^2}{\partial p^2}\left[F\left(\sum_{i=0}^{2} p^i y_i\right)\right]_{p=0} = y_2 F'(y_0) + \frac{1}{2!}\, y_1^2 F''(y_0),$$

$$H_3 = \frac{1}{3!}\frac{\partial^3}{\partial p^3}\left[F\left(\sum_{i=0}^{3} p^i y_i\right)\right]_{p=0} = u_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!}\, y_1^3 F'''(y_0).$$

Analysis of the Kamal-He's Iterative Method for the Solution of Partial Differential Equations

The Kamal-He's Iterative method is explored in this section for the solution of non-homogeneous nonlinear PDEs

$$L_t^n y(x,t) + Ry(x,t) + Ny(x,t) = g(x,t),\ t > 0,$$
...(15)

Where R and N are linear and nonlinear operators, respectively, with the initial conditions

$$y^{(k)}(x,0) = c_k,\ \ k = 0,1,\ldots,n-1 \ \ldots(16)$$

Taking the Kamal transformation, we get

$$K\{L_t^n y(x,t)\} = K\{g(x,t) - Ry(x,t) - Ny(x,t)\},\ \ t > 0,$$
...(17)

$$\frac{K\{y(x,t)\}}{v^n} - \frac{\sum_{k=0}^{n-1} y^k(x,0)}{v^{n-k-1}} = K\{g(x,t) - Ry(x,t) - Ny(x,t)\}.$$
...(18)

This is equivalent to

$$K\{y(x,t)\} = vy(x,0) + v^2 y'(x,0) + \ldots + v^n y^{(n-1)}(x,0) + v^n K\{g(x,t)\}$$
$$-K\{Ry(x,t) + Ny(x,t)\}. \qquad \ldots(19)$$

Applying the inverse of the KH of Eqn. (19), we get

$$y(x,t) = y(x,0) + ty'(x,0) + \ldots + \frac{t^n}{n!} y^{(n-1)}(x,0) + K^{-1}\left[v^n K\{g(x,t)\}\right]$$
$$-K^{-1}\left[v^n K\{Ry(x,t) + Ny(x,t)\}\right] \qquad \ldots(20)$$

The infinite series shown here reflects the KH solution of $y(x,t)$ as

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t), \qquad \ldots(21)$$

where $y_n(x,t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(x,t) = \sum_{n=0}^{\infty} H_n(t), \qquad \ldots(22)$$

By applying Eqn. (21) and Eqn. (22) in Eqn. (20)

$$\sum_{n=0}^{\infty} y_n(x,t) = y(x,0) + ty'(x,0) + \ldots + \frac{t^n}{n!} y^{(n-1)}(x,0) + K^{-1}\left[v^n K\{g(x,t)\}\right]$$
$$-K^{-1}\left[v^n K\left\{R\sum_{n=0}^{\infty} y_n(x,t) + \sum_{n=0}^{\infty} H_n(x,t)\right\}\right].$$
$$\ldots(23)$$

Comparing both sides of Eqn. (23), we

$$\begin{cases} y_0(x,t) = y(x,0) + ty'(x,0) + \ldots + \frac{t^n}{n!} y^{(n-1)}(x,0) + K^{-1}\left[v^n K\{g(x,t)\}\right] \\ y_1(x,t) = -K^{-1}\left[v^n K\{Ry_0(x,t) + H_0(x,t)\}\right] \\ y_2(x,t) = -K^{-1}\left[v^n K\{Ry_1(x,t) + H_1(x,t)\}\right] \\ y_{n+1}(x,t) = -K^{-1}\left[v^n K\{Ry_n(x,t) + H_n(x,t)\}\right], \quad n = 0,1,2,\ldots \end{cases} \qquad \ldots(24)$$

Thus, the approximate solution of Eqn. (15) is:

$$y(x,t) = y_0(x,t) + y_1(x,t) + y_2(x,t) + \dots \dots(25)$$

RESULTS

Example 1: Consider the linear Klein-Gordon equations as follows [(Kumar, 2014); (Selamat, 2020)]:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) = 0 \quad \dots(26)$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x \quad \dots(27)$$

The exact solution is

$$u(x,t) = e^t + \sin x \quad \dots(28)$$

Applying the Kamal-He's of Eqn. (26), we get

$$\frac{K(v)}{v} - u(x,0) - K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - K\{u(x,t)\} = 0 \quad \dots(29)$$

$$K(v) = vu(x,0) + vK\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + vK\{u(x,t)\} \quad \dots(30)$$

Using the inverse Kamal- He's of Eqn. (30) and applying the initial condition, we obtain

$$u(x,t) = 1 + \sin x + K^{-1}\left[vK\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] + K^{-1}[vK\{u(x,t)\}] \quad \dots(31)$$

Substituting Eqns. (9) and (10) into Eqn. (31), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + \sin x + K^{-1}\left[vK\left\{\frac{\partial^2 \sum_{n=0}^{\infty} u_n(x,t)}{\partial x^2}\right\}\right] + K^{-1}[vK\{u_n(x,t)\}] \quad \dots(32)$$

So, the recursive relation is obtained from the solution of the Eqn. (32) using the Kamal-He's Iterative method as follows

$$u_0(x,t) = 1 + \sin x \quad ...(33)$$

$$u_{n+1}(x,t) = K^{-1}\left[vK\left\{\sum_{n=0}^{\infty} u_{xxn}(x,t)\right\}\right] + K^{-1}\left[vK\{u_n(x,t)\}\right], \quad n = 0,1,2... \quad ...(34)$$

$$\begin{cases} u_1(x,t) = K^{-1}\left[vK\{u_{xx,0}(x,t)\}\right] + K^{-1}\left[vK\{u_0(x,t)\}\right] = t \\ u_2(x,t) = K^{-1}\left[vK\{u_{xx,1}(x,t)\}\right] + K^{-1}\left[vK\{u_1(x,t)\}\right] = \frac{t^2}{2!} \\ u_3(x,t) = \frac{t^3}{3!} \\ u_4(x,t) = \frac{t^4}{4!} \\ u_5(x,t) = \frac{t^5}{5!} \\ u_6(x,t) = \frac{t^6}{6!} \\ \cdots \end{cases} \quad ...(35)$$

The result is

$$u(x,t) = 1 + \sin x + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + ... \quad ...(36)$$

Table 1: Comparison of approximate and exact solutions from KHM and VIM for Example1 $(t = 0.1)$

x	EXACT u(x,t)	KHM U(x,t)	VIM U(x,t)
0	1.105170918	1.10517092	1.1051667
0.1	1.205004335	1.20500433	1.2050001
0.2	1.303840249	1.30384025	1.303836
0.3	1.400691125	1.40069112	1.4006869
0.4	1.49458926	1.49458926	1.494585
0.5	1.584596457	1.58459646	1.5845922
0.6	1.669813391	1.66981339	1.6698091
0.7	1.749388605	1.74938861	1.7493844
0.8	1.822527009	1.82252701	1.8225228
0.9	1.888497828	1.88849783	1.8884936
1	1.946641903	1.9466419	1.9466377

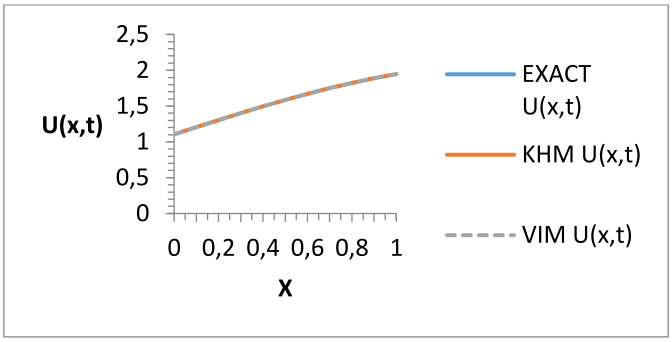

Fig. 1: Comparison of approximates and exact solutions for the linear KGE in Example 1 for fixed $(t = 0.1)$

Table 2: Approximate Error of the KHM and VIM for Example 1

X	EXACT $u(x,t)$	KHM $u(x,t)$	VIM $u(x,t)$	KHM Error	VIM Error
0	1.105170918	1.1051709	1.1051667	2.01E-11	4.25141E-06
0.1	1.205004335	1.2050043	1.2050001	2.01E-11	4.25141E-06
0.2	1.303840249	1.3038402	1.303836	2.01E-11	4.25141E-06
0.3	1.400691125	1.4006911	1.4006869	2.01E-11	4.25141E-06
0.4	1.49458926	1.4945893	1.494585	2.01E-11	4.25141E-06
0.5	1.584596457	1.5845965	1.5845922	2.01E-11	4.25141E-06
0.6	1.669813391	1.6698134	1.6698091	2.01E-11	4.25141E-06
0.7	1.749388605	1.7493886	1.7493844	2.01E-11	4.25141E-06
0.8	1.822527009	1.822527	1.8225228	2.01E-11	4.25141E-06
0.9	1.888497828	1.8884978	1.8884936	2.01E-11	4.25141E-06
1	1.946641903	1.9466419	1.9466377	2.01E-11	4.25141E-06

Example 2: Consider the second-order linear Klein-Gordon equations as follows (Kasumo, 2020):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = u(x,t) \qquad ...(37)$$

Subject to the initial conditions

$$u(x,0)=1+\sin x\,,\; u_t(x,0)=0 \qquad \text{...(38)}$$

The exact solution is

$$u(x,t)=\sin x+\cosh(t) \qquad \text{...(39)}$$

Applying the Kamal-He's of Eqn. (37), we ...(40)

$$\frac{K(v)}{v^2}-\frac{u(x,0)}{v}-u_t(x,0)-K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}-K\{u(x,t)\}=0$$

$$K(v)=vu(x,0)+v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}+v^2K\{u(x,t)\}$$

...(41)

Using the inverse Kamal- He's of Eqn. (41) and applying the initial condition, we obtain

$$u(x,t)=1+\sin x+K^{-1}\left[v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right]+K^{-1}\left[v^2K\{u(x,t)\}\right] \qquad \text{...(42)}$$

Substituting Eqns. (9) and (10) into Eqn. (42), we ge

$$\sum_{n=0}^{\infty}u_n(x,t)=1+\sin x+K^{-1}\left[v^2K\left\{\frac{\partial^2\sum_{n=0}^{\infty}u_n(x,t)}{\partial x^2}\right\}\right]+K^{-1}\left[v^2K\{u_n(x,t)\}\right] \qquad \text{...(43)}$$

So, the recursive relation is obtained from the solution of the Eqn. (43) using the Kamal-He's Iterative method as follows

$$u_0(x,t)=1+\sin x \qquad \text{...(44)}$$

$$u_{n+1}(x,t)=K^{-1}\left[v^2K\left\{\sum_{n=0}^{\infty}u_{xxn}(x,t)\right\}\right]+K^{-1}\left[v^2K\{u_n(x,t)\}\right],n=0,1,2... \qquad \text{...(45)}$$

$$\begin{cases} u_1(x,t) = K^{-1}\left[v^2 K\{u_{xx,0}(x,t)\}\right] + K^{-1}\left[v^2 K\{u_0(x,t)\}\right] = \dfrac{t^2}{2!} \\ u_2(x,t) = K^{-1}\left[v^2 K\{u_{xx,1}(x,t)\}\right] + K^{-1}\left[v^2 K\{u_1(x,t)\}\right] = \dfrac{t^4}{4!} \\ u_3(x,t) = \dfrac{t^6}{6!} \\ u_4(x,t) = \dfrac{t^8}{8!} \\ u_5(x,t) = \dfrac{t^{10}}{10!} \\ u_6(x,t) = \dfrac{t^{12}}{12!} \\ \ldots \end{cases} \quad \ldots(46)$$

The result is

$$u(x,t) = 1 + \sin x + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^{10}}{10!} + \frac{t^{12}}{12!} + \ldots \quad \ldots(47)$$

Table 3: Comparison of approximate and exact solutions from KHM and TAM for Example 2 $(t = 0.1)$

x	EXACT u(x,t)	KHM u(x,t)	TAM u(x,t)
0	1.005004168	1.005004168	1.0050042
0.1	1.104837585	1.104837585	1.1048376
0.2	1.203673499	1.203673499	1.2036735
0.3	1.300524375	1.300524375	1.3005244
0.4	1.39442251	1.39442251	1.3944225
0.5	1.484429707	1.484429707	1.4844297
0.6	1.569646641	1.569646641	1.5696466
0.7	1.649221855	1.649221855	1.6492219
0.8	1.722360259	1.722360259	1.7223603
0.9	1.788331078	1.788331078	1.7883311
1	1.846475153	1.846475153	1.8464752

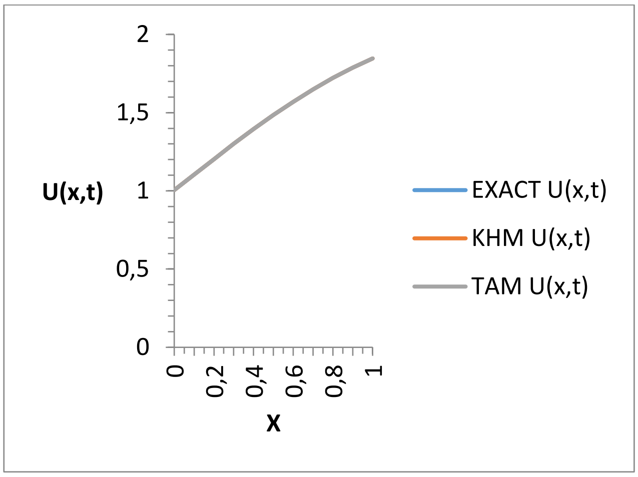

Fig. 2: Comparison of approximate and exact solutions for the linear KGE in Example 2 for fixed $(t = 0.1)$

Table 4: Approximate Error of the KHM and TAM for Example 2

x	EXACT u(x,t)	KHM u(x,t)	TAM u(x,t)	KHM Error	IM Error
0	1.005004168	1.0050042	1.0050042	2.22E-16	2.482E-13
0.1	1.104837585	1.1048376	1.1048376	2.22E-16	2.482E-13
0.2	1.203673499	1.2036735	1.2036735	2.22E-16	2.482E-13
0.3	1.300524375	1.3005244	1.3005244	2.22E-16	2.482E-13
0.4	1.39442251	1.3944225	1.3944225	2.22E-16	2.482E-13
0.5	1.484429707	1.4844297	1.4844297	2.22E-16	2.482E-13
0.6	1.569646641	1.5696466	1.5696466	2.22E-16	2.482E-13
0.7	1.649221855	1.6492219	1.6492219	2.22E-16	2.482E-13
0.8	1.722360259	1.7223603	1.7223603	2.22E-16	2.482E-13
0.9	1.788331078	1.7883311	1.7883311	2.22E-16	2.482E-13
1	1.846475153	1.8464752	1.8464752	2.22E-16	2.482E-13

Example 3: Consider the second-order linear Klein-Gordon equations as follows (Fang *et al.*, 2022):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = -u(x,t) \qquad ...(48)$$

Subject to the initial conditions

$$u(x,0) = 0, \; u_t(x,0) = x \qquad ...(49)$$

The exact solution is

$$u(x,t) = x\sin t \qquad ...(50)$$

Applying the Kamal-He's, we get

$$\frac{K(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) - K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + K\{u(x,t)\} = 0 \qquad ...(51)$$

$$K(v) = v^2 u_t(x,0) + v^2 K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - v^2 K\{u(x,t)\} \qquad ...(52)$$

Using the inverse Kamal- He's of Eqn. (52) and applying the initial condition, we obtain

$$u(x,t) = v^2 x + K^{-1}\left[v^2 K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] - K^{-1}\left[v^2 K\{u(x,t)\}\right] \qquad ...(53)$$

Substituting Eqns. (9) and (10) into Eqn. (53), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = tx + K^{-1}\left[v^2 K\left\{\frac{\partial^2 \sum_{n=0}^{\infty} u_n(x,t)}{\partial x^2}\right\}\right] - K^{-1}\left[v^2 K\{u_n(x,t)\}\right] \qquad ...(54)$$

So, the recursive relation is obtained from the solution of the Eqn. (54) using the Kamal-He's Iterative method as follows

$$u_0(x,t) = tx \qquad ...(55)$$

$$u_{n+1}(x,t) = K^{-1}\left[v^2 K\left\{\sum_{n=0}^{\infty} u_{xxn}(x,t)\right\}\right] - K^{-1}\left[v^2 K\{u_n(x,t)\}\right], \qquad n = 0,1,2... \qquad ...(56)$$

$$\begin{cases} u_1(x,t) = K^{-1}\left[v^2 K\{u_{xx,0}(x,t)\}\right] - K^{-1}\left[v^2 K\{u_0(x,t)\}\right] = -\dfrac{xt^3}{3!} \\ u_2(x,t) = K^{-1}\left[v^2 K\{u_{xx,1}(x,t)\}\right] - K^{-1}\left[v^2 K\{u_1(x,t)\}\right] = \dfrac{xt^5}{5!} \\ u_3(x,t) = -\dfrac{xt^7}{7!} \\ u_4(x,t) = \dfrac{xt^9}{9!} \\ u_5(x,t) = -\dfrac{xt^{11}}{11!} \\ u_6(x,t) = \dfrac{t^{13}}{13!} \\ \ldots \end{cases} \qquad \ldots(57)$$

The result is

$$u(x,t) = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!} + \frac{xt^9}{9!} - \frac{xt^{11}}{11!} + \frac{xt^{13}}{13!}\ldots \qquad \ldots(58)$$

Table 5: Comparison of approximate and exact solutions from KHM and NIM for Example 3 $(t = 0.1)$

x	EXACT U(x,t)	KHM U(x,t)	NIM U(x,t)
0	0	0	0
0.1	0.00998334	0.00998334	0.00998333
0.2	0.01996668	0.01996668	0.01996667
0.3	0.02995002	0.02995002	0.02995
0.4	0.03993337	0.03993337	0.03993333
0.5	0.04991671	0.04991671	0.04991667
0.6	0.05990005	0.05990005	0.0599
0.7	0.06988339	0.06988339	0.06988333
0.8	0.07986673	0.07986673	0.07986667
0.9	0.08985007	0.08985007	0.08985
1	0.09983342	0.09983342	0.09983333

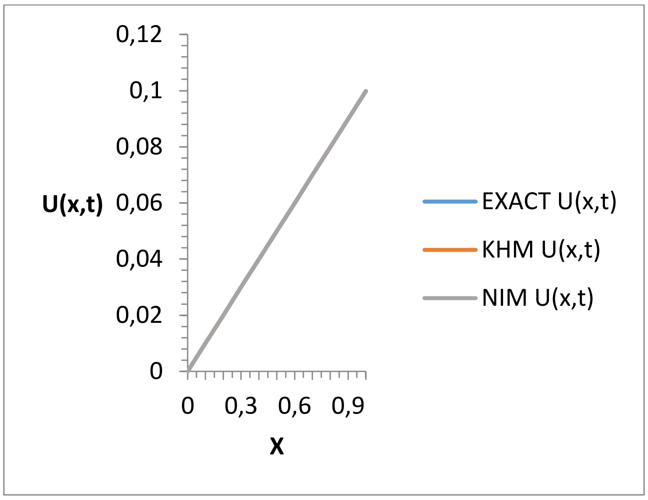

Fig. 3: Comparison of approximates and exact solutions for the linear KGE in Example 3 for fixed $(t = 0.1)$

Table 6: Approximate Error of the KHM and NIM for Example 3

x	EXACT U(x,t)	KHM U(x,t)	NIM U(x,t)	KHM Error	IM Error
0	0	0	0	0	0
0.1	0.00998334	0.009983342	0.009983333	3.469E-18	8.33E-09
0.2	0.01996668	0.019966683	0.019966667	6.939E-18	1.67E-08
0.3	0.02995002	0.029950025	0.02995	3.469E-18	2.5E-08
0.4	0.03993337	0.039933367	0.039933333	1.388E-17	3.33E-08
0.5	0.04991671	0.049916708	0.049916667	6.939E-18	4.17E-08
0.6	0.05990005	0.05990005	0.0599	6.939E-18	5E-08
0.7	0.06988339	0.069883392	0.069883333	1.388E-17	5.83E-08
0.8	0.07986673	0.079866733	0.079866667	2.776E-17	6.67E-08
0.9	0.08985007	0.089850075	0.08985	1.388E-17	7.5E-08
1	0.09983342	0.099833417	0.099833333	1.388E-17	8.33E-08

Nonlinear KGE

Here, we will study the nonlinear **KGE**

Example 4: Consider the nonlinear Klein-Gordon equations as follows (Fang *et al.*, 2022):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = 6xt(x^2 - t^2) + x^6 t^6 \quad ...(59)$$

Subject to the initial conditions

$$u(x,0) = u_t(x,0) = 0 \quad ...(60)$$

The exact solution is

$$u(x,t) = x^3 t^3 \quad ...(61)$$

Applying the Kamal-He's of Eqn. (59), we get

$$\frac{K(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) - K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + K\{u^2(x,t)\} = K\{6x^3 t - 6xt^3 + x^6 t^6\} \quad ...(62)$$

$$K(v) = v^2 K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - v^2 K\{u^2(x,t)\} + v^2 K\{6x^3 t - 6xt^3 + x^6 t^6\} \quad ...(63)$$

Using the inverse Kamal- He's of Eqn. (63) and applying the initial condition, we obtain

$$u(x,t) = K^{-1}\left[v^2 K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] - K^{-1}\left[v^2 K\{u^2(x,t)\}\right] + K^{-1}\left[v^2 K\{6x^3 t - 6xt^3 + x^6 t^6\}\right] \quad ...(64)$$

Substituting Eqns. (9) and (10) into Eqn. (64), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = K^{-1}\left[v^2 K\{u_{xxn}(x,t)\}\right] - K^{-1}\left[v^2 K\{H_n(x,t)\}\right] + K^{-1}\left[v^2 K\{6x^3 t - 6xt^3 + x^6 t^6\}\right] \quad ...(65)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = K^{-1}\left[v^2 K\{u_{xxn}(x,t)\}\right] - K^{-1}\left[v^2 K\{H_n(x,t)\}\right] + x^3 t^3 - \frac{6\times 3! xt^5}{5!} + \frac{6! x^6 t^8}{8!} \quad ...(66)$$

So, the recursive relation is obtained from the solution of the Eqn. (66) using the Kamal-He's Iterative method as follows

$$u_0(x,t) = x^3 t^3 \quad ...(67)$$

$$u_{n+1}(x,t)=K^{-1}\left[v^2K\{u_{xxn}(x,t)\}\right]-K^{-1}\left[v^2K\{H_n(x,t)\}\right]-\frac{6\times 3!xt^5}{5!}+\frac{6!x^6t^8}{8!},n=0,1,2... \quad ...(68)$$

$$\begin{cases} u_1(x,t)=K^{-1}\left[v^2K\{u_{xx,0}(x,t)\}\right]-K^{-1}\left[v^2K\{H_0(x,t)\}\right]-\frac{6\times 3!xt^5}{5!}+\frac{6!x^6t^8}{8!}=0 \\ u_2(x,t)=0 \\ u_3(x,t)=0 \\ u_4(x,t)=0 \\ u_5(x,t)=0 \\ u_6(x,t)=0 \\ ... \end{cases}$$

...(69)

The result is

$$u(x,t)=x^3t^3 \quad ...(70)$$

Example 5: Consider the nonlinear non-homogeneous Klein-Gordon equations as follows [(Kumar, 2014); (Fang *et al.*, 2022)]:

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}+u^2(x,t)=x^2t^2 \quad ...(71)$$

Subject to the initial conditions

$$u(x,0)=0, \qquad u_t(x,0)=x \quad ...(72)$$

The exact solution is

$$u(x,t)=xt \quad ...(73)$$

Applying the Kamal-He's of Eqn. (71), we get

$$\frac{K(v)}{v^2}-\frac{u(x,0)}{v}-u_t(x,0)-K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}+K\{u^2(x,t)\}=K\{x^2t^2\} \quad ...(74)$$

$$K(v)=xv^2+v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}-v^2K\{u^2(x,t)\}+v^2K\{x^2t^2\} \quad ...(75)$$

Using the inverse Kamal- He's of Eqn. (93) and applying the initial condition, we obtain

$$u(x,t)=K^{-1}\left[xv^2\right]+K^{-1}\left[v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right]-K^{-1}\left[v^2K\left\{u^2(x,t)\right\}\right]+K^{-1}\left[v^2K\left\{x^2t^2\right\}\right] \quad ...(76)$$

Substituting Eqns. (9) and (10) into Eqn. (76), we get

$$\sum_{n=0}^{\infty}u_n(x,t)=xt+K^{-1}\left[v^2K\left\{u_{xxn}(x,t)\right\}\right]-K^{-1}\left[v^2K\left\{H_n(x,t)\right\}\right]+K^{-1}\left[v^2K\left\{x^2t^2\right\}\right] \quad ...(77)$$

$$\sum_{n=0}^{\infty}u_n(x,t)=xt+\frac{x^2t^4}{12}+K^{-1}\left[v^2K\left\{u_{xxn}(x,t)\right\}\right]-K^{-1}\left[v^2K\left\{H_n(x,t)\right\}\right] \quad ...(78)$$

So, the recursive relation is obtained from the solution of the Eqn. (78) using the Kamal-He's Iterative method as follows

$$u_0(x,t)=xt \quad ...(79)$$

$$u_{n+1}(x,t)=\frac{x^2t^4}{12}+K^{-1}\left[v^2K\left\{u_{xxn}(x,t)\right\}\right]-K^{-1}\left[v^2K\left\{H_n(x,t)\right\}\right], n=0,1,2... \quad ...(80)$$

$$\begin{cases} u_1(x,t)=\frac{x^2t^4}{12}+K^{-1}\left[v^2K\left\{u_{xx,0}(x,t)\right\}\right]-K^{-1}\left[v^2K\left\{H_0(x,t)\right\}\right]=\frac{x^2t^4}{12}-\frac{x^2t^4}{12}=0 \\ u_2(x,t)=0 \\ u_3(x,t)=0 \\ u_4(x,t)=0 \\ u_5(x,t)=0 \\ u_6(x,t)=0 \\ ... \end{cases} \quad ...(81)$$

The result is

$$u(x,t)=xt \quad ...(82)$$

DISCUSSION

In this section, we validate and assess the accuracy of the KHM through graphical comparisons. Figure 1 illustrates a comparison between KHM, VIM, and the exact solution of the linear Klein-Gordon equation in Example 1. Figure 2 presents the comparison between KHM, TAM, and the exact solution in Example 2, while Figure 3 compares KHM, NIM, and the exact solution in Example 3. The absolute errors, displayed in Tables 2, 4, and 6, further highlight the comparison between other methods and the approximate solution obtained by KHM. These results affirm the high accuracy and reliability of the KHM approach. All computations and graphical representations were generated using Maple software. The plot distributions and absolute errors demonstrate that KHM is a powerful, straightforward, and easy-to-implement method for solving such linear PDEs. Additionally, we observed that KHM provides the exact solution for nonlinear Klein-Gordon equations in just a single iteration, as shown in Examples 4 and 5.

CONCLUSION

In conclusion, the results presented confirm that the KHM is a highly accurate and reliable method for solving both linear and nonlinear Klein-Gordon equations. The graphical comparisons and error analysis demonstrate the approach's effectiveness when compared to other methods such as VIM, TAM, and NIM. The KHM shows excellent performance in providing precise solutions, with the added benefit of being straightforward and easy to implement for linear PDEs. Furthermore, KHM's ability to obtain the exact solution for nonlinear Klein-Gordon equations in a single iteration underscores its computational efficiency. Overall, the KHM proves to be a powerful tool for solving a wide range of equations in mathematical physics.

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