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A Semi-Analytical Method for Nonlinear System of Delay Differential Equation via Modified He's Polynomial

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Abstract

In this work, a simple technique based on the combination of sumudu transform and variational iteration method via Modified He's polynomial is introduced to solve systems of non-linear delay differential equations. The introduced technique is simpler and shorter in its computational procedures and time than the other methods. In addition, the modified He's polynomial takes care of the nonlinear terms and hence, making the method less stressful in terms of computations. Also, this technique reduces the complexity of calculating Lagrange's multiplier values which need more computational procedures and time. These advantages make it reliable and its efficiency is demonstrated with numerical examples.

Keywords: Modified He's Polynomial, Sumudu Transform Method, Variational Iteration Method, Delay Differential Equations

Introduction

Delay differential equation (DDE) is a class of functional differential equation arising in numerous applications from different areas of sciences, engineering and many other fields of studies. Over the years, ordinary differential equations (ODEs) are essential in modeling engineering and physical phenomena. In certain time, the mathematical models of real-life problems need to consider both the present and the past process of the system behaviour. However, the standard forms of ODEs are not always effective in modeling the scientific problems with a memory effect. Thus, such phenomena can be better described by means of delay differential equations (DDEs) (Alredi& Al-Jeaid, 2023). Contrary to ordinary differential equations (ODEs) the derivatives of the independent variables in DDEs at a certain time are express in terms of the values of the function at previous time (Barde, 2019). For this purpose, DDEs are used in various applications instead of ODEs. Therefore, researchers in many fields use DDEs for modelling the physical behaviour which contains both the present information and some history of the system.

Delay differential equations (DDEs) are proved to be useful in control systems (Fridman *et al.,*2000). In many physical phenomena, lasers, traffic models (Davis, 2002), metal cutting, epidemiology, neuroscience, population dynamics (Kuange, 1993), chemical kinetics (Epstein &Luo, 1991). Different kinds of vigorous techniques have been developed in recent time to find an approximate solution for these such delay differential equations, as the Sumudu variational iteration method (SVIM) (Vilu *et al.*, 2019), the Laplace variational iteration method (LVIM) (Biala *et al.,* 2014), the Optimal homotopy asymptotic method (OHAM) (Anakira *et al.,* 2013), the Differential transform method (DTM) (Karakoc & Bereketoglu, 2009; Rashidi *et al.*, 2011), the Homotopy perturbation method (HPM) (Shakeri & Dehghan, 2008), the Adomian decomposition method (ADM) (Evans &Raslan, 2007) and the Homotopy analysis method (HAM) (Alomari *et al.* 2009; Hassan & Rashidi 2011). The method of variational iteration method (VIM) was developed for solving nonlinear problems by He (1999, 2000, 2004, 20 07) He and Wang (2007) and He and Wu (2007). Subsequent Batiha *et al.* (2007), Mahdy *et al.* (2015), Noor and Mohyu-Din (2008), Wazwaz (2009) and Wu (2013) reflect the VIM procedure's versatility, reliability and performance. A new modifed variational iteration method was found, inspired and driven by Wu's thoughts and combining with the Sumudu transform (Liu *et al.,*2016). The new approach is based on variational iteration theory and Sumudu transform. Subashini *et al.*

(2019) in their research, used the basic motivation and extend this new reliable approach for the solution of linear and nonlinear DDEs which are normally challenging to analyze due to their multifaceted nature and boundless dimensionality. In this paper, the modified He's polynomial is added to the new approach by Subashini *et al.* (2019), thereby making it easier to extend the application of the modified method to solving nonlinear systems of delay differential equations.

The Sumudu Variational Iteration Method Via Modified He's Polynomial (SVIMH)

To establish the procedure of the proposed method, consider the following form of nonlinear RDDEs.

$$v_i^{(n)}(t) = f_i\left(t, \boldsymbol{v}(t), \boldsymbol{v}'(t), \dots \boldsymbol{v}^{(n-1)}(t), \boldsymbol{v}\left(\alpha_j(t)\right), \boldsymbol{v}'\left(\alpha_j(t)\right), \dots \boldsymbol{v}^{(n-1)}\left(\alpha_j(t)\right)\right), \quad t \in [0, \infty), \qquad \dots (1)$$

Where, $i = 1,2, \dots, m$, $j = 0,1,2, \dots r$, $f_i: [0, \infty) \times \mathbb{R}^{nm} \times \mathbb{R}^{nmr} \to \mathbb{R}$are continuous functions $\boldsymbol{v}^{(k)}(t) = \left[v_1^{(k)}(t), v_2^{(k)}(t), \dots, v_m^{(k)}(t)\right]$ and $\boldsymbol{v}^{(k)}\left(\alpha_j(t)\right) = \left[v_1^{(k)}\left(\alpha_j(t)\right), v_2^{(k)}\left(\alpha_j(t)\right), \dots, v_m^{(k)}\left(\alpha_j(t)\right)\right]$,for $k = 0,1,2 \dots, n-1$ and $\alpha_j(t)$ are continuous delay functions. This research considered two forms of real time delay (proportional and constant delays) therefore, the delay function $\alpha_j(t)$ is defined as follows:

(i) Proportional delay:$\alpha_j(t) = b_j t$, for $b_j \in (0,1)$

(ii) Constant delay:$\alpha_j(t) = t - \tau_j$, where $\tau_j > 0$ is real constant.

The initial condition of Equation (28) with respect to proportional delay is defined as follows:

$$v_i(0) = \mu_{i,0}, v_i'(0) = \mu_{i,1}, \dots, v_i^{(n-1)}(0) = \mu_{i,n-1}, \qquad \dots (2)$$

where $\mu_{i,0}, \mu_{i,1}, \dots, \mu_{i,n-1}$ are constants real numbers. While the initial condition with constant delay is given as:

$$y_i(t) = p_i(t), t \in [-D, 0], i = 1, 2, \dots, m, \qquad \dots (3)$$

where $p_i(t), p_i'(t), \ldots, p_i^{(n-1)}(t)$ are continuous functions on $[-D, 0]$ and $D = \max_{j=1,2,\ldots,r}[\tau_j]$

In this work, we extend the idea in finding the unknown Lagrange multiplier. Then the equation (1) is converted into an algebraic equation below:

$$\frac{d^n[v_i(t)]}{dt^n} + R_i[\mathbf{v}(\mathrm{t})] + F_i[(\mathbf{v}(\boldsymbol{t}))] = g_i(t) \ , \qquad \ldots (4)$$

with Equations (2) and (3) as initial condition, $\boldsymbol{v}(t) = (v_1(t), v_2(t), \ldots, v_m(t))$, R_i are linear operators and $N_i$are nonlinear operators represent the nonlinear terms.Apply Sumudu transform with its fundamental properties to (4) to obtain the following equation.

$$u^{-m}v_i(u) - u^{-m}v_i(0) - \cdots - u^{-1}v^{m-1}{}_i(0) + S(R_i[\mathrm{V}] + F_i[(\mathrm{V})]) = \mathrm{S}(g_i(t)) \quad \ldots (5)$$

Then by using the concept Variational Iteration Method gives the correction functional as

$$\tilde{\mathrm{v}}_{i(nt1)}(u) = \tilde{\mathrm{v}}_{i(n)}(\mathrm{u}) + \lambda(\mathrm{u})\big[u^{-m}\tilde{\mathrm{v}}_i(u) - u^{-m}v_i(0) - \cdots - u^{-1}v^{m-1}{}_i(0) + S(R_i[\mathrm{V}] + F_i[(\mathrm{V})]) - \mathrm{S}(g_i(t))\big] \qquad \ldots (6)$$

Considering the terms $S(R_i[\tilde{\mathrm{v}}_n] + F_i[(\tilde{\mathrm{v}}_n)]) - \mathrm{S}(g_i(t))$ as restricted variations, we let (6) be stationary with respect to $\tilde{\mathrm{v}}_n$

$$\delta\tilde{\mathrm{v}}_{i(nt1)}(u) = \delta\tilde{\mathrm{v}}_n(u) + \lambda(\mathrm{u})\left(\frac{1}{u^m}\,\delta\tilde{\mathrm{v}}_n(u)\right) \qquad \ldots (7)$$

From (3), we define Lagrange multiplier as

$$\lambda(\mathrm{u}) = -u^m \qquad \ldots (8)$$

Succeeding approximations can then be attained with the application of inverse Sumudu transform S^{-1} into (5) and equation (6) becomes

$$v_{i(nt1)}(u) = v_{i(n)}(\mathrm{u}) - S^{-1}\Big[u^m\Big\{u^{-m}v_i(u) - u^{-m}v_i(0) - \cdots - u^{-1}v^{m-1}{}_i(0) + S\Big(R_i[\mathrm{V}] + H_{i,n}(\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \ldots, \boldsymbol{v}_m(t))\Big) - (g_i(t))\Big\}\Big] \qquad \ldots (9)$$

Where the nonlinear operator$F_i[(\mathrm{v})]$ are computed using the modified He's polynomial $H_{i,n}(\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \ldots, \boldsymbol{v}_m(t))$ and the initial approximation is given as:

$$v_0(t) = S^{-1}\left(v(0) + \cdots + u^{m-1}v^{m-1}(0)\right) = v(0) + v'(0) + \cdots + \frac{t^{m-1}v^{m-1}(0)}{(m-1)!} \quad (10)$$

Now, as $n \to \infty$, $v_n(t)$converges to the exact solutionsv_i.

Numerical Applications

In this section the proposed method shall be applied to solve some problems of nonlinear system of DDEs

Example 1: Consider the first order nonlinear system of delay differential equation [Subashini *et al.*, 2019]

$$y'(t) = 1 - 2y^2\left(\frac{t}{2}\right), \qquad y(0) = 0 \quad \ldots (11)$$

The exact solution is known as , $y(t) = \sin(t)$

Taking the sumudu Transform we obtain;

$$\frac{y(u)}{u} - \frac{y(0)}{u} = S\left[1 - 2y^2\left(\frac{t}{2}\right)\right] \quad \ldots (12)$$

The iteration formula thus is;

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + \lambda(\mathrm{u})\left[\frac{y(u)}{u} - \frac{y(0)}{u} + S\left(-1 + H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right)\right] \ldots (13)$$

And its Lagrange multiplier

$$\lambda(\mathrm{u}) = -u^m$$
$$= -u \quad \ldots (14)$$

Applying inverse Sumudu Transform gives

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{-\mathrm{u}\left[\frac{y(u)}{u} - \frac{y(0)}{u} + S\left(-1 + H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right)\right]\right\} \ldots (15)$$

$$= y_{i(n)}(\mathrm{u}) + S^{-1}\left\{\mathrm{u}\left[S\left(1 - H_{i,n}\left(2y^2\left(\frac{t}{2}\right)\right)\right)\right]\right\}$$

Therefore

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{\mathrm{u}\left[S\left(1 - H_{i,n}\left(2y^2\ \left(\frac{t}{2}\right)\right)\right)\right]\right\} \quad \dots (16)$$

With initial approximation $y_0(\mathrm{t}) = 0$ and applying the iteration formula (16) above, we obtain

$$y_1(\mathrm{t}) = t$$

$$y_2(\mathrm{t}) = t - \frac{t^3}{6} \quad \dots (17)$$

$$y_3(\mathrm{t}) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{8064}$$

From these approximations, it can be seen that the solution tends to form the Taylor series expansion of sin (t). So, as $n \to \infty$, $y_n(t)$converges to the exact solutions sin(t). In order to attest numerically whether or not the suggested approach maintains the accurateness, numerical solutions of the approximation up to y_3 were evaluated. The table and graph of the approximate solutions were plotted and compared with that of the exact solution. Table 1 shows the comparison between exact solution, SVIMH and SVIM and Figure 1 displays behavior of the graphs of Example 1. The outcome is in good agreement with each other.

Table 1: The comparison between exact solution y(t), SVIMH and SVIM

T	EXACT	SVIMH(n=3)	SVIM(n=4)
0	0	0	0
0.1	0.099833	0.099833417	0.0998334
0.2	0.198669	0.198669331	0.1986693
0.3	0.29552	0.295520207	0.2955202
0.4	0.389418	0.389418342	0.3894183
0.5	0.479426	0.479425533	0.4794255
0.6	0.564642	0.564642446	0.5646424
0.7	0.644218	0.644217577	0.6442176
0.8	0.717356	0.717355723	0.7173557
0.9	0.783327	0.78332585	0.7833258
1	0.841471	0.841468254	0.8414683

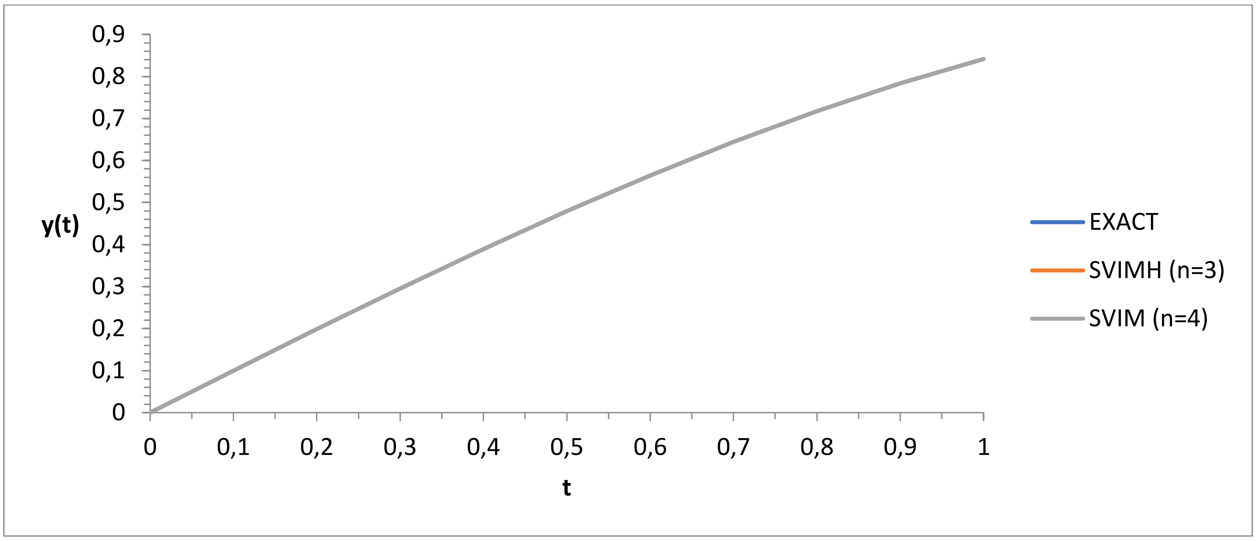

Figure 1: Graphs of exact solution y(t), SVIMH and SVIM.

Example 2: Consider the first order nonlinear system of delay differential equation [Mohammed, 2021, Barde, 2019 and Rostam *et. al*, 2010]

$$y_1'(t) = y_2^2(t)$$

$$y_2'(t) = \frac{1}{2}y_1\left(\frac{t}{2}\right) \qquad \dots(18)$$

$$y_3'(t) = e^{\frac{5t}{2}}y_2(t) + 9e^{2t}y_3\left(\frac{t}{3}\right)$$

With initial condition

$$y_1(0) = 1, y_2(0) = 1, y_3(0) = 0 \qquad \dots(19)$$

Equation (18), together with the initial conditions equation (19), has the exact solution

$$y_1(t) = e^t, y_2(t) = e^{\frac{t}{2}}, y_3(t) = te^{3t} \qquad \dots(20)$$

Taking Sumudu Transform

$$\frac{y_1(u)}{u} - \frac{y_1(0)}{u} = S[y_2^2(t)]$$

$$\frac{y_2(u)}{u} - \frac{y_2(0)}{u}$$
$$= S\left[\frac{1}{2}y_1\left(\frac{t}{2}\right)\right] \qquad \dots (21)$$

$$\frac{y_3(u)}{u} - \frac{y_3(0)}{u} = S\left[\ e^{\frac{5t}{2}}y_2(t) +\ 9e^{2t}y_3\left(\frac{t}{3}\right)\right]$$

The iteration formulae thus is;

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + \lambda(\mathrm{u})\left[\frac{y_1(u)}{u} - \frac{y_1(0)}{u} - S\left(H_{i,n}(y_2^2(t)\,)\right)\right]$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + \lambda(\mathrm{u})\left[\frac{y_2(u)}{u} - \frac{y_2(0)}{u} - S\left(H_{i,n}\left(\frac{1}{2}y_1\left(\frac{t}{2}\right)\right)\right)\right] \qquad \dots (22)$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + \lambda(\mathrm{u})\left[\frac{y_3(u)}{u} - \frac{y_3(0)}{u} - S\left(H_{i,n}\left(e^{\frac{5t}{2}}y_2(t) +\ 9e^{2t}y_3\left(\frac{t}{3}\right)\right)\right)\right]$$

And its Lagrange multiplier

$$\lambda(\mathrm{u}) = -u^m = -u \qquad \dots (23)$$

Applying inverse Sumudu Transform to (22) gives

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{-\mathrm{u}\left[\frac{y_1(u)}{u} - \frac{y_1(0)}{u} - S\left(H_{i,n}(y_2^2(t)\,)\right)\right]\right\}$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{-\mathrm{u}\left[\frac{y_2(u)}{u} - \frac{y_2(0)}{u} - S\left(H_{i,n}\left(\frac{1}{2}y_1\left(\frac{t}{2}\right)\right)\right)\right]\right\} \qquad \dots (24)$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{-\mathrm{u}\left[\frac{y_3(u)}{u} - \frac{y_3(0)}{u} - S\left(H_{i,n}\left(e^{\frac{5t}{2}}y_2(t) +\ 9e^{2t}y_3\left(\frac{t}{3}\right)\right)\right)\right]\right\}$$

Therefore

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{\mathrm{u}\left[S\left(H_{i,n}\left(y_2^2(t)\right)\right)\right]\right\}$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{\mathrm{u}\left[S\left(H_{i,n}\left(\frac{1}{2}y_1\left(\frac{t}{2}\right)\right)\right)\right]\right\} \quad \ldots (25)$$

$$y_{i(nt1)}(u) = y_{i(n)}(\mathrm{u}) + S^{-1}\left\{\mathrm{u}\left[S\left(H_{i,n}\left(e^{\frac{5t}{2}}y_2(t) + 9e^{2t}y_3\left(\frac{t}{3}\right)\right)\right)\right]\right\}$$

With initial approximation $y_1(0) = 1, y_2(0) = 1, y_3(0) = 0$ and applying the iteration formula (25) above, we obtain

$y_{1,0}(t) = 1,$ $y_{2,0}(t) = 1,$ $y_{3,0}(t) = 0$

$y_{1,1}(t) = 1 + t\,,$ $y_{2,1}(t) = 1 + \frac{t}{2},$ $y_{3,1}(t) = t\left[1 + \frac{5t}{4} + \frac{25t^2}{24} + \frac{125t^3}{192} + \cdots\right]$

$y_{1,2}(t) = 1 + t + \frac{t^2}{2},$ $y_{2,2}(t) = 1 + \frac{t}{2} + \frac{t^2}{8},$ $y_{3,2}(t) = t\left[1 + 3t + \frac{495t^2}{216} + \frac{7875t^3}{5184} + \cdots\right]$

$y_{1,3}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6},$ $y_{2,3}(t) = 1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48},$

From these approximations above, it can be seen that the solutions to equation (18) with initial conditions in equation (19) are as follows:

$$y_{1,3}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$$

$$y_{2,3}(t) = 1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \cdots$$

$$y_{3,2}(t) = t\left[1 + 3t + \frac{495t^2}{216} + \frac{7875t^3}{5184} + \cdots\right]$$

Thus as $n \to \infty$ the approximate solutions tend to converge as follows:

$$y_1(t) = e^t, y_2(t) = e^{\frac{t}{2}}, y_3(t) = te^{3t}$$

Table 2: The comparison between exact solution y1(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

T	EXACT	SVIMH(n=3)	ADM (n=6)	HAM(n=5)
0	1	1	1	1
0.1	1.105170918	1.105170918	1.105170917	1.105167
0.2	1.221402758	1.221402756	1.221402667	1.221333
0.3	1.349858808	1.349858763	1.34985775	1.3495
0.4	1.491824698	1.491824356	1.491818667	1.490667
0.5	1.648721271	1.648719618	1.648697917	1.645833
0.6	1.8221188	1.8221128	1.822048	1.816
0.7	2.013752707	2.013734818	2.013571417	2.002167
0.8	2.225540928	2.225494756	2.225130667	2.205333
0.9	2.459603111	2.459496363	2.45875825	2.4265
1	2.718281828	2.718055556	2.716666667	2.666667

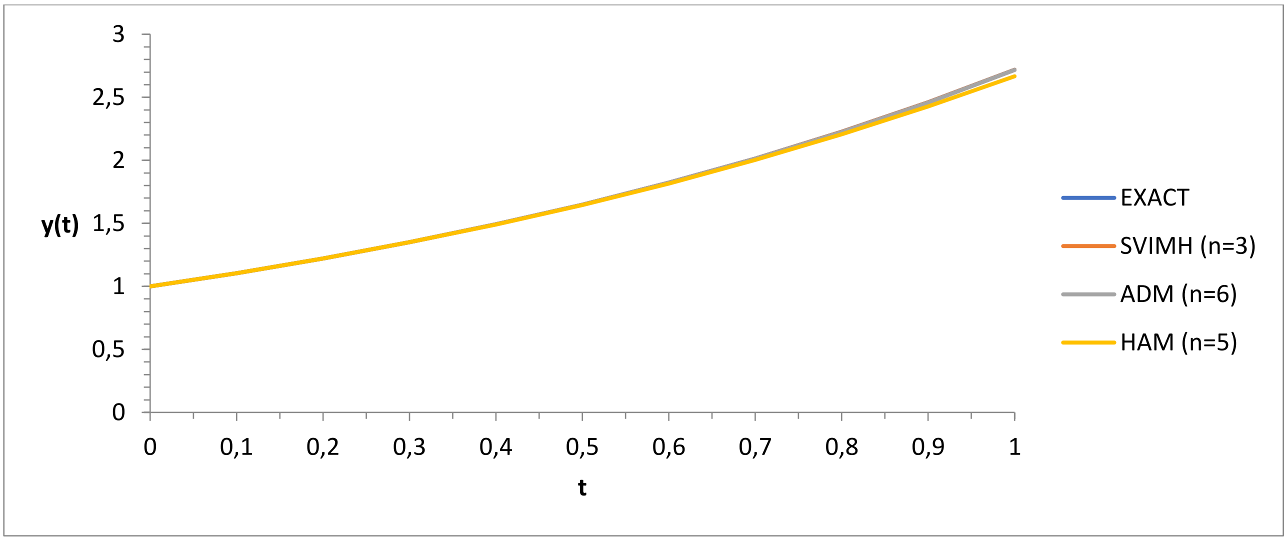

Figure 2: Graphs of exact solution y1(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 2 and figure 2 above, the results obtained from our proposed method were in good agreement with the results from ADM, HAM and the exact solutions. This shows that the proposed method is effective and efficient.

Table 3: The comparison between exact solution y2(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

t	EXACT	SVIMH(n=3)	ADM(n=7)	HAM(n=6)
0	1	1	1	1
0.1	1.051271	1.05127109	1.0512711	1.05127083
0.2	1.105171	1.10517083	1.1051708	1.10516667
0.3	1.161834	1.16183359	1.1618336	1.1618125
0.4	1.221403	1.2214	1.2214	1.22133333
0.5	1.284025	1.28401693	1.2840169	1.28385417
0.6	1.349859	1.3498375	1.3498375	1.3495
0.7	1.419068	1.41902109	1.4190211	1.41839583
0.8	1.491825	1.49173333	1.4917333	1.49066667
0.9	1.568312	1.56814609	1.5681461	1.5664375
1	1.648721	1.6484375	1.6484375	1.64583333

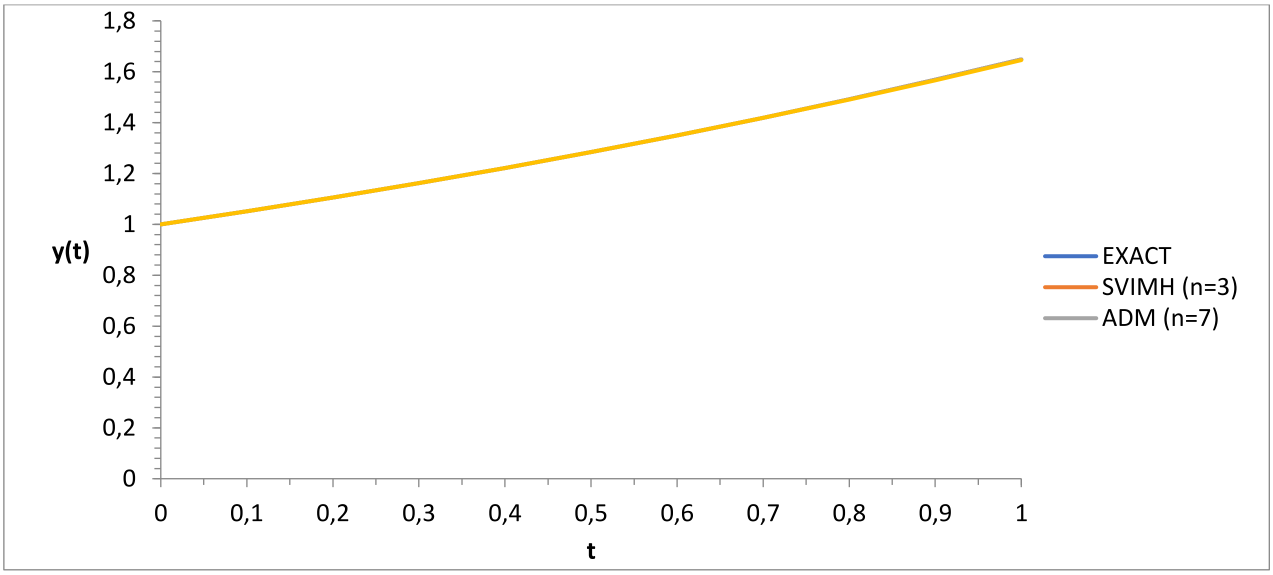

Figure 3: Graphs of exact solution y2(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 3 and figure 3 above, the values demonstrate that the outcomes are in excellent agreement with those of the other approaches. This shows that the proposed method is effective and efficient.

Table 4: The comparison between exact solution y3(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

t	EXACT	SVIMH(n=3)	ADM(n=5)	HAM(n=4)
0	0	0	0	0
0.1	0.134986	0.134985881	0.1345	0.13495
0.2	0.364424	0.364423671	0.356	0.3632
0.3	0.737881	0.737877379	0.6915	0.72795
0.4	1.328047	1.327997659	1.168	1.2832
0.5	2.240845	2.240464565	1.8125	2.09375
0.6	3.629788	3.627750254	2.652	3.2352
0.7	5.716319	5.707824322	3.7135	4.79395
0.8	8.818541	8.789099374	5.024	6.8672
0.9	13.39176	13.30308941	6.6105	9.56295
1	20.08554	19.84642857	8.5	13

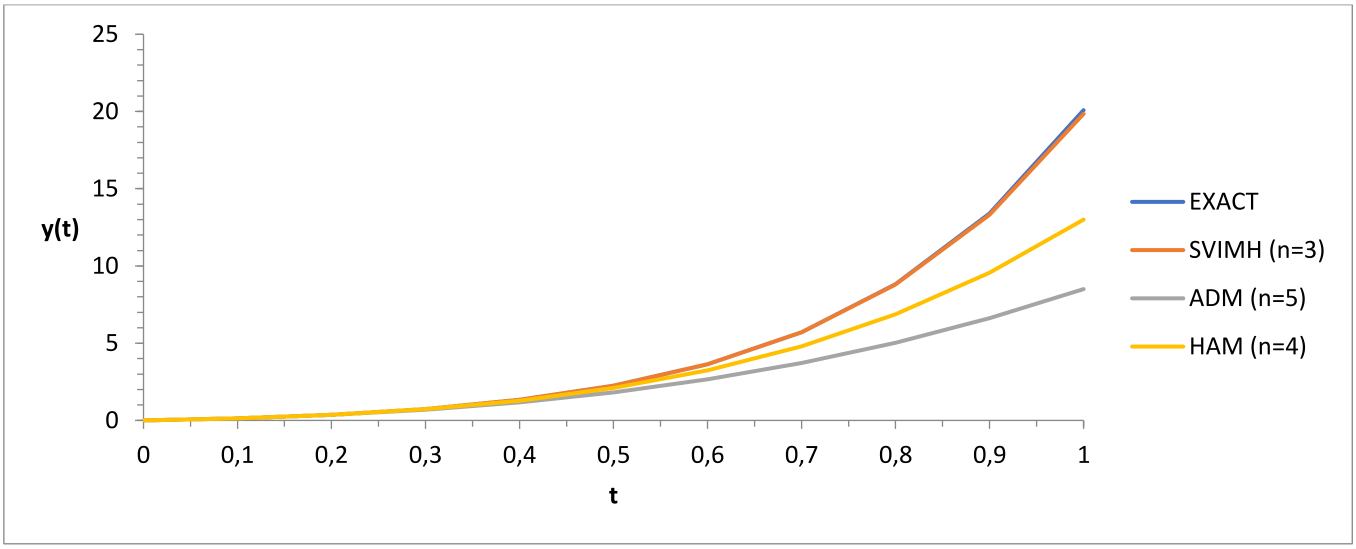

Figure 4: Graphs of exact solution y3(t), the approximate solution using SVIMH, ADM, and HAM of example 2.

From table 4 and figure 4 above, the values demonstrate that the approximate solutions obtained through the proposed method is closer to exact than other approaches. This shows that the proposed method is effective and efficient.

Example 3: Consider the second order nonlinear system of delay differential equation [Barde, 2019 and Rebenda *et. al,* 2015]

$$y_1''(t) - y_1'(t) - 2y_1\left(\frac{t}{2}\right)y_2(t-1) - 4 = -t^4 - 2t^3 x - t^2 - 4t$$

$$y_2''(t) - y_2'(t-2) - y_2\left(\frac{t}{3}\right)y_1\left(\frac{t}{2}\right) - 6 = -\frac{t^4}{18} - 2t \quad \dots (53)$$

With initial condition

$$y_1(0) = y_1'(0) = 0, \quad y_2(0) = y_2'(0) = 0 \quad \dots (54)$$

From the given initial conditions, the initial approximations can be chosen as

$$y_{1,0}(t) = 2t^2, \quad y_{2,0}(t) = t^2 \quad \dots (55)$$

And using our method, the next iterations gives the following results;

$$y_{1,1}(t) = 2t^2 + \frac{2t^3}{3} - \frac{t^4}{12}$$

$$y_{2,1}(t) = t^2 + \frac{t^3}{6}$$

$$y_{1,2}(t) = 2t^2 + \frac{2t^3}{3} - \frac{5t^4}{72} - \frac{17t^5}{720} - \frac{37t^6}{17280} - \frac{155t^7}{12096} - \frac{5t^8}{4032} - \frac{7t^9}{72576}$$

$$y_{2,2}(t) = t^2 + \frac{t^3}{2} - \frac{t^6}{270} - \frac{t^7}{3402} - \frac{t^8}{870912} - \frac{t^9}{2239488}$$

Table 5: The comparison between exact solution y1(t) and the approximate solution using SVIMH for Example3.

t	EXACT	SVIMH (n=3)
0	0	0
0.1	0.02	0.020659463
0.2	0.08	0.085213123
0.3	0.18	0.197361557
0.4	0.32	0.360537167
0.5	0.5	0.577813946
0.6	0.72	0.851779839
0.7	0.98	1.184363234
0.8	1.28	1.576604537
0.9	1.62	2.028363064
1	2	2.5379488

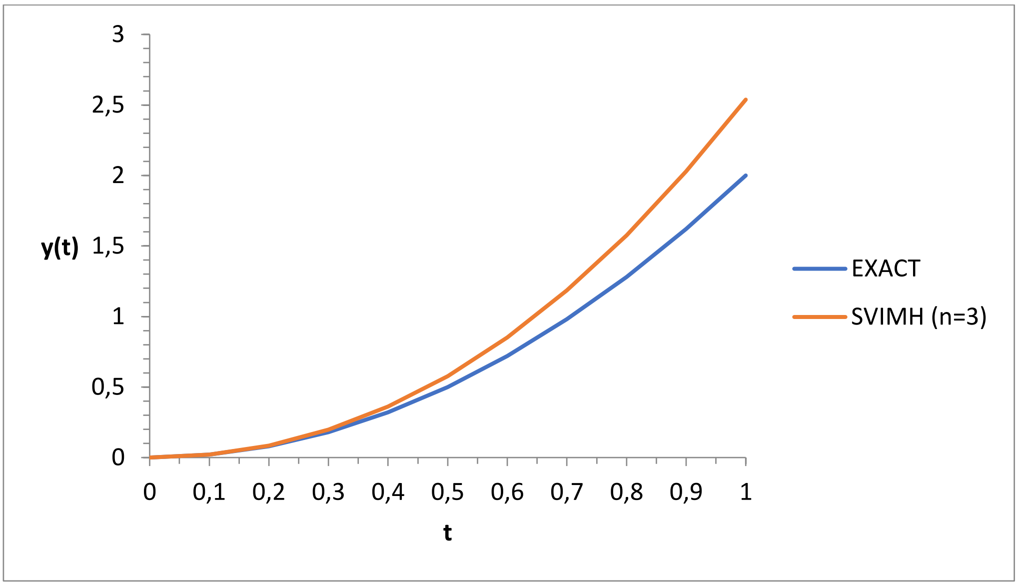

Figure 5: Graphs of exact solution y1(t) and the approximate solution using SVIMH for Example3.

Table 6: The comparison between exact solution y2(t) and the approximate solution using SVIMH for Example3.

t	EXACT	SVIMH(n=3)
0	0	0
0.1	0.01	0.0105
0.2	0.04	0.04399976
0.3	0.09	0.10349723
0.4	0.16	0.19198434
0.5	0.25	0.31243978
0.6	0.36	0.46781873
0.7	0.49	0.66103921
0.8	0.64	0.89496493
0.9	0.81	1.17238443
1	1	1.4959864

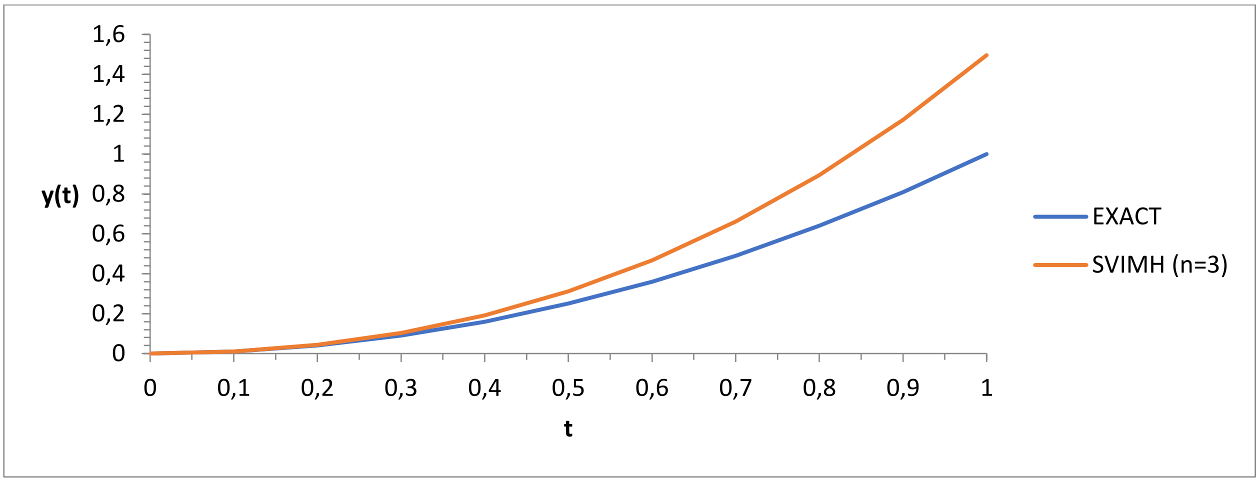

Figure 6 Graphs of exact solution y2(t) and the approximate solution using SVIMH for Example3.

From table 5 and 6, figure 5 and 6 above, the proposed method was able to solve a problem of system of delay differential equation with constant delay which is one of the objective of this research. The approximate solution has a good agreement with the exact solution.

Conclusion

The Sumudu transform is a simple variant of the Laplace transform and is essentially identical with the Laplace. It has many interesting properties that make it easy to visualize and hence making the process of the solution simpler. Sumudu Lagrange multiplier is derived from Sumudu transform plus integrating with procedures of VIM to obtain the approximate solutions to DDEs. Lagrange multipliers, $\lambda(u)$could be known optimally with this diferent approach of variational theory. A new modifcation of the VIM was attained. This recommended procedure was obtained by not having applied any linearization, discretization, or impractical rules. This new technique provides more convincing or accurate sequence of results that converges quickly in physical problems. In this article, the SVIMH was successfully applied in solving nonlinear system of DDEs. In order to attest numerically whether or not the suggested approach maintains the accurateness, numerical solutions of the approximation up to y3 were evaluated. The results were compared in the tables 1,2,3,4,5 and 6 while Figures 1, 2, 3,4,5 and 6 display graphs of the compared values between our methods and others in Example 1 to Example 3. The outcomes are in good agreement with each other. It is worth stating that this method is efficacious in minimizing number of computations contrasting with traditional methods in spite of sustaining great accurateness of the numerical outcome.

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