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EXPANSIVE TYPE RATIONAL CONTRACTION IN METRIC SPACE AND COMMON FIXED POINT THEOREMS

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Abstract

The field of expansive mappings in fixed-point theory is one of the most fascinating areas in mathematics. In this theory, contraction is one of the main tools used to prove a fixed point's existence and uniqueness. For all of the analyses, the fixed point theorem proposed by Banach's contraction theory is highly popular and widely used to prove that a solution to the operator equation Tx=x exists and is unique. Through the present article, we utilize rational expressions in metric spaces to deliver unique common stable (fixed) point results in expansive mapping. The main outcomes of numerous relevant innovations in the newest research are built upon them.

Keywords: Fixed point, Complete Metric space, Rational expressions, Expansive mapping

Introduction

Specific fixed point conclusions on expansive mappings in metric spaces were found byWang et al. in 198.likewise using processes, Khan et al. (1986) improved over Wang et al. (1984)'s result. Several fixed-point theories for expansion maps were developed by Park and Rhoades in 988. Furthermore, for pair mappings, Taniguchi (1989) and Rhoades (1985) improved on the results of 1984. Moreover, Taniguchi (1989), Rhoades (1985), and Khan et al. (1986) findings are expanded upon and generalized by Kang and Rhoades (1992) and Kang (1993). The author established several theorem of common fixed points in complete metric spaces for two self mappings and provided a pair of mappings in 1992 for an expansion condition. Pathak et al.(1996) demonstrated several fixed-point theorems for expansion maps. Sometime ago, Shrivasatava, et al. (2014), proved some fixed point theorems for expansion mappings in complete metric space.

They also provided fixed point and common fixed point results for these mappings in 2-metric spaces within the same year. Subsequently, in 2017, Gornicki presented a novel concept of F-contraction and derived some new fixed point results, predominantly in a complete G-metric space. Very recently,Yesikaya and Aydin's were proved theorem of fixed point for expansive mappings over metric space. During the same year, Shakuntala and Tiwari (2020) using rational expressions and obtained special fixed point theorems over metric spaces for expansive mappings.

Our findings renew, generalize, and improve previous work of Shakuntala and Tiwari (2020) and establish common fixed-point theorems for expansive mapping in metric space.

Preliminaries Notes

The characteristics of metric spaces and their definitions are provided below:

Definition 1 [Shrivastava et al. (2014.)]: Assume that $\Im \neq \phi$and $d: \Im \times \Im \rightarrow [0, \infty)$ represent a mapping that satisfying the following specification:

(i) for all $\varphi, \phi \in \Im$ then , $d(\varphi, \phi) \geq 0$,

(ii) for all $\varphi, \phi \in \Im$; then $d(\varphi, \phi) = 0 \Leftrightarrow \varphi = \phi$,

(iii) for all $\varphi, \phi \in \Im$;then $d(\varphi, \phi) = d(\phi, \varphi)$;

(iv) $d(\varphi, \phi) \leq d(\varphi, \psi) + d(\psi, \phi)$, for all $\varphi, \phi, \psi \in \Im$.

Then the pair $(\Im, d)$ can be saying to as a metric space, if d is metric of $\Im$.

Definition 2[Shrivastava et al. (2014.)]: A sequence $\{\varphi_n\}$ is referred to Cauchy sequence at $(\mathfrak{I}, d)$, if for some $\varepsilon > 0 \ \exists \ n_0 \in N$, by through, $\forall \ m, n > n_0$

$$d(x_n, x_m) < \varepsilon. i.e. \lim_{n\to\infty} d(x_n, x_m) < \varepsilon.$$

Definition 3 [Shrivastava et al. (2014.)]: Assume a sequence $\{\varphi_n\}$ is convergent to $\varphi \in \mathfrak{I}$ in $(\mathfrak{I}, d)$, if $\lim_{n\to\infty} d\varphi = 0$, or $\varphi_n \to \varphi$.

Definition 4[Shrivastava et al. (2014)]: A metric space is considered complete when everything the Cauchy sequences converges in $(\mathfrak{I}, d)$.

Definition 5: Presume $S: \mathfrak{I} \to X\mathfrak{I}$ be a mapping on metric space $(\mathfrak{I}, d)$ is called expansive mapping, if for every $\varphi, \phi \in \mathfrak{I}$, there exists a number $r > 1$ such that

$$d(S\varphi, S\phi) \geq rd(\varphi, \phi)$$

Main Results

We will conduct a detailed proof, broaden the scope, and build upon the results presented by Shakuntla and Twari in 2020. Through this endeavor, we aim to derive a shared fixed point applicable to expansive mappings that abide by rational expressions.

Theorem 6: Presume that $S, T: \mathfrak{I} \to \mathfrak{I}$ are two surjective function in $(\mathfrak{I}, d)$ complete that satisfies inequality as under

$$d(S\varphi, T\phi) \geq P\left[\frac{d(\phi,S\varphi)+d(\varphi,T\phi)}{1+d(\phi,T\phi).d(\varphi,T\phi)}\right] + Q\left[\frac{d(\phi,S\varphi)+d(\varphi,T\phi)}{d(\varphi,\phi)+d(\phi,T\phi)}\right].d(\phi, T\phi) + Rd(\varphi, \phi)$$

(6.1)

For all $\varphi, \phi \in X$, where $P, Q, R \geq 0$ are real constants and $Q + R > 1 + 2P, R > 1 + 2P$. Then prove that $Sv = u = Tv$ will unique common fixed point in $\mathfrak{I}$.

Proof: Choose $\varphi_0 \in \mathfrak{I}$. We construct two iterative sequences $\{\varphi_{2k}\}$ and $\{\varphi_{2k-1}\}$, $k \in \mathrm{N}$ as follows

$$\varphi_{2k} = S\varphi_{2k+1}, \text{ some } k = 0,1,2,3 \ldots$$

and

$$\varphi_{2k+1} = T\varphi_{2k+2}, \text{some } k = 0,1,2,3 \ldots.$$

Presently we put $\varphi = \varphi_{2k+1}$ and $\phi = \varphi_{2k+2}$, in (6.1), we find

$$d(S\varphi_{2k+1}, T\varphi_{2k+2}) \geq P\left[\frac{d(\varphi_{2k+2}, \ S\varphi_{2k+1})+d(\varphi_{2k+1},T\varphi_{k+2})}{1+d(\varphi_{2k+2},T\varphi_{2k+2}).d(\varphi_{2k+1}, T\varphi_{2k+2})}\right]$$

$$+Q\left[\frac{d(\varphi_{2k+2},\ S\varphi_{2k+1})+d(\varphi_{2k+1},T\varphi_{2k+2})}{d\left(\varphi_{2k+1},\ \varphi_{2k+2}\right)+d(\varphi_{2k+2},T\varphi_{2k+2})}\right]d(\varphi_{2k+2},T\varphi_{2k+2})$$

$$+\,Rd(\varphi_{2k+1},\varphi_{2k+2})$$

$$d(\varphi_{2k},\varphi_{2k+1})\geq Pd(\varphi_{2k+2},\varphi_{2k})+\frac{Q}{2}d(\varphi_{2k+2},\varphi_{2k})+R\,d(\varphi_{2k+1},\varphi_{2k+2})$$

$$\geq P[d(\varphi_{2k},\varphi_{2k+1})+d(\varphi_{2k+1},\varphi_{2k+2}]$$

$$+\frac{Q}{2}[d(\varphi_{2k},\varphi_{2k+1})+d(\varphi_{2k+1},\varphi_{2k+2})]$$

$$+\,R\,d(\varphi_{2k+1},\varphi_{2k+2}).$$

Therefore,

$$\left(1-P-\frac{Q}{2}\right)d(\varphi_{2k},\varphi_{2k+1})\ \geq\left(P+\frac{Q}{2}+R\right)\ d(\alpha_{i+1},\alpha_{i+2})$$

$$\Rightarrow d(\varphi_{2k+1},\varphi_{2k+2})\ \leq\ \left[\frac{1-P-\frac{Q}{2}}{P+\frac{Q}{2}+R}\right]d(\varphi_{2k},\varphi_{2k+1})$$

$\Rightarrow d(\varphi_{2k+1},\varphi_{2k+2})\ \leq \zeta d(\varphi_{2k},\varphi_{2k+1})$, where, $\zeta=\left[\frac{1-P-\frac{Q}{2}}{P+\frac{Q}{2}+R}\right]$.

In general, we can write

$$d(\varphi_{2k+1},\varphi_{2k+2})\leq\ \zeta^{k}d(\varphi_{1},\varphi_{0}).$$

Since $0\leq\zeta<1$ as $i\rightarrow\infty,\zeta^{k}\ \rightarrow 0.$ So, we have $d(\varphi_{2k+1},\varphi_{2k+2})\rightarrow 0.$

Accordingly, $\{\varphi_{2k}\}$ is Cauchy sequence in $\Im\ \exists\ \ u\in\Im$ like that $\varphi_{2k}\rightarrow u.$ Since S is continuous. So, we have $Su=\lim_{k\rightarrow 1}\varphi_{2k+1}=u.$ In this manner, u will fixed point of S.

Correspondingly, we are able to demonstrate $Tu=\lim_{k\rightarrow 1}\varphi_{2k+1}=u.$

Hence, as result $Su=u=Tu,$ thus, both S and T have u as their common fixed points.

Since S and T are a onto map and hence, $\exists$ a another aspect (point) $v\in X$ through $u=Sv=Tv.$

Now, let us look at

$$d(\varphi_{2k},u)=d(S\varphi_{2k+1},Tv)$$

$$\geq P\left[\frac{d(v,S\varphi_{2k+1})+d\left(\varphi_{2k+1},\,Tv\right)}{1+d(v,Tv).d(\varphi_{2k+1},Tv)}\right]+Q\left[\frac{d(v,S\varphi_{2k+1})+d(\varphi_{2k+1},Tv)}{d(\varphi_{2k+1},v)+d(v,Tv)}\right].d(v,Tv)$$

$$+Rd(\varphi_{2k+1},v)$$

$$=\ P\left[\frac{d(v,\varphi_{2k})+d\left(\varphi_{2k+1},\,Tv\right)}{1+d(v,Tv).d(\alpha_{i+1},Sv)}\right]+Q\left[\frac{d(v,\,\varphi_{2k})+d(\varphi_{2k+1},Tv)}{d(\varphi_{2k+1},v)+d(v,Tv)}\right].d(v,Tv)$$

$$+Rd(\varphi_{2k+1},v).$$

Since $\{\varphi_{2k+1}\}$ is a sub sequence of $\{\alpha_{2k}\}$, so, $\{\alpha_{2k}\}\rightarrow v\Rightarrow\ \{\alpha_{2k+1}\}\rightarrow v,$ when $k\rightarrow\infty.$

Therefore,

$$0 \geq P\left[\frac{d(v,v)+d(v,u)}{1+d(v,v).d(v,u)}\right] + Q\left[\frac{d(v,v)+d(v,u)}{d(v,v)+d(u,v)}\right].d(v,u) + Rd(v,v)$$

$$\Rightarrow 0 \geq (P+Q)d(u,v)$$

$$\Rightarrow \ d(u,v) = 0 \Rightarrow u = v.$$

As a result, there exists u single fixed point that is common to both S and T. The verification process for the theorem has been completed.

Corollary 7: Presume that $S, T: \Im \to \Im$ are two surjective function in $(\Im, d)$ complete that satisfies inequality as under

$$d(S\varphi, T\phi) \geq P\left[\frac{d(\phi,S\varphi)+d(\varphi,T\phi)}{1+d(\phi,T\phi).d(\varphi,T\phi)}\right] + Qd(\varphi,\phi) \ldots \qquad (7.1)$$

For all $\varphi, \phi \in X$, where $P, Q, \geq 0$ are real constants and $P + Q > 1$. Then prove that $Sv = u = Tv$ will unique common fixed point in $\Im$.

Proof: Setting $Q = 0$ in Theorem 6, then required above corollary.

If $S = T$ in Corollary 7, then we get corollary as below:

Corollary 8: Presume that $S, : \Im \to \Im$ is a surjective function in $(\Im, d)$ that satisfies inequality as under

$$d(S\varphi, S\phi) \geq P\left[\frac{d(\phi,S\varphi)+d(\varphi,S\phi)}{1+d(\phi,S\phi).d(\varphi,S\phi)}\right] + Qd(\varphi,\phi) \ldots \qquad (7.1)$$

For all $\varphi, \phi \in X$, where $P, Q, \geq 0$ are real constants and $P + Q > 1$. Then, we have S has a unique fixed point in $\Im$.

Remark: If $S = T$ and $P = A, Q = B$ and $R = C$, in the Theorem 6, then we get the Theorem 3.1 of Shakunla and Tiwari (2020).

Theorem 9: Presume that $S, T: \Im \to \Im$ are two surjective function in $(\Im, d)$ that satisfies inequality as under

$$d(S\varphi, T\phi) \geq \ \alpha d(\varphi,\phi) + \ \beta\frac{d(\varphi,S\varphi).d(\phi,T\phi)}{d(\varphi,\phi)} + \gamma\frac{d(\phi,T\phi).d(\varphi,T\phi)}{d(\varphi,\phi)+d(\phi,T\phi)} \ \ldots (9.1)$$

for all $\varphi, \phi \in X$, where $\alpha, \beta, \gamma > 0$ are all real constants $\beta + \gamma > 1 + 2\alpha, \gamma > 1, \gamma > 1 + 2\alpha$. Then , we claim that $Sv = u = Tv$ have unique common fixed point.

Proof: Imply that $\varphi_0 \in \Im$. There is φ_1, φ_2 in $\Im$ such that

$$S(\varphi_1) = \ \varphi_0, \text{ and } T\varphi_2 = \varphi_1.$$

In this way, we define two sequences $\{\varphi_{2k}\}$ and $\{\varphi_{2k+1}\}$ as through:

$\varphi_{2k} = S\varphi_{2k+1}$, some $k = 0,1,2,3 \dots$

and

$\varphi_{2k+1} = T\varphi_{2k+2}$, some $k = 0,1,2,3 \dots.$

If $\varphi_{2k} = \varphi_{2k+1}$, for some $k \geq 1$,implies that φ_{2k} is a fixed point of S and T.

Now, we put $\varphi = \varphi_{2k+1}$ and $\phi = \varphi_{2k+2}$, in (6.1), we find

$$d(\varphi_{2k}, \varphi_{2k+1}) = d(S\varphi_{2k+1,} T\varphi_{2k+2})$$

$$\geq \alpha d(\varphi_{2k+1}, \varphi_{2k+2}) + \beta \frac{d(\varphi_{2k+1,} S\varphi_{2k+1}).d(\varphi_{2k+2,} T\varphi_{2k+2})}{d(\varphi_{2k+1,} \varphi_{2k+2})}$$

$$+ \gamma \frac{d(\varphi_{2k+2,} T\varphi_{2k+2}).d(\varphi_{2k+1,} T\varphi_{k+2})}{d(\varphi_{2k+1,} \varphi_{2k+2}) + d(\varphi_{2k+2}, Tp_{k+2})}$$

$$= \alpha d(\varphi_{2k+1}, \varphi_{2k+2}) + \beta \frac{d(\varphi_{2k+1,} \varphi_{2k}).\ d(\varphi_{2k+2,} \varphi_{2k+1})}{d(\varphi_{2k+1}, \varphi_{2k+2})}$$

$$+ \gamma \frac{d(\varphi_{2k+2}, \varphi_{2k+1}).d(\varphi_{2k+1}, \varphi_{2k+1})}{d(\varphi_{2k+1,} \varphi_{2k+2}) + d(\varphi_{2k+2,} \varphi_{2k+1})}$$

$$\geq \alpha d(\varphi_{2k+1}, \varphi_{2k+2}) + \beta d(\varphi_{2k+1}, \varphi_{2k})$$

Therefore,

$$\Rightarrow\ d(\varphi_{2k+1}, \varphi_{2k+2}) \leq h\ d(\varphi_{2k}, \varphi_{2k+1}), \text{ where } \frac{1-\beta}{\alpha} = h.$$

In general, we can write

$$\Rightarrow\ d(\varphi_{2k+1}, \varphi_{2k+2}) \leq h^n\ d(\varphi_0, \varphi_1).$$

Since $0 \leq h < 1$ as $k \to \infty, h^n \to 0$, we have

$$d(\varphi_{2k+1}, \varphi_{2k+2}) \to 0.$$

Hence $\{\varphi_{2k}\}$ in $\Im$ is Cauchy sequence that complete $\exists$ aspent point $u \in X$ through that $\{\varphi_{2k}\} \to u$, since S is a continuous, we have

$$Su = \lim_{k\to\infty} p_{k+1} = u.$$

In this manner, u will fixed point of S. Correspondingly, we are able to demonstrate $Tu = \lim_{k\to 1} \varphi_{2k+1} = u$.

Hence, as result $Su = u = Tu$, thus, both S and T have u as their common fixed points.

Since S and T are a onto map and hence, $\exists$ a another aspect (point) $v \in X$ through $u = Sv = Tv$.

Consider,

$$d(\varphi_{2k}, u) = d(Sp_{k+1}, Tv)$$

$$\geq\ \alpha d(\varphi_{2k+1}, v) +\ \beta \frac{d(\varphi_{2k+1,} S\varphi_{2k+1}).d(v,Tv)}{d(\varphi_{2k+1,} v)} + \gamma \frac{d(v,Tv).d(\varphi_{k+1,} Tv)}{d(\varphi_{2k+1,} v) + d(v,Tv)}$$

$$= \alpha d(\varphi_{2k+1}, v) + \beta \frac{d(\varphi_{2k+1},\varphi_{2k}).d(v,Tv)}{d(\varphi_{2k+1},v)} + \gamma \frac{d(v,Tv).d(\varphi 2_{k+1},Tv)}{d(\varphi_{2k+1},v)+d(v,Tv)}.$$

Since $\{\varphi_{2k+1}\}$ is a subsequence of $\{\varphi_{2k}\}$. So, $\{\varphi_{2k}\} \to u, \{\varphi_{2k+1}\} \to u,$ when $k \to \infty$. So,

$$0 \geq \alpha d(u,v) + \beta \frac{d(u,u).d(v,u)}{d(u,v)} + \gamma \frac{d(v,u).d(u,u)}{d(u,v)+d(v,u)}$$

$$0 \geq \alpha d(u,v)$$

$$\Rightarrow d(u,v) \leq 0.$$

As a result, there exists u unique common fixed point that is common to both S and T. The verification process for the theorem has been completed.

Theorem 10: Presume that $S,T:\Im \to \Im$ are two surjective function in $(\Im, d)$ that satisfies inequality as under

$$d(Sp,Tq) \geq r_1 d(p,q) + r_2 d(p,Sp) + r_3 d(q,Tq) + r_4 d(p,Tq)$$
$$+r_5 \frac{d(p,Sp).d(q,Tq)}{d(p,q)} + r_6 \frac{[1+d(p,Sp)].d(p,Tq)}{d(p,q)} + r_7 \frac{d(q,Sq).d(p,Tq)}{d(p,q)}. \quad (10.1)$$

For all $p,q \in X$, where $r_i \geq 0, r_1 + r_2 + r_3 + r_5 > 1$and $r_1 + r_2 > 0$. Then , we claim that $Sv = u = Tv$ have unique common fixed point.

Proof: Imply that $p_0 \in X$. There is p_1 and p_2 in X such that

$$S(P_1) = p_0 \text{ and } T(p_2) = p_1.$$

In this way, we define two sequence $\{p_{2k}\}$ and $\{p_{2k+1}\}$ such that:

$$p_{2k} = Sp_{2k+1}, \text{ for } k = 0,1,2 \ldots \ldots$$

And

$$p_{2k+1} = Tp_{2k+2}, \text{ for } k = 0,1,2, \ldots$$

If $p_{2k} = p_{2k+1}$, for some $k \geq 1$, then it is common fixed point of S and T.Therefore, Presently, we put $p = p_{2k1}$ and $q = p_{2k+2}$ in (10.1), we gain

$$\Rightarrow d(p_{2k+1}, p_{2k+2}) \leq \frac{1-(r_2+r_5)}{(r_1+r_3)} d(p_{2k}, p_{2k+1})$$

$$\Rightarrow d(p_{2k+1}, p_{2k+2}) \leq \delta d(p_{2k}, p_{2k+1}),$$

$$\text{where } \delta = \frac{1-(r_2+r_5)}{(r_1+r_3)} \text{ as } r_1 + r_2 + r_3 + r_5 > 1.$$

So, in general we can write

$$\Rightarrow d(p_{2k+1}, p_{2k+2}) \leq \delta^n d(p_0, p_1).$$

Since $0 \leq \delta < 1$ as $n \to \infty, \delta^n \to 0$, we have

$$d(p_{n+1}, p_{n+2}) \to 0.$$

Hence $\{p_{2k}\}$ in $\mathfrak{I}$ is a Cauchy sequence that complete,∃ a point $x \in X$ through $\{p_{2k}\} \to x$. Since S is a continuous, we have

$$Sx = \lim_{n\to\infty} p_{n+1} = x.$$

In this way, x will fixed point of S. Correspondingly, we are able to demonstrate $Tx = \lim_{k\to 1} p_{2k+1} = x$. Hence, as result $Sx = x = Tx$, thus, both S and T have x as their common fixed points. Since S and T are two surjective map and hence, ∃ an other point $y \in X$ such that $x = Sy = Ty$. We have

$$d(p_n, x) = d(Sp_{n+1}, Sy)$$

$$\geq r_1 d(p_{2k+1}, y) + r_2\, d(p_{2k+1}, Sp_{2k+1}) + r_3\, d(y, Sy) + r_4 d(p_{2k+1}, Sy)$$

$$+ r_5 \frac{d(p_{2k+1}, Sp_{2k+1}).d(y,Sy)}{d(p_{n+1},y)} + r_6 \frac{[1+d(p_{2k+1}, Sp_{2k+1})]d(p_{2k+1},Sy)}{d(p_{2k+1},y)}$$

$$+ r_7 \frac{d(y,Sy)\, d(p_{2k+1},Sy)}{d(p_{2k+1},y)}$$

$$= r_1 d(p_{2k+1}, y) + r_2\, d(p_{2k+1}, p_{2k}) + r_3\, d(y, Sy) + r_4 d(p_{2k+1}, Sy)$$

$$+ r_5 \frac{d(p_{2k+1},2k).d(y,Sy)}{d(p_{2k+1},y)} + r_6 \frac{[1+d(p_{2k+1},p_{2k})]d(p_{2k+1},Sy)}{d(p_{2k+1},y)} + r_7 \frac{d(y,Sy)\, d(p_{2k+1},Sy)}{d(p_{2k+1},y)}.$$

Since $\{p_{2k+1}\}$ is a subsequence of $\{p_{2k}\}$. So, $\{p_{2k}\} \to x, \{p_{2k+1}\} \to x$, when $k \to \infty$.

So,

$$0 \geq r_1 d(x, y) + r_2\, d(x, x) + r_3\, d(y, x) + r_4 d(x, x) + r_5 \frac{d(x,x).d(y,x)}{d(x,y)}$$

$$+ r_6 \frac{[1+d(x,x)]d(x,x)}{d(x,y)} + r_7 \frac{d(y,x)\, d(x,x)}{d(x,y)}$$

$$\Rightarrow 0 \geq (r_1 + r_3)\, d(x, y \;\Rightarrow d(x, y) = 0 \Rightarrow x = y.$$

As a result, there exists u unique common fixed point that is common to both S and T. The verification process for the theorem has been completed.

Theorem 11: Presume that $S, T: \mathfrak{I} \to \mathfrak{I}$ are two surjective function in $(\mathfrak{I}, d)$ that satisfies inequality as under

$$d(Sx, Ty) \geq q\, max \left\{ d(x, y), \frac{d(x,Sx).d(y,Ty)}{d(x,y)}, \frac{[1+d(x,Sx)].d(x,Ty)}{d(x,y)}, \frac{d(y,Ty).d(x,Ty)}{d(x,y)} \right\}$$

(11.1)

$\forall\ x, y \in \Im$ and $q > 1$. Then , we claim that $Sv = u = Tv$.

Proof: Construct two sequences $\{\alpha_{2k}\}$ and $\{\alpha_{2k+1}\}$ as in proof of theorem 6, we claim that inequality 11.1, for put $x = \alpha_{2k+1}$ and $y = \alpha_{2k+2}$. Then we have

$$d(S\alpha_{2k+1}, T\alpha_{k+2}) \geq q\ max \left\{ \begin{array}{c} d(\alpha_{2k+1}, \alpha_{2k+2}), \dfrac{d(\alpha_{2k+1,}S\alpha_{2k+1}).\, d(\alpha_{2k+2}, T\alpha_{2k+2})}{d(\alpha_{2k+1}, \alpha_{2k+2})}, \\ \dfrac{[1 + d(\alpha_{2k+1}, S\alpha_{2k+1})]\ d(\alpha_{k+1}, T\alpha_{2k+2})}{d(\alpha_{2k+1}, \alpha_{2k+2})}, \\ \dfrac{d(\alpha_{k+2}, T\alpha_{k+2}) d(\alpha_{2k+1}, T\alpha_{k+2})}{d(\alpha_{2k+1}, \alpha_{2k+2})} \end{array} \right\}$$

$$d(\alpha_k, \alpha_{k+1}) \geq q\ max \left\{ \begin{array}{c} d(\alpha_{2k+1}, \alpha_{2k+2}), \dfrac{d(\alpha_{2k+1,}\alpha_{2k}).d(\alpha_{2k+2,}\alpha_{2k+1})}{d(\alpha_{k+1,}\alpha_{k+2})}, \\ \dfrac{[1+d(\alpha_{2k+1}, \alpha_{2k})]\ d(\alpha_{2k+1,}\alpha_{2k+1})}{d(\alpha_{2k+1,}\alpha_{2k+2})}, \\ \dfrac{d(\alpha_{2k+2,}\alpha_{2k+1}) d(\alpha_{2k+1,}\alpha_{2k+1})}{d(\alpha_{2k+1,}\alpha_{2k+2})} \end{array} \right\}$$

$$= q\ max\{d(\alpha_{k+1}, \alpha_{k+2}), d(\alpha_{k+1}, \alpha_k)\}$$

Case I:

$$d(\alpha_{2k}, \alpha_{2k+1}) \geq q\ max\{d(\alpha_{2k}, \alpha_{2k+1})\} \Rightarrow 1 \geq q, \text{ This is contradiction.}$$

Case II: $d(\alpha_{2k}, \alpha_{2k+1}) \geq q d(\alpha_{2k+1}, \alpha_{2k+2})$

$$\Rightarrow d(\alpha_{2k+1}, \alpha_{2k+2}) \leq \frac{1}{q} d(\alpha_{2k}, \alpha_{2k+1})$$

$$\Rightarrow d(\alpha_{2k+1}, \alpha_{2k+2}) \leq \lambda\, d(\alpha_{2k}, \alpha_{2k+1}), \text{ where } \frac{1}{q} = \lambda < 1 \text{ as } q > 1.$$

So, in general we have

$$d(\alpha_{2k}, \alpha_{2k+1}) \leq \lambda d(\alpha_{2k-1}, \alpha_{2k}), \text{ for } k = 0,1,2 \ldots.$$

Therefore,

$$d(\alpha_{2k}, \alpha_{2k+1}) \leq \lambda^{2k} d(\alpha_0, \alpha_1) \tag{11.2}$$

We find that $\{\alpha_{2k}\}$ is Cauchy sequence using (11.2) as proved theorem 6. That $\Im$ complete metric space there exists a point $x^* \in X$ by through $\{\alpha_{2k}\} \to x^*$.

Since S and T are two surjective self-map and hence there exists an anther fixed point $y^* \in X$ such that

$$x^* = Sy^* = Ty^*$$

Now, Consider,

$$d(\alpha_{2k}, x^*) = d(S\alpha_{k+1}, Ty^*)$$

$$\geq q\, max \begin{Bmatrix} d(\alpha_{2k+1}, y^*), \frac{d(\alpha_{2k+1}, S\alpha_{2k+1}) d(y^*, Sy^*)}{d(\alpha_{k+1}, y^*)}, \\ \frac{[1+d(\alpha_{k+1}, S\alpha_{2k+1})] d(\alpha_{2k+1}, Sy^*)}{d(\alpha_{2k+1}, y^*)}, \\ \frac{d(y^*, Sy^*) d(\alpha_{2k+1}, Sy^*)}{d(\alpha_{2k+1}, y^*)} \end{Bmatrix}$$

$$\geq q\, max \begin{Bmatrix} d(\alpha_{2k+1}, y^*), \frac{d(\alpha_{2k+1}, \alpha_{2k}) d(y^*, x^*)}{d(\alpha_{2k+1}, y^*)}, \\ \frac{[1+d(\alpha_{2k+1}, \alpha_{2k})] d(\alpha_{2k+1}, x^*)}{d(\alpha_{2k+1}, y^*)} \\ \frac{d(y^*, x^*) d(\alpha_{2k+1}, x^*)}{d(\alpha_{2k+1}, y^*)} \end{Bmatrix}$$

Since $\{\alpha_{2k+1}\}$ is a subsequence of $\{\alpha_{2k}\}$. So, $\{\alpha_{2k}\} \to x^*, \{\alpha_{2k+1}\} \to x^*$, when $k \to \infty$. Therefore

$$d(x^*, x^*) \geq q\, max \begin{Bmatrix} d(x^*, y^*), \frac{d(x^*, x^*) d(y^*, x^*)}{d(x^*, y^*)}, \\ \frac{[1 + d(x^*, x^*)] d(x^*, x^*)}{d(x^*, y^*)}, \frac{d(y^*, x^*). d(x^*, x^*)}{d(x^*, y^*)} \end{Bmatrix}$$

$$0 \geq q d(x^*, y^*) \quad \Rightarrow q d(x^*, y^*) = 0 \Rightarrow x^* = y^*.$$

As a result, there exists u unique common fixed point that is common to both S and T. The verification process for the theorem has been completed.

Conclusion

we discuss and improve the existing literature of Shakuntala and Tiwari (2020) and obtained unique common fixed point Theorems for expansive type Rational contraction n in metric spaces. Our results are Theorem 6, with Corollary 7,and 8 as well as Theorem 9,10, and 11 .

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