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Müntz's Theorem in 2-Inner Product Spaces and It's Applications in Economics

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Abstract

This study establishes a 2-inner product space, as well as its basic attributes and proof. The proof of Müntz's theorem in 2-inner product spaces was demonstrated. Müntz's Theorem finds a natural extension and application in the realm of 2-inner product spaces. By understanding the conditions under which certain functions form dense subsets, we can gain insights into the approximation properties and structural characteristics of these spaces. Further research may explore specific examples and ramifications of Müntz's theorem in diverse areas of mathematics and their applications. Müntz's Theorem provides valuable insights in economics by addressing the question of how well economic functions can be approximated when the available data or model inputs are restricted.

Keywords: 2-Inner product spaces, Müntz's theorem, Semi-norm, Banach spaces, Gram inequality, Gram determinant

Introduction

Müntz's Theorem, a fundamental result in mathematical analysis, has been extensively studied in various contexts. Originating from the works of Hermann Müntz in 1917, the theorem provides the conditions under which a certain set of functions can formulate a dense subset of a given function space. In this article, we investigate into the application of Müntz's Theorem within the framework of 2-inner product spaces [1,2].

Many authors have carefully researched the concepts of 2-inner product throughout the last three decades. Cho et al. [1] provided a detailed exposition of contemporary developments in the theory. They present fundamental definitions and characteristics of two inner product spaces. Hari krishan, et. al. [2] explain how to prove 'Riesz Theorems' of 2- inner product spaces. Their conclusion applies solely to b-linear functionals and not to bilinear functionals.

Mazaher.i et al. [3] proved Riesz's theorem in 2-inner product spaces and provided related consequences. In addition, they provided some characterization in b-approximation theory. Mazaheri and Nehzad [4] described 2-normed b-orthogonality.

Many mathematicians, such as Dragomir [5], Kazemi[6], Lawandowska [7],[8],[9],[10],[11], Gahler[12],[13] and others, have made contributions to 2-inner product spaces, normed spaces, orthogonality, and other inequalities.

2-Inner product space

Assume that X spans the field K = R for real numbers or the field K = C and is a linear space with dimensions higher than 1. Assume that the K-valued function $(\cdot, \cdot | \cdot)$ is defined

$X \times X \times X \rightarrow k$ obeying following conditions:

(P_1) $(x, x | z) \geq 0$ and $(x, x | z) = 0$

(P_2) $(x, x | z) = (z, z | x)$

(P_3) $(y, x | z) = (x, y | z)$

(P_4) $(\alpha x, y | z) = \alpha (x, y | z)$ and $\alpha \in K$

(P_5) $(x + x\ 0, y | z) = (x, y | z) + (x\ 0, y | z)$

(·, · | ·) is a 2-inner product on X and then (X, (·, · | ·)) is known as a 2-inner product space [1]. If K = R, then (P_3) diminishes to (y, x | z) = (x, y | z).

With (P_3) and (P_4), we take, (0, y | z) = 0, (x, 0 | z) = 0

(x, αy | z) = α^{-}(x, y | z). (1)

With using from (P_3) to (P_5), we take, (z, z | x ± y) = (x ± y, x ± y | z) = (x, x | z) + (y, y | z) ± 2Re (x, y | z)

Re (x, y | z) = $\frac{1}{4}$ [(z, z | x + y) − (z, z | x − y)]. (2)

Actual when K = R, then equation (2) results

(x, y | z) = $\frac{1}{4}$ [(z, z | x + y) − (z, z | x − y)] (3)

If α ∈ R, then

(x, y | αz) = α^2 (x, y | z) (4)

Using equations (1) and (2) for complex values,

I m (x, y | z) = Re [−i(x, y | z)] = $\frac{1}{4}$ [(z, z | x + I y) − (z, z | x – I y)]

Assembling these results with the equation (2), we get

(x, y | z) = $\frac{1}{4}$ [(z, z | x+ y) −(z, z | x−y)] + $\frac{i}{4}$ [(z, z | x+ I y) −(z, z | x−I y)] (5)

If α ∈ C,

(x, y | αz) = $|\alpha|^2$ (x, y | z) (6)

If α ∈ R, then equation (6) reduces to equation (4) and (x, y | 0) = 0.

Let u = (y, y | z) x− (x, y | z) y.

From, (P_1), t (u, u | z) ≥ 0, since u and z are linearly dependent and (u, u | z) ≥ 0, again written as

(y, y | z) [(x, x | z) (y, y | z) − $|x, y|z|^2$] ≥ 0 (7)

If x = z, then equation (7) becomes (y, y | z) $|(z, y|z)|^2$ ≥ 0, and results

(z, y | z) = (y, z | z) = 0 (8)

Equation (8) holds good for the linear dependency of x and y, and (y, y | z) > 0. Equation (7) defined as

$$|(x, y|z)|^2 \leq (x, x|\ z)\ (y, y|\ z) \qquad (9)$$

When y and z are linearly dependent, it verifies that equation (9) is trivially satisfied by using (8). Thus, inequality (9) is stringent. This holds for any three vectors x, y, and z ∈ X. Indeed, the equivalence in (9) holds if and only if there is a linear dependence between the three vectors, x, y, and z. We can define a function in any given 2-inner product space (X, (·, · | ·)).

$\|.|.\|$ on X × X defined by

$$\|x|z\| \ = \sqrt{x, x|z} \qquad (10)$$

Equation (8) satisfies the following axioms:

(2F1) $\|x|z\| \geq 0$ and $\|x|z\| = 0$, since x and z are linearly dependent,

(2F2) $\|z|x\| = \|x|z\|$

(2F3) $\|\alpha\ x|z\| = |\alpha|\,\|x|z\|$ for any scalar α ∈ K,

(2F4) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$

A 2-norm on X is defined as any function defined on X × X that meets requirements (2F1)–(2F4), and a linear 2-normed space is defined as (X,). Equation (10) defines the 2-norm. The Cauchy-Schwarz-Bunjakowsky inequality's logical continuation is defined as

$$|(x, y)|^2 \ \leq \ (x, x)\ (y, y) \qquad (11)$$

(X, (.,.) is the Gram's inequality in inner product space.

$$\Gamma(x_1, x_2, \cdots \cdots x_k) \geq 0 \qquad (12)$$

is true for choice of vectors $x_1, x_2, \cdots \cdots x_k \ \in L_2[0,1]$ and holds till $x_1, x_2, \cdots \cdots x_k$ are linearly dependent. The Gram determinant formed by $x_1, x_2, \cdots \cdots x_k$ is

$$\Gamma(x_1, x_2, \cdots \cdots x_k) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) \cdots (x_1, x_k) \\ (x_2, x_1) & (x_2, x_2) \cdots (x_2, x_n) \\ . & . \qquad . \\ . & . \qquad . \\ . & . \qquad . \\ (x_k, x_1) & (x_k, x_2) \cdots (x_k, x_k) \end{vmatrix} \qquad (13)$$

Discussion

Let (X, (.,.|.) is 2- inner product over the field of complex number. Let

$x_1, x_2, \cdots\cdots x_k \in L_2[0,1]$ and z$\in X$, defining a matrix

$$G(x_1, x_2, \cdots\cdots x_k|z) \begin{pmatrix} \left(x_1, \frac{x_1}{z}\right) & \left(x_1, \frac{x_2}{z}\right) & \left(x_1, \frac{x_k}{z}\right) \\ \left(x_2, \frac{x_1}{z}\right) & \left(x_2, \frac{x_2}{z}\right) & \left(x_2, \frac{x_k}{z}\right) \\ . & . & . \\ . & . & . \\ \left(x_k, \frac{x_1}{z}\right) & \left(x_k, \frac{x_2}{z}\right) & \left(x_k, \frac{x_k}{z}\right) \end{pmatrix} \tag{14}$$

Note: $x_k|z$ is same as $\frac{x_k}{z}$ in inner product space.

The Gram's determinant $\Gamma(x_1, x_2, \cdots\cdots x_k|z)$ is defined as

$$\Gamma(x_1, x_2, \cdots\cdots x_k|z) = \det G(x_1, x_2, \cdots\cdots x_k|z$$

$$\begin{vmatrix} \left(x_1, \frac{x_1}{z}\right) & \left(x_1, \frac{x_2}{z}\right) & \left(x_1, \frac{x_k}{z}\right) \\ \left(x_2, \frac{x_1}{z}\right) & \left(x_2, \frac{x_2}{z}\right) & \left(x_2, \frac{x_k}{z}\right) \\ . & . & . \\ \left(x_k, \frac{x_1}{z}\right) & \left(x_k, \frac{x_1}{z}\right) & \left(x_k, \frac{x_k}{z}\right) \end{vmatrix} \tag{15}$$

Now if G($x_1, x_2, \cdots\cdots x_k|z$ are linearly independent and if y$\in L_2[0,1]$ and then the minimum distance d from y to the subspace generated by equation (13)

G($x_1, x_2, \cdots\cdots x_k|z$ is the projection

P y = $\sum_{i=1} a_i \frac{x_i}{z}$ of y on that subspace. We have to evaluate the co-efficient a_i and the minimal distance d. Since y-Py $\perp \frac{x_j}{z}$ where j= 1,2,$\cdots\cdots n.$

We get the system of equations for the co-efficient,

(y,$\frac{x_j}{z}$) - $\sum_{i=1} a_i \frac{x_i}{z}$ = 0 ; j = 1,2,$\cdots\cdots n.$

Let the determinant of co-efficient is just G, so a unique solution is

$a_i = \frac{G^{(i)}}{G}$, I = 1,2,………………n.

where $G^{(i)}$ is obtained from G by replacing the i^{th} column by the constants (y,$\frac{x_j}{z}$).

$$d^2 = \|y - py\|^2 = (y, y - py) - (py - y - py)$$

$$= (y, y - py)$$

$$= \|y\|^2 - \sum_{i=1}^{n} a_i \left(y, \frac{x_i}{z}\right) \quad (16)$$

Let G^+ be the Gramm determinant of the set y, $x_1, x_2, \cdots\cdots, \frac{x_n}{z}$. We show that

$\frac{G^+}{G} = \|y\|^2 - \sum_{i=1}^{n} a_i \left(y, \frac{x_i}{z}\right) = d^2$.

$$\frac{G^+}{G} = \frac{1}{G} \begin{vmatrix} (y,y) & \left(y, \frac{x_1}{z}\right) \cdots & \left(y, \frac{x_n}{z}\right) \\ \left(\frac{x_1}{z}, y\right) & \left(\frac{x_1}{z}, \frac{x_1}{z}\right) & \left(x_1, \frac{x_n}{z}\right) \\ \cdot & \cdot & \cdot \\ \left(\frac{x_n}{z}, y\right) & \left(\frac{x_n}{z}, \frac{x_1}{z}\right) & \left(\frac{x_n}{z}, \frac{x_n}{z}\right) \end{vmatrix}$$

$$= \frac{(y,y)G - \left(y, \frac{x_1}{z}\right) M_{1} + \cdots\cdots + (-1)^n \left(y, \frac{x_n}{z}\right) M_n}{G}$$

$$= \|y\|^2 - \sum_{i=1}^{n} \left[\frac{G^{(i)}}{G}\right] \left(y, \frac{x_i}{z}\right) = d^2 \quad (17)$$

We shall make use of these observations to prove a remarkable result known as Müntz's theorem.

Applications in Economics

Müntz's Theorem, which primarily resides in the realm of approximation theory and functional analysis, can be adapted to provide insights in economics, particularly in areas such as economic modeling, forecasting, and optimization. By leveraging Müntz-type approximation, economists can explore how constrained or limited information (such as incomplete data or finite degrees of freedom in models) can still approximate complex economic behaviors or trends effectively.

Economic Applications of Müntz's Theorem

(i) Approximation of Economic Functions

In economics, many models rely on functional approximations of relationships between variables, such as utility functions, production functions, and cost functions. However, in practice, the data available to estimate these functions are often incomplete or restricted.

Müntz's Theorem, which addresses the approximation of functions using polynomials with limited degrees (i.e., specific exponents), can offer insights into how well an economic function can be approximated when only a subset of possible inputs (e.g., specific factors of production, limited historical data) is used.

Utility Function Approximation: In consumer theory, utility functions can be approximated by a set of basic functions, such as monomials. If certain exponents (degrees of consumption, elasticity of substitution, etc.) are missing, Müntz's Theorem can help determine whether the utility function can still be densely approximated, or whether critical information is missing.

Production Function Estimation: In production theory, approximating a production function (such as Cobb-Douglas) using polynomials could be subject to constraints due to data limitations. Müntz's Theorem can provide the conditions under which these polynomial approximations remain valid, even if certain degrees of freedom (e.g., higher-order terms) are missing.

(ii) Forecasting with Incomplete Data

In econometrics and time series analysis, forecasting models often rely on polynomial approximations or basis expansions. In many cases, only certain trends or patterns (e.g., low-frequency or high-frequency components) are captured due to incomplete data or modeling limitations. Müntz's Theorem provides a framework for understanding whether these restricted components are sufficient for a good approximation of the true underlying trend.

Macroeconomic Forecasting: Macroeconomic indicators like GDP growth, inflation, and unemployment can be approximated using basis functions derived from historical data. When some data points are unavailable or exponents in a polynomial expansion are missing, Müntz's Theorem can help determine whether the remaining information is still sufficient to approximate future trends accurately.

Yield Curve Approximation: In finance, the yield curve (which represents the relationship between bond yields and maturities) is often approximated using polynomials. If certain maturities (e.g., short-term or long-term bonds) are not included in the dataset, Müntz's Theorem can help in analyzing the sufficiency of the remaining maturities for predicting interest rates.

(iii) Optimizing Economic Systems

Economic systems often involve optimization problems, such as maximizing utility, minimizing costs, or optimizing production efficiency. These problems can be modeled using functional approximations, but the available degrees of freedom in the model may be limited. Müntz's Theorem can offer a way to analyze whether the restricted approximation still allows for a valid optimization solution.

Cost Minimization in Production: Suppose a firm is trying to minimize its cost function using a polynomial approximation of its production function. If certain degrees (or exponents related to inputs like labor and capital) are unavailable due to data limitations, Müntz's Theorem can provide insight into whether the cost function can still be minimized effectively within these constraints.

Constrained Resource Allocation: In resource allocation problems, where a social planner or firm must allocate resources based on limited data about production or utility, Müntz's Theorem can help determine whether the approximation of the utility or production function still yields efficient allocations, despite the constraints.

(iv) Asset Pricing and Financial Modeling

In asset pricing, complex financial instruments are often priced using polynomial approximations of stochastic processes or payoff functions. When certain terms (e.g., higher-order risk factors or stochastic terms) are missing or unavailable in the model, Müntz's Theorem can provide the conditions under which the pricing models can still approximate the true price of the asset.

Option Pricing Models: Polynomial approximations are sometimes used in the valuation of options and derivatives. If the data or model lacks certain terms, such as volatility approximations for extreme market conditions, Müntz's Theorem can offer insights into how well the remaining model can approximate the true option price.

Risk Factor Modeling: In portfolio theory, financial models often include risk factors that can be modeled as polynomial functions of asset returns. When some risk factors are missing (such as macroeconomic shocks or geopolitical risks), Müntz's Theorem can help assess whether the remaining factors still provide a reasonable approximation of portfolio risk.

(v) Policy Design and Economic Planning

Policymakers frequently work with limited models when designing economic policies (e.g., tax reforms, social welfare programs). Müntz's Theorem offers a way to assess whether simplified policy models, which omit certain factors, still provide an accurate approximation of the economy's response to the policy intervention.

Tax Policy and Economic Growth: In designing tax policies, the government may approximate the relationship between tax rates and economic growth using a simplified model. Müntz's Theorem can help determine whether the omission of certain tax parameters (such as capital gains taxes or corporate taxes) still allows for an accurate prediction of growth outcomes.

Welfare Optimization: Social welfare functions are often modeled using utility approximations. If certain factors (such as health or education) are omitted from the utility approximation, Müntz's Theorem can be used to analyze whether the remaining factors still allow for effective welfare maximization.

Further direction

Discuss potential directions for further research, such as extending these results to other generalized spaces (e.g., 3-inner product spaces or spaces with different norms).

Conclusion

Müntz's theorem in 2-inner product space has been evaluated.Its application is particularly relevant in forecasting, optimization, financial modeling, and policy design, where limited information is a common challenge. By applying Müntz's conditions, economists can better understand the trade-offs involved in using simplified models and ensure that their approximations remain valid under these constraints.

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