MIKAILALSYS
Journal of Multidisciplinary Sciences

e-ISSN : 2987-2286
p-ISSN : 2987-3924

Index : Dimensions, Scilit, Crossref, Garuda, Semantic, Lens,Google Scholar,Base, etc

https://doi.org/10.58578/mikailalsys.v3i2.5317

An Enhanced Variational Iteration Method for Solving Ordinary and Partial Differential Equations

Araga Hassan[1], M. Y. Adamu[2], A. G. Madaki[3], Yohanna Nehemiah[4], Michael Cornelius[5], U. M. Nasir[6]
[1,2,3,4,6]Abubakar Tafawa Balewa University, Bauchi, Nigeria
[5]Gombe State University, Gombe, Nigeria
jerry4complex@gmail.com

Article Info:

Submitted:	Revised:	Accepted:	Published:
Mar 5, 2025	Mar 22, 2025	Apr 2, 2025	Apr 7, 2025

Abstract

The Variational Iteration Method (VIM) has proven to be a powerful technique for solving both ordinary and partial differential equations. However, its reliance on Lagrange multipliers for each type of equation has posed significant limitations, complicating its application and reducing its efficiency. This study introduces a Modified Variational Iteration Method (MVIM) that eliminates the need for Lagrange multipliers, addressing these challenges. The MVIM reformulates the correctional functional, simplifying the solution process and enhancing computational efficiency. The method is applied to both linear and nonlinear ordinary and partial differential equations, demonstrating its ability to provide accurate and fast-converging solutions. Numerical examples show that the MVIM outperforms traditional VIM in terms of computational time and convergence speed, and compares favourably with other methods such as the Adomian Decomposition Method (ADM) and New Iteration Method (NIM). The results highlight the potential of MVIM as a versatile and efficient tool for solving complex differential equations in a variety of scientific and engineering applications.

Keywords: Variational Iteration Method, Ordinary Differential Equations, Partial Differential Equations

Introduction

The Variational Iteration Method (VIM) was introduced by He in 2007 (J. H. He & Wu, 2007) and has since become a widely used technique in various scientific disciplines. It has been demonstrated by several researchers (e.g., Araghi et al., 2011; A. Wazwaz, 2007; A. M. Wazwaz, 2009) to be a reliable and efficient method for solving both linear and nonlinear problems. Compared to other methods such as the Adomian decomposition method and perturbation methods, VIM has proven to be more powerful and versatile. Notably, unlike the Adomian method, which requires computational algorithms to handle nonlinear terms, VIM directly addresses the nonlinearities without the need for restrictive assumptions that could alter the physical nature of the solutions, as is often the case with linearization (Wazwaz, 2011; Scheiber & Brasov, 2019).

VIM is particularly effective for solving differential equations, both partial and ordinary. A key feature of the method is the use of Lagrange multipliers, which play a crucial role in its formulation and application. In many instances, these multipliers are tailored to address specific problems (Goswami & Alqahtani, 2016). Over the past decade, VIM has become one of the most frequently employed analytical methods (Wu, 2015). However, its success largely depends on the accurate identification of the Lagrange multipliers (Al-Saar, 2019). Additionally, studies (e.g., Abassy et al., 2007; Yildirim, 2013; Benattia & Belghaba, 2019; Zongo *et al.*, 2024; Al-Mdallal et al., 2016) have highlighted a key drawback: applying VIM to nonlinear differential equations can lead to the calculation of extraneous terms, requiring repeated computations and increasing the overall time spent on series solutions. Despite its proven effectiveness, the Variational Iteration Method (VIM) has several inherent limitations, including the necessity for repeated calculations, the evaluation of extraneous terms, and the labor-intensive process of determining Lagrange multipliers (Zongo *et al.*, 2024; Yildirim, 2013). To mitigate these challenges, this study introduces a Modified Variational Iteration Method (MVIM), which seeks to enhance the efficiency of the solution process by addressing these shortcomings (Al-Mdallal et al., 2016; Zongo *et al.*, 2024).

Basics of the Variational Iteration Method (VIM)

The variational iteration method (VIM) established by Ji-Huan He [(He & Wu, 2007)], is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components wise as in the case of Adomian's method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems. To illustrate the basic idea of this method, we consider the following equation:

$$Lu(x) + Nu(x) = g(x) \qquad \dots (1)$$

$$\frac{d^k u(0)}{dt^k} = c_k\,, \quad k = 0,1,2,\dots,n-1,$$

Where L is a linear operator, N is a nonlinear operator, $g(x)$ is a known continuous function. The basic characteristic of the method is constructionof the following correctional functional for Eqn. (1) as proposed by Ji-Huan He in (2006):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\big(Lu(t) + N\tilde{u}_n(t) - g(t)\big)dt \qquad \dots (2)$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ as a restricted variation which means $\delta\tilde{u}_n = 0$. It is to be noted that the Lagrange's multiplier λ can be a constant or a function.

The variational iteration method consists of the following two essential steps.

I. It is required first to determine the Lagrange's multiplier λ that can be identified optimally via integration by parts and using the restricted variation.

II. Having determined λ, an iteration formula, without restricted variation should be used for determination of the successive approximation $u_{n+1}(x), n \geq 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function. Consequently, the result is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \qquad \ldots(3)$$

Generally, in solving initial value problems of differential equations by VIM, the crucial point is identifying the Lagrange multipliers (Goswami & Alqahtani, 2016).

The Refined VIM for Solving Ordinary Differential Equations

To demonstrate the approach, we examine Eqn. (1) under specified initial conditions. Utilizing the refined VIM, the correction functional is formulated based on the principles of the original VIM as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}(Lu_k(t)+Nu_k(t)) - g(t)\right]dt \qquad \ldots(4)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

Is obtained from the given initial condition(s)

Now using the recursive relations we have the following:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x [(t-x)^{q-1}(Lu_0(t) + Nu_0(t)) - g(t)]dt$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} (Lu_k(t) + Nu_k(t)) - g(t)\right] dt$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} (Lu_k(t) + Nu_k(t)) - g(t)\right] dt$$

$$u_4(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{3} (Lu_k(t) + Nu_k(t)) - g(t)\right] dt$$

….

Where q is determined by the order of the equation under consideration

Results on PDEs

The present section examines the application of the proposed algorithms on partial differential equations (PDEs).

Nonlinear PDEs

Example 1: Consider the nonlinear non-homogeneous PDE as follows [(Kumar, 2014); (Fang *et al.*, 2022)]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = x^2 t^2 \qquad \text{...(5)}$$

Subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = x \qquad \text{...(6)}$$

The exact solution is

$$u(x,t) = xt \qquad \text{...(7)}$$

We rewrite Eqn. (5) as follows based on the correctional functional in Eqn. (4):

$$u_{n+1}(x,t) = c_0 + tc_1 + \frac{(-1)^q}{(q-1)!} \int_0^t (s-t)^{q-1} \left[\sum_{\mathrm{k}=0}^{\infty} \left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 u_n(x,s)}{\partial x^2} + u^2(x,s) - x^2 s^2 \right) \right] ds \qquad \text{...(8)}$$

Where

$$\frac{(-1)^q}{(q-1)!}=1 \qquad ...(9)$$

And

$$c_0+tc_1=u(x,0)+u_t(x,0)=tx \qquad ...(10)$$

Considering the given initial values, we can select $u_0(x,0)=tx$. Using this selection in Eqn. (8) yields the following successive approximations:

$$\begin{cases} u_0(x,0)=tx \\ u_1(x,t)=\int_0^t (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2}-\frac{\partial^2 u_0(x,s)}{\partial x^2}+u_0{}^2(x,s)-x^2s^2\right]ds=0 \\ u_2(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\sum_{n=0}^{1}\frac{\partial^2 u_n(x,s)}{\partial x^2}+\sum_{n=0}^{1}u_n{}^2(x,s)-x^2s^2\right]ds=0 \\ u_3(x,t)=\int_0^t (s-t)\left[\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial s^2}-\sum_{n=0}^{2}\frac{\partial^2 u_n(x,s)}{\partial x^2}+\sum_{n=0}^{2}u_n{}^2(x,s)-x^2s^2\right]ds=0 \\ u_4(x,t)=0 \\ u_5(x,t)=0 \\ u_6(x,t)=0 \\ u_7(x,t)=0 \end{cases}$$

...(11)

And so on. Thus, the solution is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t)+u_6(x,t)+u_7(x,t)+...$$

$$u(x,t)=tx \qquad ...(12)$$

This coincides with the exact solution

Example 2: Consider the nonlinear non-homogeneous PDE as follows (R. Nuruddeen, *et. al* 2018):

$$\frac{\partial u(x,t)}{\partial t}-u\frac{\partial u(x,t)}{\partial x}=0 \qquad ...(13)$$

Subject to the initial condition

$$u(x,0) = x. \quad ...(14)$$

The exact solution is

$$u(x,t) = \frac{x}{1-t} \quad ...(15)$$

We rewrite Eqn. (13) as follows:

$$u_{n+1}(x,t) = c_0 + \frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial u_n(x,s)}{\partial s} - u\frac{\partial u_n(x,s)}{\partial x}\right)\right]ds \quad ...(16)$$

Where

$$\frac{(-1)^q}{(q-1)!} = -1 \quad ...(17)$$

And

$$c_0 = u(x,0) = x \quad ...(18)$$

Considering the given initial values, we can select $u_0(x,0) = x$. Using this selection in Eqn. (16) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = x \\ u_1(x,t) = -\int_0^t (s-t)^0\left[\frac{\partial u_0(x,s)}{\partial s} - u\frac{\partial u_0(x,s)}{\partial x}\right]ds = xt \\ u_2(x,t) = -\int_0^t (s-t)^0\left[\sum_{n=0}^{1}\frac{\partial u_n(x,s)}{\partial s} - \sum_{n=0}^{1}\frac{\partial u_n(x,s)}{\partial x}\right]ds = \frac{1}{3}xt^3 + xt^2 \\ u_3(x,t) = -\int_0^t (s-t)^0\left[\sum_{n=0}^{2}\frac{\partial u_n(x,s)}{\partial s} - \sum_{n=0}^{2}\frac{\partial u_n(x,s)}{\partial x}\right]ds = \frac{1}{63}xt^7 + \frac{1}{9}xt^6 + \frac{1}{3}xt^5 + \frac{2}{3}xt^4 \\ \ldots \end{cases}$$

...(19)

And so on. Thus, the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots =$$

$$xt + xt^3 + xt^2 + \frac{1}{63}xt^7 + \frac{1}{9}xt^6 + \frac{1}{3}xt^5 + \frac{2}{3}xt^4 \quad ...(20)$$

Table 1: Comparison of the approximate solution from enhanced VIM, NIM, ADM and exact solutions at $t = 0.1.$ **for Example 2**

x	EXACT	EVIM	NIM	ADM
0	0	0	0	0
0.1	0.111111	0.111107	0.1111	0.1111
0.2	0.222222	0.222214	0.2222	0.2222
0.3	0.333333	0.333321	0.3333	0.3333
0.4	0.444444	0.444428	0.4444	0.4444
0.5	0.555556	0.555535	0.5555	0.5555
0.6	0.666667	0.666642	0.6666	0.6666
0.7	0.777778	0.777749	0.7777	0.7777
0.8	0.888889	0.888856	0.8888	0.8888
0.9	1	0.999963	0.9999	0.9999
1	1.111111	1.11107	1.111	1.111

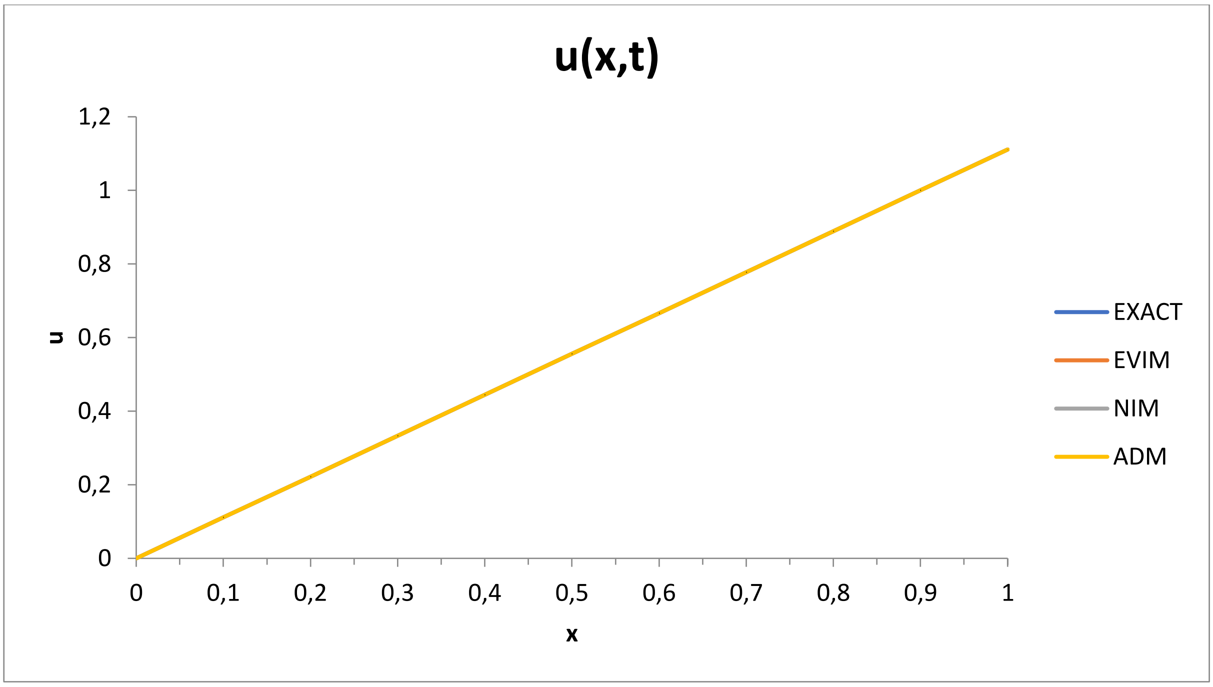

Figure 1: Approximate solutions obtained in comparison with the exact solution, NIM and ADM for Example 2

Linear PDEs

Example 3: Consider the evolution equations as follows [(Bhalekar & Daftardar-Gejji, 2010)]

$$u_t(x,t) + u_{xxxx}(x,t) = 0, \qquad t > 0 \qquad ...(21)$$

With initial conditions

$$u(x,0) = sinx \qquad ...(22)$$

The correction functional for Eqn. (21) is

$$u_{n+1}(x,t) = c_0 + \frac{(-1)^q}{(q-1)!}\int_0^t (s-t)^{q-1}\left[\sum_{k=0}^{\infty}\left(\frac{\partial u_n(x,s)}{\partial s} + \frac{\partial^4 u_n(x,s)}{\partial x^4}\right)\right]ds \qquad ...(23)$$

Where

$$\frac{(-1)^q}{(q-1)!} = -1 \qquad ...(24)$$

And

$$c_0 = u(x,0) = sinx \qquad ...(25)$$

Considering the given initial values, we can select $u_0(x,0) = sinx$. Using this selection in Eqn. (23) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = sinx \\ u_1(x,t) = -\int_0^t (s-t)^0 \left[\frac{\partial u_0(x,s)}{\partial s} + \frac{\partial^4 u_0(x,s)}{\partial x^4} \right] ds = -tsinx \\ u_2(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{1} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{1} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = \frac{1}{2} t^2 sinx \\ u_3(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{2} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{2} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = -\frac{1}{6} t^3 sinx \\ u_4(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{3} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{3} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = \frac{1}{24} t^4 sinx \\ u_5(x,t) = -\int_0^t (s-t)^0 \left[\sum_{n=0}^{4} \frac{\partial u_n(x,s)}{\partial s} + \sum_{n=0}^{4} \frac{\partial^4 u_n(x,s)}{\partial x^4} \right] ds = -\frac{1}{120} t^5 sinx \end{cases} \quad ...(26)$$

And so on. Thus, the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) + ...$$

$$u(x,t) = sinx - tsinx - \frac{1}{6} t^3 sinx + \frac{1}{24} t^4 sinx - \frac{1}{120} t^5 sinx \quad ...(27)$$

That leads to the exact solution

$$u(x,t) = e^{-t} sinx \quad ...(28)$$

which is in high agreement with the exact solution $e^{-t} sinx$ and matches with the VIM and NIM solution

Table 2: Approximate solution is compared with exact solution for *t* = 0.9

x	EXACT	EVIM	ERROR
0	0	0	0
0.1	0.040589	0.040524	6.52E-05
0.2	0.080773	0.080643	0.00013
0.3	0.12015	0.119957	0.000193
0.4	0.158326	0.158071	0.000254
0.5	0.19492	0.194607	0.000313
0.6	0.229566	0.229198	0.000369
0.7	0.261919	0.261499	0.000421
0.8	0.291655	0.291187	0.000468
0.9	0.318477	0.317966	0.000511
1	0.342117	0.341567	0.000549

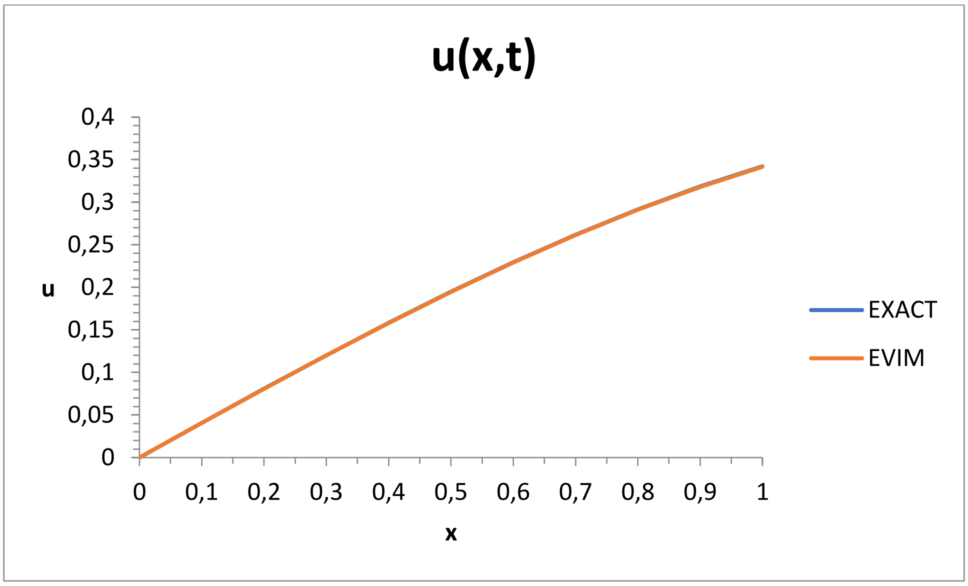

Figure 2: Approximate solutions obtained in comparison with the exact solution for Example 3

Table 3: Comparison of results between exact solution, EVIM, NIM and VIM for Example 1 ($t = 0.9$)

x	EXACT	EVIM (n=5)	NIM (n=7)	VIM (n=6)
0	0	0	0	0
0.1	0.040589	0.040524056	0.038286115	0.04101531
0.2	0.080773	0.080643209	0.076189688	0.08162081
0.3	0.12015	0.119956602	0.113331999	0.12141078
0.4	0.158326	0.158071428	0.149341934	0.15998766
0.5	0.19492	0.194606856	0.183859694	0.19696599
0.6	0.229566	0.229197838	0.216540389	0.2319763
0.7	0.261919	0.26149875	0.247057483	0.26466878
0.8	0.291655	0.291186853	0.275106061	0.29471678
0.9	0.318477	0.317965513	0.30040587	0.32182007
1	0.342117	0.341567167	0.322704123	0.34570784

Example 4: We now consider the wave-like equations as follows: [(Bhalekar & Daftardar-Gejji, 2010)]

$$u_{tt}(x,t) = \frac{x^2}{2} u_{xx}(x,t), \ t > 0 \qquad \text{...(29)}$$

With initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = x^2 \qquad \text{...(30)}$$

The correction functional for Eqn. (29) is

$$u_{n+1}(x,t) = c_0 + tc_1 + \frac{(-1)^q}{(q-1)!} \int_0^t (s-t)^{q-1} \left[\sum_{k=0}^{\infty} \left(\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{x^2}{2} \frac{\partial^2 u_n(x,s)}{\partial x^2} \right) \right] ds \qquad \text{...(31)}$$

Where

$$\frac{(-1)^q}{(q-1)!} = 1 \qquad \text{...(32)}$$

And

$$c_0 + tc_1 = u(x,0) + u_t(x,0) = tx^2 \qquad \text{...(33)}$$

Considering the given initial values, we can select $u_0(x,0) = tx^2$. Using this selection in Eqn. (31) yields the following successive approximations:

$$\begin{cases} u_0(x,0) = tx^2 \\ u_1(x,t) = \int_0^t (s-t) \left[\frac{\partial^2 u_0(x,s)}{\partial s^2} - \frac{x^2}{2} \frac{\partial^2 u_0(x,s)}{\partial x^2} \right] ds = \frac{1}{6} t^3 x^2 \\ u_2(x,t) = \int_0^t (s-t) \left[\sum_{n=0}^{1} \frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{x^2}{2} \sum_{n=0}^{1} \frac{\partial^2 u_n(x,s)}{\partial x^2} \right] ds = \frac{1}{120} t^5 x^2 \\ u_3(x,t) = \int_0^t (s-t) \left[\sum_{n=0}^{2} \frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{x^2}{2} \sum_{n=0}^{2} \frac{\partial^2 u_n(x,s)}{\partial x^2} \right] ds = \frac{1}{5040} t^7 x^2 \\ u_4(x,t) = \int_0^t (s-t) \left[\sum_{n=0}^{3} \frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{x^2}{2} \sum_{n=0}^{3} \frac{\partial^2 u_n(x,s)}{\partial x^2} \right] ds = \frac{1}{362880} t^9 x^2 \\ u_5(x,t) = \int_0^t (s-t) \left[\sum_{n=0}^{4} \frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{x^2}{2} \sum_{n=0}^{4} \frac{\partial^2 u_n(x,s)}{\partial x^2} \right] ds = \frac{1}{39916800} t^{11} x^2 \end{cases} \qquad \text{...(34)}$$

And so on. Thus, the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) + \ldots$$

$$u(x,t) = tx^2 + \frac{1}{6}t^3x^2 + \frac{1}{5040}t^7x^2 + \frac{1}{362880}t^9x^2 + \frac{1}{39916800}t^{11}x^2 + \ldots \quad \ldots(35)$$

That leads to the exact solution

$$u(x,t) = x^2 sinht \qquad \ldots(36)$$

Table 4: Approximate solution is compared with exact solution for Example 4 (*t* = 1)

x	EXACT	EVIM	ERROR
0	0	0	0
0.1	0.011752	0.011752	7.68E-15
0.2	0.047008	0.047008	3.07E-14
0.3	0.105768	0.105768	6.91E-14
0.4	0.188032	0.188032	1.23E-13
0.5	0.2938	0.2938	1.92E-13
0.6	0.423072	0.423072	2.76E-13
0.7	0.575849	0.575849	3.76E-13
0.8	0.752129	0.752129	4.91E-13
0.9	0.951913	0.951913	6.22E-13
1	1.175201	1.175201	7.67E-13

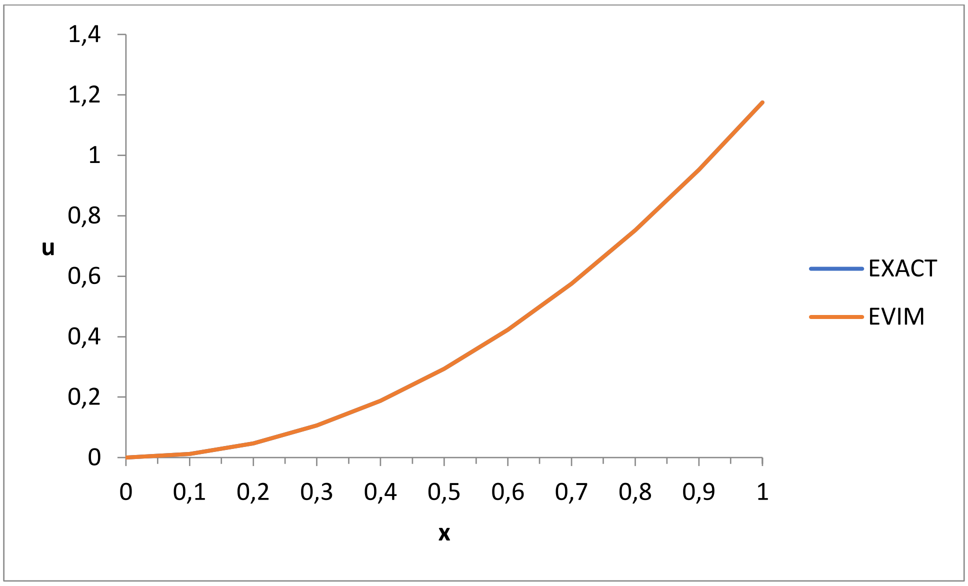

Figure 3: Approximate solutions obtained in comparison with the exact solution for Example 4

Results on ODEs

The present section examines the application of the proposed algorithms on ordinary differential equations (ODEs).

Nonlinear ODEs

Example 5: we consider the following nonlinear differential equation (Sumiati *et al.*, 2020)

$$u''(x) = \frac{1}{2}u(x)[1-u(x)] \qquad x \geq 0 \qquad \ldots(37)$$

Subject to the initial condition

$$u(0) = \frac{1}{2} \qquad \ldots(38)$$

with exact solution

$$u(x) = \frac{e^{\frac{1}{2}x}}{1+e^{\frac{1}{2}x}} \qquad \ldots(39)$$

In view of Eqn. (5), the correctional functional for Eqn. (37) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\left(u''(t) - \frac{1}{2}u(t)[1-u(t)]\right)\right]dt \quad \dots(40)$$

Now, we rewrite the correctional function Eqn. (40) in recursive relations as follows:

$$u_0(x) = u(0) = \frac{1}{2} \quad \dots(41)$$

$$u_{n+1}(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(u_k''(t) - \frac{1}{2}u_k(t)[1-u_k(t)]\right)\right]dt \quad \dots(42)$$

$n \geq 0.$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\left(u_0''(t) - \frac{1}{2}u_0(t)[1-u_0(t)]\right)\right]dt = \frac{1}{8}x \quad \dots(43)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{1}\left(u_k''(t) - \frac{1}{2}u_k(t)[1-u_k(t)]\right)\right]dt = -\frac{1}{384}x^3 \quad \dots(44)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{2}\left(u_k''(t) + \frac{1}{2}u_k^2(t) - \frac{1}{2}u_k(t)\right)\right]dt = \frac{1}{15360}x^5 \quad \dots(45)$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{1}{2} + \frac{1}{8}x - \frac{1}{384}x^3 + \frac{1}{15360}x^5 \quad \dots(46)$$

Table 5: EVIM Solutions in Comparisons with the ADM and VIM for and the Exact Solution for Example 5

x	EXACT	EVIM (n=3)	VIM (n=6)	ADM (n=5)	NIM (n=5)
0.0	0.5000	0.500000000	0.500000000	0.500000000	0.500000000
0.1	0.5125	0.512497396	0.524979187	0.512497396	0.512497396
0.2	0.525	0.524979187	0.549833997	0.524979187	0.524979187
0.3	0.5374	0.537429846	0.574442517	0.537429846	0.537429846
0.4	0.5498	0.549833999	0.59868766	0.549834000	0.549833999
0.5	0.5622	0.562176510	0.622459332	0.562176514	0.562176510
0.6	0.5744	0.574442549	0.645656314	0.574442563	0.574442549
0.7	0.5866	0.586617673	0.668187813	0.586617713	0.586617673
0.8	0.5987	0.598687898	0.689974656	0.598688000	0.598687898
0.9	0.6106	0.610639774	0.710950130	0.610640006	0.610639774
1.0	0.6225	0.622460453	0.731060543	0.622460938	0.622460453

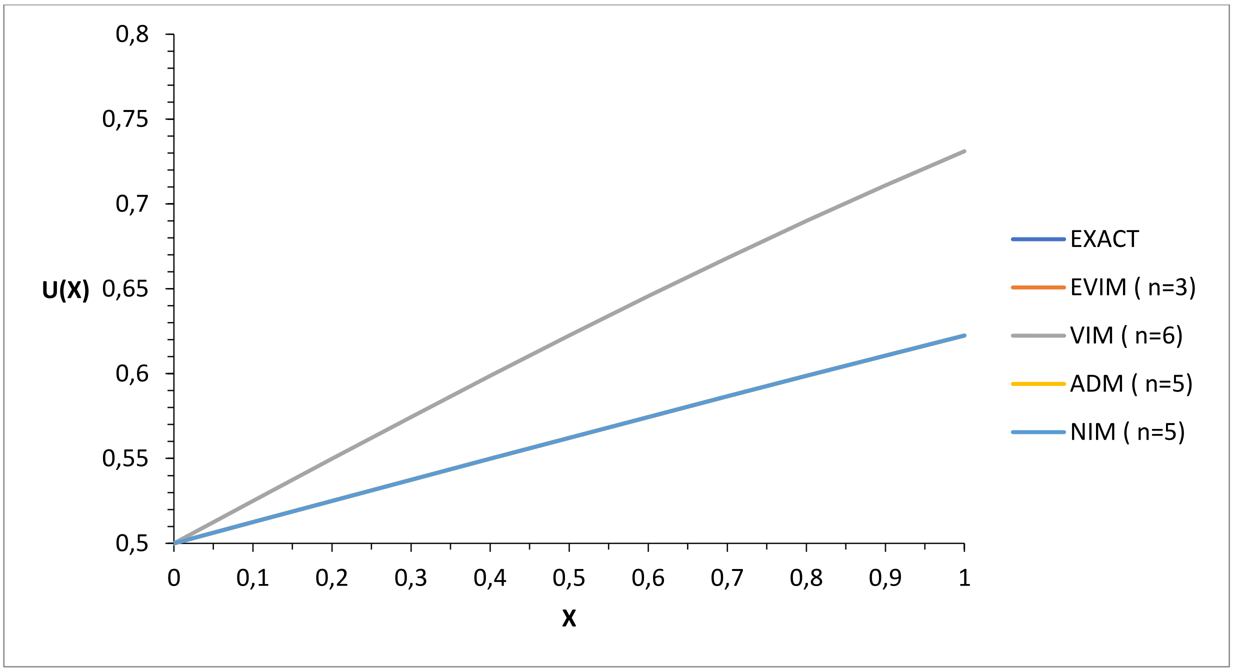

***Figure 4**: Graphical Plots of Approximate Solutions Obtained by the EVIM with the Exact Solution, VIM, ADM and NIM for Example 5*

Discussion of Results

The primary aim of this study was to enhance the Variational Iteration Method (VIM) by proposing a Modified Variational Iteration Method (MVIM) that removes the dependence on Lagrange multipliers, thus increasing computational efficiency and general applicability. Throughout the research, the proposed method was applied to both ordinary differential equations (ODEs) and partial differential equations (PDEs), demonstrating significant improvements in solution accuracy and computational performance.

The results for nonlinear and linear PDEs, as well as for ODEs, illustrated the effectiveness of the MVIM. For nonlinear PDEs, the approximation error was consistently low when compared to traditional methods such as the Adomian Decomposition Method (ADM) and the Natural Iteration Method (NIM). The enhanced method achieved better convergence and less computational effort, demonstrating that it can handle more complex problems with higher precision. For example, the comparison for Example 2 (nonlinear PDE) showed that the EVIM (Enhanced VIM) outperformed the NIM and ADM in terms of proximity to the exact solution, with negligible errors even at higher values of x.

When applied to linear PDEs, the MVIM yielded results that were in close agreement with the exact solutions and showed reduced computational time compared to traditional methods. The example involving the evolution equation (Example 3) demonstrated that the MVIM, with its higher efficiency, converged more rapidly than the classical VIM, NIM, and ADM, while maintaining an error that is orders of magnitude smaller.

The application of the MVIM to nonlinear ODEs also highlighted the method's strength. In Example 5, the MVIM provided highly accurate results for a nonlinear ODE with an exact solution. The method's ability to handle nonlinearities without the need for specific Lagrange multipliers was a key advantage, as it eliminated the need for problem-specific adjustments and allowed for faster, more versatile calculations. Additionally, when compared to ADM and NIM, the EVIM showed superior accuracy and fewer iterations, confirming its efficiency.

The iterative process, which involves the use of recursive relations for successive approximations, proved to be stable across various test cases. This robustness, coupled with the method's ability to be easily generalized across different types of differential equations, positions MVIM as a promising tool for solving complex mathematical models.

In comparison with the standard VIM, the EVIM demonstrated a clear improvement by eliminating the dependence on Lagrange multipliers. This reformulation not only simplified the overall solution process but also contributed to faster convergence, making it highly suitable for problems in fields such as engineering, physics, and applied mathematics where both nonlinearities and boundary complexities are prevalent.

While the study showed promising results for both ODEs and PDEs, future work could explore further optimization of the EVIM for multi-dimensional problems and its application to a broader class of equations, including fractional differential equations. Additionally, extending the method to handle real-world data and more complicated boundary conditions could further solidify its utility in practical applications.

In conclusion, the Enhanced Variational Iteration Method offers significant improvements in the solution of ordinary and partial differential equations. Its flexibility, computational efficiency, and accuracy make it an effective tool for tackling complex mathematical problems across various scientific and engineering disciplines.

Conclusion

This study presents an enhanced approach to the Variational Iteration Method (VIM) by proposing a Modified Variational Iteration Method (MVIM) aimed at eliminating the reliance on Lagrange multipliers. The traditional VIM, while successful in solving many types of linear and nonlinear differential equations, has been hampered by the need for specific Lagrange multipliers for different equations, which hinders its broad applicability and increases computational complexity.

By reformulating the correctional functional to remove the dependence on Lagrange multipliers, the MVIM simplifies the iterative process and improves its efficiency. The proposed method was successfully applied to both ordinary differential equations (ODEs) and partial differential equations (PDEs), with results showing notable improvements in accuracy and convergence. The MVIM demonstrated a significant reduction in computational time compared to traditional VIM methods, making it a promising tool for solving complex differential equations.

The results from various numerical examples, including nonlinear and linear cases, confirm that the MVIM not only produces solutions comparable to exact solutions but also offers faster convergence and reduced errors, especially when compared with other well-established methods such as the Adomian Decomposition Method (ADM) and the New Iteration Method (NIM).

Overall, the findings of this research highlight the effectiveness of the Modified Variational Iteration Method in solving a wide range of differential equations, thereby enhancing the practical applicability of VIM for both theoretical and computational purposes in mathematics and engineering fields. Future research could focus on further improving the method's adaptability to more complex boundary conditions and exploring its potential in higher-dimensional problems.

References

Abdou, M. A., Soliman, A. A., & Aty, M. A. A. (2020). On a discussion of Volterra – Fredholm integral equation with discontinuous kernel. *Journal of the Egyptian Mathematical Society*, *28*(11), 1–10.

Al-Mdallal, Q. A., Al-Khazali, H. A., & Abbas, S. M. (2016). *Modified variational iteration method for solving nonlinear equations. Journal of Computational and Applied Mathematics*, 290, 89-101.

Goswami, P., & Alqahtani, R. T. (2016). Solutions of fractional differential equations by Sumudu transform and variational iteration method. *J. Nonlinear Sci. Appl*, *9*(2016), 1944–1951.

Hamoud, A. A., & Ghadle, K. P. (2016). *On* The Numerical Solution of Nonlinear Volterra-Fredholm Integral Equations by Variational Iteration Method. *June 2017*.

He, J. (2007). Variational iteration method — Some recent results and new interpretations. *207*, 3–17. https://doi.org/10.1016/j.cam.2006.07.009

He, J. H., & Wu, X. H. (2007). Variational iteration method: New development and applications. *Computers and Mathematics with Applications*, *54*(7–8), 881–894. https://doi.org/10.1016/j.camwa.2006.12.083

Mamun, A. Al, & Tao, W. (2019). Solution of Volterra ' s Integro-Differential Equations by Using Variational Iteration Method. *May*, 1–9. https://doi.org/10.20944/preprints201905.0164.v1

Mechee, M. S., Ramahi, A. M. Al, & Kadum, R. M. (2016). Applications of Variational Iteration Method for Solving A Class of Volterra Integral Equations. *9*.

Porshokouhi, M. G., & Ghanbari, B. (2011). Variational Iteration Method for Solving Volterra and Fredholm Integral Equations of the Second Kind. *2*(1), 143–148.

Saleh, M. H. (2016). Variational Iteration Method for Solving Two Dimensional Volterra - Fredholm Nonlinear Integral Equations. *152*(3), 29–33.

Scheiber, E., & Brasov, U. T. (2019). On the Variational Iteration Method for the Nonlinear Volterra Integral Equation. *AMS Researchgate*, *1*(July), 1–8.

Wu, G. (2015). Challenge in the variational iteration method – A new approach to identification of the Lagrange multipliers. *Journal of King Saud University - Science*, *25*(2), 175–178. https://doi.org/10.1016/j.jksus.2012.12.002

Yildirim, E. (2013). *An overview of the variational iteration method and its applications*. *Mathematical Methods in the Applied Sciences*, 36(11), 1439-1450.

Zongo, G., So, O., & Barro, G. (2024). Variational Iteration Method on Linear and Nonlinear Schrödinger'S Equations. *International Journal of Numerical Methods and Applications*, *24*(2), 181–192. https://doi.org/10.17654/0975045224012