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MODELING PLANETARY AND STELLAR MOTION USING DIFFERENTIAL EQUATIONS

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Abstract

The report aims to explore the application of differential equations in modeling the motion of planets and stars within our universe, serving as an introduction to the captivating realm of celestial mechanics. We utilize differential equations to represent the movement and positions of celestial bodies within a gravitational field, grounding our analysis in Newton's laws of motion and gravitation. Moreover, we employ Kepler's laws of planetary motion to elucidate the orbits of planets around the sun. It is important to note that this report offers a simplified perspective, designed for educational purposes. In reality, celestial mechanics can be exceedingly intricate, involving n-body problems, relativistic effects, and a multitude of other factors.

Keywords: Celestial mechanics, Relativistic effects, Gravitational constant, N-body problems

Introduction

Johannes Kepler, a German astronomer, made groundbreaking contributions to celestial mechanics in the early 17^{th} century. He formulated three laws of planetary motion based on careful observations made by his mentor, Tycho Brahe. Kepler's laws describe how planets move in the solar system and are fundamental to understanding celestial mechanics. They are:

1. Kepler's first law (Law of Ellipses): Planetary orbits are ellipses with the sun at one of the two foci. This law replaced earlier notation of perfectly circular orbits and introduced the concept of elliptical orbits, with the sun not at the center but at one of the foci.
2. Kepler's second law (Law of Equal areas): A line segment joining a planet and the sun sweeps out equal areas in equal times. This law explains how a planet's speed varies as it moves along its elliptical orbit. When closer to the sun (perihelion), the planet moves faster, and when farther (aphelion), it moves more slowly.
3. Kepler's third law (Law of Harmonies): The Square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. This law establishes a mathematical relationship between a planet's distance from the sun and its orbital period. It helps relate the size of a planet's orbit to the time it takes to complete one orbit. Mathematically, it can be expressed as:

$$T^2 \propto a^3$$

   Where:

   - T is the orbital period of the planet.
   - a is the semi-major axis of the planet's elliptical orbit.

Kepler's laws were groundbreaking and marked a significant departure from the previous geocentric model of the universe. These laws describe the empirical aspects of planetary motion, and they played a crucial role in paving the way for Isaac Newton's theory of universal gravitation, which provided a physical explanation for why planets follow these elliptical paths around the Sun. Kepler's laws describe the empirical aspects of planetary motion but do not provide mechanism for why planets move the way they do. That's where the Newton's work comes in.

Newton's Law of Universal Gravitation:

Isaac Newton, an English physicist and mathematician, introduced the theory of universal gravitation in his book "Philosophiae Naturalis Principia Mathematica" (Mathematical Principles of Natural Philosophy), commonly known as the Principia, published in 1687. Newton's law of universal gravitation is a fundamental principle in explaining planetary motion:

- Every point mass attracts every other point mass by a force along the line intersecting both points. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
  Mathematically,
  $$F = \frac{G.m1.m2}{r^2}$$
  Where:
  F is the force of attraction between two objects.
  G is the universal gravitational constant.
  m1 and m2 are the masses of two objects.
  r is the distance between the centers of the two objects.

Newton's law of universal gravitation provided a theoretical framework for understanding why planets move as described by Kepler's laws. The gravitational force from the Sun, which follows Newton's law, is responsible for the elliptical orbits and the equal areas swept out by planets, as well as the precise mathematical relationship between a planet's orbital period and its distance from the sun.

Johannes Kepler and Sir Isaac Newton made significant contributions to our understanding of planetary motion, laying the foundation for modern celestial mechanics. Their theories are fundamental to explaining how planets move in our solar system.

Numerous researchers and organizations were actively working on modeling planetary motion, both in terms of understanding the fundamental principles of celestial mechanics and for practical purposes such as space missions and astronomical observations. Here are some insights into organizations and areas where research related to planetary motion was likely ongoing:

1. Space agencies: Space agencies like NASA, ESA, and other national space organizations regularly conduct research and modeling of planetary motion to plan

and execute space missions. They also analyze the dynamics of spacecraft in the solar system.

2. Astronomical observatories: Astronomers and astrophysicists at observatories and research institutions study planetary motion to improve our understanding of the solar system and beyond. They make observations and develop models to predict the positions of planets and other celestial objects.
3. Astronomical societies and conference: Astronomical societies, conferences and publications regularly feature research on planetary motion. Leading astronomers and scientists share their work and insights in these forums.

Celestial Mechanics:

Celestial mechanics is a branch of astronomy and physics that focuses on the study of the motion of celestial bodies in space, particularly the motion of planets, moons, stars, comets, and other objects within our universe. It seeks to explain and predict the positions and behaviors of these objects based on the principles of physics and mathematics. Celestial mechanics is fundamental to our understanding of the dynamics of the solar system and the universe as a whole.

The primary objectives of celestial mechanics include:

1. Understanding Planetary Orbits: Celestial mechanics provides insights into the shapes, sizes, and orientations of planetary orbits, as well as how they change over time.
2. Determining Positions and Motions: It enables the precise prediction of celestial objects positions and movements at any given time, which is essential for astronomical observations and space exploration.
3. Studying Gravitational Interactions: Celestial mechanics explores the gravitational interactions that govern the behavior. The law of universal gravitation, formulated by Isaac Newton, is a cornerstone of this field.
4. Evaluating the Effects of perturbations: Celestial mechanics takes into account the effects of perturbations, which are gravitational influences from the other celestial bodies. These perturbations can have a significant impact on orbits.

Central force concept:

Central force concept is a fundamental concept in physics, particularly in the context of celestial mechanics and the motion of objects in a central gravitational field, where gravity is the dominant force. This concept is crucial for understanding how objects like planets, moons, and other celestial bodies move under the influence of gravity. Central force is a force that acts on an object and is directed toward or away from a fixed point, known as the center of force. In the context of celestial mechanics, gravity is often the dominant central force. For example, when studying the motion of planets in the solar system, the gravitational force exerted by the Sun is considered as central force. The gravitational force acts toward the center of the Sun, and it is responsible for keeping planets in their orbits. The central force concept is often mathematically described using polar coordinates. In polar coordinates, the motion of an object is described in terms of its radial distance from the center of force (r) and its angular position (θ).

Need for differential equations in modeling planetary and stellar motion:

1. Dynamics Systems: Planets and stars are dynamic systems, meaning their positions and velocities change continuously. Differential equations provide a framework to describe how these quantities change with time.
2. Change over time: Differential equations are essential for modeling objects motion because they describe how physical quantities (position, velocity, and acceleration) change over time or other relevant independent variables (e.g., distance from a central body or time since a certain event).
3. Interaction Forces: Planetary and stellar motion is primarily governed by gravitational forces. Differential equations allow us to describe how objects interact under these forces. Newton's law of universal gravitation, for example, is expressed as a differential equation, providing a mathematical description of these interactions.
4. Predictive power: By solving differential equations, we can predict the future positions and velocities of celestial bodies. This predictive power is vital for astronomical observations, space exploration, and the understanding of long-term cosmic phenomena.
5. Numerical simulations: While some problems can be solved analytically, many celestial mechanics problems, especially those involving multiple bodies or complex

gravitational perturbations, are too challenging to solve directly. Numerical simulations using differential equations and computational methods are essential for accurate modeling.

6. Accounting for perturbations: Celestial bodies experience perturbations from other nearby objects, such as the gravitational influence of other planets or stars. Differential equations are used to account for these perturbations and determine their effects on the motion of the objects.
7. Realistic orbits: Real orbits are often not perfect circles, and they can be highly elliptical or subject to other distortions. Differential equations allow us to understand these more complex orbital shapes.
8. Accounting for Relativistic Effects: In some cases, such as when modeling the orbit of Mercury, relativistic effects (as described by Einstein's theory of general relativity) need to be considered. Differential equations can be adapted to account for these effects.
9. Scientific understanding: The study of planetary and stellar motion is central to our understanding of the universe. Differential equations provide a precise and quantitative way to study and analyze these motions, leading to insights into fundamental physical laws.

Acceleration due to gravity:

Acceleration due to gravity, often denoted as g, is a fundamental concept in physics. It represents the acceleration that a mass experiences due to the gravitational force of a celestial body, such as a planet or a star. This acceleration is directed toward the center of the celestial body and is responsible for the force of gravity.

Mathematically,

$$g = \frac{F}{m}$$

Where:

g is the acceleration due to gravity.

F is the gravitational force acting on an object.

m is the mass of the object.

The value of g depends on the mass of the celestial body and the distance from its center. On Earth, the average acceleration due to gravity is approximately 9.81 m/s^2. This means that an object with a mass of 1 kg experiences a gravitational force of about 9.81 N toward the center of the Earth.

Applying Acceleration due to gravity to a celestial body:

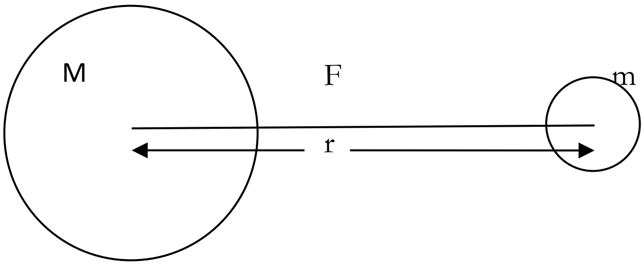
To apply the concept of acceleration due to gravity to a celestial body, let's use the example of earth. On Earth, the acceleration due to gravity is approximately 9.81 m/s^2. Here's how we can apply it to an object on or near Earth:

1. Weight calculation: The weight of an object on earth is the force of gravity acting on it. It can be calculated using the formula:
   $W = m.g$
   Where:
   W is the weight of the object.
   m is the mass of the object.
   g is the acceleration due to gravity on Earth.
2. Planetary motion: Celestial bodies, such as planets, experience gravitational acceleration from the central body, which keeps them in orbit. For example, the acceleration due to gravity on the surface of Earth keeps the Moon in orbit around our planet

Deriving the differential equation

To derive a second-order differential equation that describes the motion of a planet or star under the influence of gravity, we can start with Newton's law of universal gravitation. The equation should include the position vector, mass, and the gravitational constant. The motion of a celestial body under gravity can be described in terms of polar coordinates, where we consider the radial distance from the central body and the angular position. Here's how we can derive the equation:

Consider a celestial body (planet or star) of mass m and M is the mass of central body (Sun for planets). Let r be the distance between their centers and $\vec{r}$ be the position vector of the celestial body relative to the central body. Let G be the universal gravitational constant and F be the gravitational force between two point masses.



According to Newton's law of universal gravitation,

$$F = \frac{G.m.M}{|\vec{r}|^2}.\hat{r}$$

Where $|\vec{r}|$ is the magnitude of the position vector, and $\hat{r}$ is the unit vector pointing from the celestial body to the central body.

Also, the acceleration $\vec{a}$ experienced by the celestial body due to gravity is given by Newton's second law:

$$\vec{a} = \frac{\vec{F}}{m}$$

Substituting the gravitational force equation:

$$\vec{a} = \frac{G.M}{|\vec{r}|^2}.\hat{r}$$

In polar coordinate system, we can decompose the position vector $\vec{r}$ into its radial component (r) and angular component (θ).

The radial equation for motion under gravity is derived from the equation for acceleration as follows:

$$m.\frac{d^2r}{dt^2} = -\frac{G.m.M}{r^2} + m.r.\left(\frac{d\theta}{dt}\right)^2$$

This equation describes the radial motion of the planet or star as it orbits the central body under the influence of gravity. The first term on right side represents the gravitational attraction and the second term represents the centrifugal force.

The angular motion of a planet or star is typically not influenced directly by gravity and follows a simpler path. Therefore, the angular equation is

$$m\,.\,r^2\,.\left(\frac{d^2\theta}{dt^2}\right)^2 = 0$$

This equation states that the angular acceleration is zero, meaning the angular velocity remains constant (in the absence of other forces).

These two equations describe the motion of a planet or star under the influence of gravity. The radial equation accounts for the changes in radial distance from the central body, while the angular equation addresses the angular motion.

Circular orbits are simplified but important case in celestial mechanics. In a circular orbit, a celestial body (such as a planet or a satellite) moves in a path that traces a perfect circle around a central body (e.g., a star like the Sun). Circular orbits are particularly useful in simplifying the mathematics of celestial mechanics and provide a basis for understanding more complex elliptical orbits.

Consider a celestial body of mass m in a circular orbit around a central body of mass M (e.g., a planet orbiting the sun) the position vector $\vec{r}$ points from the central body to the celestial body. Gravity provides the necessary centripetal force to keep the celestial body in circular orbit. According to Newton's second law,

$$F = m\,.\,a$$

Where a is centripetal acceleration which is directed radially inward and is determined by the angular velocity $(\omega)$and the radius of the orbit (r). it is given by

$$a = \omega^2\,.\,r$$

Gravitational force pulling the celestial body towards the central body is given by

$$F = \frac{G.m.M}{r^2}$$

Equating the centripetal force to the gravitational force:

$$m\,.\,\omega^2\,.\,r = \frac{G.m.M}{r^2}$$

We have,

$$\omega = \frac{2\pi}{T}$$

Where T is orbital time period.

Substituting ω , we get

$$r^3 = \frac{G.M.T^2}{4\pi^2}$$

Also, centripetal force required for a circular orbit is given by:

$$Fcentripetal = m.\frac{v^2}{r}$$

Where r and v are the radius and velocity of the celestial body in the circular orbit respectively.

So, $\frac{G.m.M}{r^2} = m.\frac{v^2}{r}$

Or, $v^2 = \frac{G.M}{r}$

For example: we can use this derived equation for circular orbits to find orbital parameters of Earth's orbit around the Sun.

Solution:

Radius of Earth's circular orbit (r) = 1.496×$10^{11}m\ \ (approx)$

Mass of the Sun (M) =1.989×10^{30}kg

Velocity of Earth in its circular orbit can be calculated as:

$$v^2 = \frac{G.M}{r}$$

Or, $$v^2 = \frac{(6.67\times10^{-11}).(1.989\times10^{30})}{(1.496\times10^{11})}$$

$$\therefore v = 2.9 \times 10^4 m/s$$

This is an oversimplified representation of Earth's orbit, as Earth's actual orbit is an ellipse with variations in velocity throughout the year due to Kepler's laws and orbital dynamics. However, for basic calculations, a circular orbit approximation.

Challenges for solving differential equations analytically:

Solving differential equations analytically can be challenging especially for complex equations like motion of planets and stellar due to the following reason:

1. Complexity of the equations: equations for motion of celestial bodies can become extremely complex. Analytical solutions may not exist or may be exceedingly difficult to obtain, especially in case involving more than two bodies.
2. Nonlinearity: many differential equations describing planetary motion involve nonlinear terms, making them challenging to solve analytically. Nonlinearity can lead to lack of closed-form solutions.
3. Initial and Boundary conditions: Analytical solutions often require precise initial and boundary conditions, which can be hard to determine accurately, especially for real-world celestial systems with various perturbations.
4. Perturbations: Real planetary motion is influenced by perturbations from other celestial bodies. These perturbations add complexity to the equations, making analytical solutions less practical.
5. Relativistic effects: For high-precision applications, such as GPS systems or spacecraft navigation, relativistic effects need to be considered, which further complicate analytical solutions.

Introduction to numerical methods – Euler's method:

Numerical methods are essential for approximating solutions to differential equations, especially when analytical solutions are difficult or impossible to obtain. One of the simplest numerical methods is Euler's method, which is used to approximate the solution of first-order ordinary differential equations. It works by discretizing the domain into small time steps and iteratively updating the solution.

Euler's method approximates the solution y(t) of a first-order ordinary differential equation $dy/dt = f(t, y)$ using the following iterative formula:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

where:

y_{n+1} is the updated solution at time $t_{n+1} = t_n + h$.

y_n is the current solution at time t_n.

h is the step size (time increment)

$f(t_n , y_n)$ is the value of the derivative at time t_n and solution y_n.

Here is an example illustrating the position of earth around sun in its orbit:

Assumption:

1. Earth revolve around the Sun in it circular simplified orbit but in reality the Earth's orbit is elliptical, this example is just for demonstration purposes.
2. Earth does not experience any other force due to other celestial bodies.
3. We will use a simplified example with 2D coordinates.

   To find the position of Earth around the Sun at different intervals of time using the Euler method with the given formula:

   $y_{n+1} = y_n + h \cdot f(t_n , y_n)$

where:

y_n represents the position of Earth at time tn.

h denotes the interval.

$f(t_n, y_n)$ is the derivative of the position with respect to time.

Here is a C program for iterative operation up to 365 days to demonstrate the x- and y-coordinate:

```
#include <stdio.h>
#include <math.h>
// Function to calculate the derivative based on the circular motion
void f(double t, double x, double y, double *dxdt, double *dydt)
 {
   double r = sqrt(x * x + y * y);
   double G = 6.67430e-11; // Gravitational constant
   double M = 1.989e30;   // Mass of the Sun
   double F = G * M / (r * r);
   *dxdt = -F * x / r;
   *dydt = -F * y / r;
```

```
}
int main()
{
    double t = 1;        // Initial time
    double x = 1.496e11;   // Initial x-coordinate (Earth's distance from the Sun)
    double y = 0.0;        // Initial y-coordinate (assuming Earth starts on the x-axis)
    double h = 3600.0;     // Time step (in seconds)
    int numsteps = 365;   // Number of steps to simulate
while(t<numsteps)
 {
        double dxdt, dydt;
        f(t, x, y, &dxdt, &dydt);
        x = x + h * dxdt;
        y = y + h * dydt;
        t = t + 1;
        printf("Time: %.2f seconds, x: %.2f meters, y: %.2f meters\n", t, x, y);
    }
    return 0;
}
```

By running this program, we can find the position of Earth around the Sun in 2D from 1 to 365 days and it was found as 1.496$\times 10^{11}$ meters and 0 in x- and y-coordinates respectively.

Some mission and space exploration where such model are used:

1. To locate geostationary satellite for telecommunication, military defense.
2. Missions such as the Hubble space Telescope are positioned in orbits that provide optimal viewing angles for observing distant astronomical objects. These orbits are carefully chosen based on Earth's position relative to the Sun and other celestial objects.
3. Missions like lunar landings, spacewalks, satellites monitoring earth climate etc.

Conclusion

Using differential equation, the position of celestial bodies can easily be found using Euler's method rather than complex analytical method and perturbations as the position of earth around the Sun in simplified circular orbit (assumption) is determined.

Further exploration

Differential equations can further be used in advanced topics and techniques contributing space related researches and exploration. The theories may involve the position of celestial bodies under perturbations (i.e. three body problems) or relativistic effects in celestial mechanics.

It can be further used to determine its velocity at different interval of time in its elliptical path observing the duration of earth moving away from the sun and proceeding towards the sun in its elliptical orbit.

The research may include the influence of gravitational force among different celestial bodies orbiting the Sun in its own separate orbit.

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