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Log-Linearization Technique in DSGE Modeling: A Quick Guide

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Abstract

This paper presents a concise yet effective technique for simplifying complex nonlinear equations by converting them into a log-linear form, with particular application to Dynamic Stochastic General Equilibrium (DSGE) models. The method, known as log-linearization, leverages Taylor series approximation to express nonlinear functions as linear relationships in logarithmic deviations around a steady-state equilibrium. By doing so, the approach enhances analytical tractability and facilitates the interpretation and solution of DSGE models. The paper provides illustrative examples, including applications to Real Business Cycle (RBC) models, to demonstrate the practical implementation and benefits of the technique in macroeconomic modeling.

Keywords: Log-Linearization; Taylor Approximation; Log-Deviations; DSGE Models; RBC Models

Introduction

Economic modeling is a crucial tool that economists and policymakers use to analyze, understand, and predict the behavior of complex economic systems. It plays a significant role in answering critical questions related to fiscal policy, monetary policy, and the overall performance of economies. Since these models aim to capture the complex behaviors of economic agents and the relationships between economic variables, they often become highly intricate and challenging.

Complexity economics models use mathematical tools, such as nonlinear equations and stochastic processes. However, often, the intricate nature of monitoring the decision-making processes of several agents necessitates the use of computers. (Arthur, 2021)

A Dynamic Stochastic General Equilibrium (DSGE) is one of these models that offers a more realistic representation of the economy (Smets & Wouters, 2007). They consider the interactions between agents and incorporate stochastic elements to account for uncertainty. However, these features also contribute to their complexity. DSGE models often involve systems of nonlinear equations with expectations and intertemporal dynamics, making them more complex. Conversely, another class of models uses a social accounting matrix (SAM) called CGE models. A SAM provides a comprehensive snapshot of an economy at a point in time, detailing transactions between different economic agents (such as households, firms, and the government) and sectors. This matrix captures the production, income, and expenditure flows, allowing for a detailed representation of the economic interactions and dependencies among various actors and sectors.

DSGE models incorporate time dynamics, stochastic elements, optimization, micro foundations, market-equilibrium conditions, parameter estimation, nonlinear equations, calibration, and computational intensity. These models are based on microeconomic principles and involve modeling the decision-making processes of individual economic agents. Achieving market clearing, where supply equals demand in various markets, requires complex modeling of how prices adjust. Parameter estimation and calibration involve setting initial conditions and matching parameters to real-world data. The computational complexity arises from solving many equations and simulating economic dynamics. While DSGE models aim to capture economic intricacies, their simplifying assumptions can add to their complexity.

However, one technique stands out as essential for simplifying the DSGE models and making them more manageable is log-linearization. It come after the First-order condition (FOC) in the model.

The log-linearization method offers a solution to the issue of lowering computing complexity for systems of numerically defined equations that need simultaneous solving. Log-linearization is a technique used to transform a non-linear equation into a linear equation by expressing the variables as logarithmic deviations from their steady state values. Moreover, it provides a means to evaluate the responses of various economic variables, making it an invaluable tool for central banks and policymakers when formulating and assessing monetary and fiscal policies.

The paper is structured as follows: After the introduction, Section 2 outlines the main problem this technique tries to solve; Section 3 presents the Taylor approximation technique; Section 4 discusses log-linearization in DSGE modeling; Section 5 gives some real examples from different DSGE models; and Section 6 wraps up the paper.

The problem

In this section, we introduce the log linearization technique into DSGE modeling through two simple examples. The first example comes from a New Classical Model (NCM), and the second comes from a New Keynesian Model (NKM). (StataCorp, 2023)

1) New Classical model:

The new classical economics emerged in the 1970s as a reaction to the dominance of Keynesian economics, which dominated economic theory in the post-World War II period. Keynesian economics emphasized the role of government intervention in managing aggregate demand and stabilizing the economy.

By the 1970s, however, many economists and policymakers were disappointed with the effectiveness of Keynesian policies, especially in addressing problems such as stagflation, which is characterized by a combination of high inflation and high unemployment. The new classical economics emerged in response to these perceived limitations of the existing Keynesian framework and reflected a desire for a theoretically more rigorous and grounded approach to macroeconomics.

Production, consumption, and other variables are determined by state variables related to production and demand. This model is similar to that of (King & Rebelo, 1999) and is commonly referred to as the real business cycle (RBC) model.

The standard form of the model can be writing as:

$$\frac{1}{C_t} = \beta\, E_t \left\{ \frac{1}{C_{t+1}} (1 + R_{t+1} - \delta) \right\} \qquad (1)$$

Equation (1) defines consumption at time t (the present) C_t as a function of the expected future consumption $E_t(C_{t+1})$ and the interest rate $E_t(\mathrm{R}_{t+1})$ and E_t is the expected future value for t+1 in time t.

$$H_t^{\eta} = \frac{W_t}{C_t} \qquad (2)$$

This equation specifies the labor supply, relating it to wage W_t and consumption C_t.

$$Y_t = C_t + X_t + G_t \qquad (3)$$

In the closed economy, the national income accounting identity equals the sum of consumption C_t, investment X_t, and government spending G_t.

$$Y_t = K_t^{\alpha} (Z_t H_t)^{1-\alpha} \qquad (4)$$

A Cobb-Douglas production expresses the quantity Y_t of output as a function of capital K_t, labor H_t, and productivity Z_t.

$$W_t = (1 - \alpha) \frac{Y_t}{H_t} \qquad (5)$$

Equation (5) is for labor demand.

$$R_t = \alpha \frac{Y_t}{K_t} \qquad (6)$$

Equation (6) is for capital demand.

Since this model is dynamic, it is necessary to define a capital accumulation equation:

$$K_{t+1} = (1 - \delta) K_t + X_t \qquad (7)$$

In general, nonlinear systems like this cannot be solved analytically. However, a corresponding set of linear equations can approximate their solution very well.

By introducing log-linearization around a steady-state (SS) path where all real variables grow at the same rate, the model can be written as follows:

$$c_t = E_t(c_{t+1}) - (1 - \beta + \beta\delta)\, E_t(r_{t+1}) \qquad (1')$$

$$\eta h_t = w_t - c_t \qquad (2')$$

$$\phi_1 x_t = y_t - \phi_2 c_t - \phi_3 g_t \qquad (3')$$

$$y_t = (1-\alpha)(x_t + h_t) + \alpha k_t \qquad (4')$$

$$w_t = y_t - h_t \qquad (5')$$

$$r_t = y_t - k_t \qquad (6')$$

$$k_{t+1} = \delta x_t + (1-\delta) k_t \qquad (7')$$

$$z_{t+1} = \rho_z z_t + \xi_{t+1} \qquad (8')$$

$$g_{t+1} = \rho_g g_t + \varepsilon_{t+1} \qquad (9')$$

ξ_{t+1} and ε_{t+1} are shocks to the state variables.

Equations (8') and (9') are modeled as autoregressive $AR(1)$, with the variables ε_{t+1} and ξ_{t+1} being shocks to the state variables.

It is evident that all the equations in this model follow a linear form, which is straightforward to manipulate and easy to control.

2) New Keynesian model:

In this example, we presented a nonlinear model similar to that of (Woodford, 2003). The equations of this model describe the behaviors of agents (households, businesses and the central bank), and their interactions generate a model of inflation, interest rates and economic growth. This type of model is widely used in academic and policy research to characterize monetary policy.

The optimization of household produces an equation that links present output with the expected value of a function based on output Y_{t+1}, inflation Π_{t+1}, and the current nominal interest rate R_t,

$$\frac{1}{Y_t} = \beta\, E_t\left\{\frac{1}{Y_{t+1}} \frac{R_t}{\Pi_{t+1}}\right\} \qquad (A)$$

where β is the representative household's rate of time preference.

Firms' optimization generates an equation that specifies a link between the present deviation of inflation from its steady state $\Pi_t - \Pi$, the anticipated deviation of inflation in t+1 $E_t(\Pi_{t+1} - \Pi)$, and the ratio of real output Y_t to the natural output level Z_t.

$$(\Pi_t - \Pi) + \frac{1}{\phi} = \phi\left(\frac{Y_t}{Z_t}\right) + \beta E_t(\Pi_{t+1} - \Pi) \qquad (B)$$

Where ϕ is a coefficient that links the pricing decisions of firms, however, they are impacted only by the deviation of inflation $(\pi_{t+1} - \pi)$.

Finally, the central bank adjusts interest rates in response to inflation and other factors influencing the central bank's policy are not considered in this model. Therefore, the equation can be expressed as follows:

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi}\right)^{\frac{1}{\beta}} U_t \qquad (C)$$

Where R, π are the SS-values of the interest rate and inflation respectively. U_t is a state variable that captures all interest rate movements that are not caused by inflation.

Before proceeding with parameter estimation, the model equations undergo several adjustments. As outlined by Woodford (2003), the model initially presented in sections $(A) - (C)$ is modified by introducing $X_t = \frac{Y_t}{Z_t}$ to represent the output gap. This revision is adopted in our analysis as well. By incorporating into the framework, we transition to a revised three-equation system.

$$1 = \beta E_t\left(\frac{X_t}{X_{t+1}} \frac{1}{G_t} \frac{R_t}{\Pi_{t+1}}\right) \qquad (A')$$

$$(\Pi_t - \Pi) + \frac{1}{\phi} = \phi X_t + \beta E_t(\Pi_{t+1} - \Pi) \qquad (B')$$

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi}\right)^{\frac{1}{\beta}} U_t \qquad (C')$$

where $G_t = \frac{Z_{t+1}}{Z_t}$ is a state variable that captures changes in Z_t.

The model is finalized by including equations that depict the progression of the state variables G_t and U_t. We express them as autoregressive models in logarithmic form.

$$\log(G_{t+1}) = (1 - \rho_g)\log(G) + \rho_g \log(G_t) + \varepsilon_{t+1} \qquad (D')$$

$$\log(U_{t+1}) = (1-\rho_u)\log(U) + \rho_u \log(U_t) + \xi_{t+1} \qquad (E')$$

The variables ε_{t+1} and ξ_{t+1} are shocks to the state variables.

The model in $(A') - (E')$ is nonlinear. By apply the log linearization technique, we can write the model in its corresponding linearized form. The linearized versions of $(A') - (E')$ are:

$$x_t = E_t x_{t+1} - (r_t - E_t \pi_{t+1} - g_t) \qquad (a'')$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \qquad (b'')$$

$$r_t = \frac{1}{\beta}\pi_t + u_t \qquad (c'')$$

$$g_{t+1} = \rho_g g_t + \varepsilon_{t+1} \qquad (d'')$$

$$u_{t+1} = \rho_u \mathrm{u}_t + \xi_{t+1} \qquad (e'')$$

Equations $(a'') - (e'')$ specify a canonical New Keynesian model of inflation π_t, the output gap x_t, and the interest rate r_t.

As we can observe from these two examples, the linear form follows the log-linearization. The coefficients in this version are easy to interpret, allowing us to calculate the Impulse Response Functions (IRF) response easily.

Now, the question is, how can we apply the log-linearization in the non-linear model? The response is by employing a Taylor approximation around the steady state.

Taylor approximation:

1) The Taylor Series:

The Scottish mathematician James Gregory discovered the concept of a Taylor series, which was formally introduced by the English mathematician Brook Taylor in 1715. The study of numerical methods and the implementation of numerical algorithms are significantly influenced by Taylor's series. A Taylor series is a mathematical representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The series is predicated on Taylor's Theorem, which posits that the polynomial can be used to approximate any smooth function, f(x), in the vicinity of an expansion point a:

$$\boldsymbol{f(x) = f(a) + \frac{(x - a)}{1!} f^{(1)}(a) + \frac{(x - a)^2}{2!} f^{(2)}(a) + \cdots + \frac{(x - a)^n}{n!} f^{(n)}(a) + \cdots}$$

In terms of sigma notation, the Taylor series can be written as

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}$$

where $\boldsymbol{f^{(k)}(a)}$ is the k^{th} derivative, evaluated at $x = a$, and k! is the factorial of k.

If $f\ (x)$ equals the sum of its Taylor series (about a) at x, then we have:

$$\boldsymbol{\lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f\ (x)}$$

The series is also referred to as a Maclaurin series (or power series) if it is centered at zero (i.e., the expansion point a is set to zero). The term "Maclaurin series" is derived from the Scottish mathematician Colin Maclaurin, who extensively employed this distinctive form of the Taylor series in the 18th century. Subsequently, the power series is as follows

$$\boldsymbol{f(x) = f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots}$$

a) Some examples:

- Let's calculate the Taylor series at $x = 0$ for $f(x) = \exp(x)$

First, we will find the derivatives of the given function.

$f(x) = \exp(x) \Rightarrow f(0) = \exp(0) = 1$

$f'(x) = \exp(x) \Rightarrow f'(0) = 1$

$f''(x) = \exp(x) \Rightarrow f''(0) = 1$

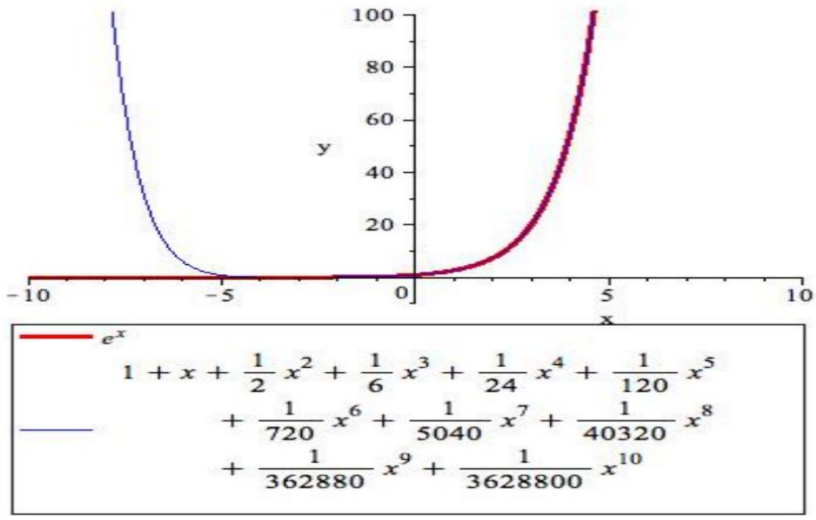
$f'''(x) = \exp(x) \Rightarrow f'''(0) = 1$

Therefore, the required series is:

$$\exp(x) \approx \sum_{n=0}^{\infty} \frac{\boldsymbol{f^{(n)}(0)}}{\boldsymbol{n!}} (x)^n \approx \boldsymbol{f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \cdots}$$

$$\approx \mathbf{0} + \boldsymbol{x} + \frac{\boldsymbol{x}^2}{\mathbf{2}} + \frac{\boldsymbol{x}^3}{\mathbf{6}} + \frac{\boldsymbol{x}^4}{\mathbf{24}} + \frac{\boldsymbol{x}^5}{\mathbf{120}} + \frac{\boldsymbol{x}^6}{\mathbf{720}} + \frac{\boldsymbol{x}^7}{\mathbf{5040}} + \frac{\boldsymbol{x}^8}{\mathbf{40320}} + \frac{\boldsymbol{x}^9}{\mathbf{362880}} + \frac{\boldsymbol{x}^{10}}{\mathbf{3628800}} = \boldsymbol{g}(\boldsymbol{x})$$

Figure 1: expo function presentation



source: the authors

The figure above illustrates estimates of the function $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{exp}(\boldsymbol{x})$ within the interval from -4 to 4. These estimates are obtained using both the fourth and tenth-degree Taylor polynomials. Increasing the polynomial degree leads to more accurate approximations.

- Let's calculate the Taylor series at $x = 0$ for $f(x) = \ln(x + 1)$

First, we will find the derivatives of the given function.

$$\mathrm{f}(x) = \ln(x+1) \Rightarrow f(0) = \ln(1) = 0$$

$$f'(x) = \ln(x+1)' = \frac{1}{x+1} \Rightarrow f'(0) = 1$$

$$f''(x) = -(x+1)^{-2} \Rightarrow f''(0) = -1$$

$$f'''(x) = 2(x+1)^{-3} \Rightarrow f'''(0) = 2$$

Therefore, the required series is:

$$\ln(x+1) \approx \sum_{n=0}^{\infty} \frac{\boldsymbol{f}^{(n)}(\mathbf{3})}{\boldsymbol{n}!}(\boldsymbol{x}-\boldsymbol{a})^n$$

$$\approx \boldsymbol{f}(\mathbf{0}) + \frac{(\boldsymbol{x}-\mathbf{0})}{\mathbf{1}!}\boldsymbol{f}^{(1)}(\mathbf{0}) + \frac{(\boldsymbol{x}-\mathbf{0})^2}{\mathbf{2}!}\boldsymbol{f}^{(2)}(\mathbf{0}) + \frac{(\boldsymbol{x}-\mathbf{0})^3}{\mathbf{3}!}\boldsymbol{f}^{(3)}(\mathbf{0}) + \boldsymbol{\varepsilon}(\boldsymbol{n}+\mathbf{1})$$

$$\approx \mathbf{0} + \boldsymbol{x} - \frac{(\boldsymbol{x})^2}{\mathbf{2}} + \frac{(\boldsymbol{x})^3}{\mathbf{3}} + \boldsymbol{\varepsilon}(\boldsymbol{n}+\mathbf{1}) = \boldsymbol{g}(\boldsymbol{x})$$

- Let's calculate the Taylor series at $x = 3$ for $f(x) = x^3 + x^2 + 1$

First, we will find the derivatives of the given function.

$f(x) = x^3 - 10x^2 + 6 \Rightarrow f(3) = -57$

$f'(x) = 3x^2 - 20x \Rightarrow f'(3) = -33$

$f''(x) = 6x - 20 \Rightarrow f''(3) = -2$

$f'''(x) = 6 \Rightarrow f'''(3) = 6$

$f''''(x) = 0$

Therefore, the required series is:

$$x^3 - 10x^2 + 6 \approx \sum_{n=0}^{\infty} \frac{\boldsymbol{f}^{(n)}(\mathbf{3})}{\boldsymbol{n}!}(\boldsymbol{x}-\boldsymbol{a})^n$$

$$\approx \boldsymbol{f}(\mathbf{3}) + \frac{(\boldsymbol{x}-\mathbf{3})}{\mathbf{1}!}\boldsymbol{f}^{(1)}(\mathbf{3}) + \frac{(\boldsymbol{x}-\mathbf{3})^2}{\mathbf{2}!}\boldsymbol{f}^{(2)}(\mathbf{3}) + \frac{(\boldsymbol{x}-\mathbf{3})^3}{\mathbf{3}!}\boldsymbol{f}^{(3)}(\mathbf{3}) + \mathbf{0}$$

$$\approx -\mathbf{57} - \mathbf{33}(\boldsymbol{x}-\mathbf{3}) - (\boldsymbol{x}-\mathbf{2})^2 + (\boldsymbol{x}-\mathbf{3})^3$$

2) The Taylor Series for multivariable functions:

For a function of two variables $f(x, y)$ whose partials all exist to the nth partials at the point (a, b), the nth -degree Taylor polynomial of $f(x, y)$ near the point (a, b) is:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{a},\boldsymbol{b}) + \frac{(\boldsymbol{x}-\boldsymbol{a})}{\mathbf{1}!}\boldsymbol{f}_x^{(1)}(\boldsymbol{a},\boldsymbol{b}) + \frac{(\boldsymbol{y}-\boldsymbol{b})}{\mathbf{1}!}\boldsymbol{f}_y^{(1)}(\boldsymbol{a},\boldsymbol{b}) + \frac{(\boldsymbol{x}-\boldsymbol{a})^2}{\mathbf{2}!}\boldsymbol{f}_x^{(2)}(\boldsymbol{a},\boldsymbol{b}) + \frac{(\boldsymbol{y}-\boldsymbol{b})^2}{\mathbf{2}!}\boldsymbol{f}_y^{(2)}(\boldsymbol{a},\boldsymbol{b}) + (\boldsymbol{x}-\boldsymbol{a})(\boldsymbol{y}-\boldsymbol{b})\,\boldsymbol{f}_{xy}(\boldsymbol{a},\boldsymbol{b}) + \cdots + \frac{(\boldsymbol{x}-\boldsymbol{a})^n}{\boldsymbol{n}!}\boldsymbol{f}_x^{(n)}(\boldsymbol{a},\boldsymbol{b}) + \frac{(\boldsymbol{y}-\boldsymbol{b})^n}{\boldsymbol{n}!}\boldsymbol{f}_y^{(n)}(\boldsymbol{a},\boldsymbol{b}) + \cdots$$

We can write this formulation above as:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{P}_{\mathrm{n(x,y)}} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\frac{d^{(i+j)}}{\partial x^i \partial y^j} f(a,b)}{i!\,j!}(x-a)^i\,(y-b)^j$$

a) **Some examples:**

- **Let's consider the function:** $\boldsymbol{f(x,y) = x^3y^2 + y \ et \ (a,b) = (1,1)}$

$$f(1+h, 1+k) = f(1,1) + h\frac{\partial f}{\partial x}(1,1) + k\frac{\partial f}{\partial y}(1,1) + \sqrt{h^2 + k^2}\varepsilon(h,k)$$

We have: $\frac{df}{dx}(x,y) = 3x^2y^2 \ \ and \ \ \frac{df}{dx}(1,1) = 3$

$\frac{df}{dy}(x,y) = 2x^3y + 1 \ and \ \ \frac{df}{dy}(1,1) = 3$

So, the first Taylor approximation of $f(x,y)$ can be write as:

$$f(1+h, 1+k) = 2 + 3h + 3k + \sqrt{h^2 + k^2}\varepsilon(h,k)$$

With: $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$

- **Let's consider the function:** $\boldsymbol{f(x,y) = x^3y^2 + y \ et \ (a,b) = (1,1)}$

$\frac{df}{dx}(x,y) = 3x^2y^2 \ \ and \ \ \frac{df}{dx}(1,1) = 3$

$\frac{d^2f}{dx^2}(x,y) = 6xy^2 \ \ and \ \ \frac{d^2f}{dx^2}(1,1) = 6$

$\frac{df}{dy}(x,y) = 2x^3y + 1 \ and \ \ \frac{df}{dy}(1,1) = 3$

$\frac{d^2f}{dy^2}(x,y) = 2x^3 \ and \ \ \frac{df}{dy}(1,1) = 2$

$\frac{df}{dydx}(x,y) = 6x^2y \ and \ \ \frac{df}{dy}(1,1) = 6$

So, the second Taylor approximation of $f(x,y)$ can be write as:

$$f(1+h, 1+k) = 2 + 3h + 3k + \frac{6}{2}h^2 + \frac{2}{2}k^2 + 6h.k + (h^2 + k^2)\varepsilon(h,k)$$

With: $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$

Log-Linearization technique in DSGE models:

1- **Solving DSGE Models:**

Solving a Dynamic Stochastic General Equilibrium (DSGE) model involves a systematic process (Figure 2). First, we derive the first-order necessary conditions (FONC), representing the equations governing agent behavior. These conditions represent the equations governing the behavior of economic agents (e.g., households, firms, government). The agents maximize their objectives (e.g., utility for households, profit for firms) subject to constraints (e.g., budget constraints, production functions). This process typically involves setting up a Lagrangian (Fernández-Villaverde et al., 2016) and taking partial derivatives with respect to each choice variable and the Lagrange multipliers. The resulting equations form a system of nonlinear equations that describe the optimal behavior of agents in the economy.

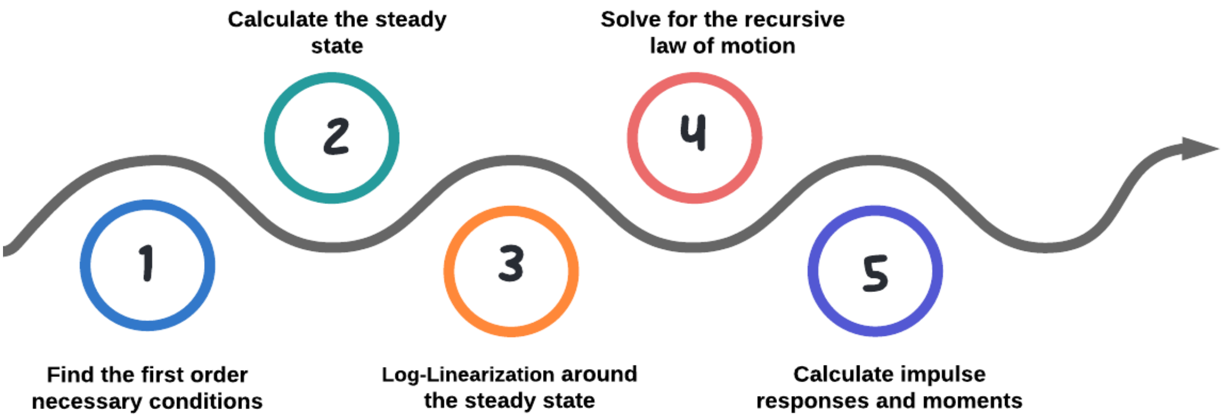
Then, we calculate the steady state, a reference point for analyzing dynamics. The steady state of the model is a situation where all variables grow at constant rates and the economy is in long-term equilibrium. To find the steady state, we set the time derivatives (or differences in the case of discrete time) of the variables to zero and solve the resulting system of equations. The steady state serves as a reference point for analyzing the dynamics of the model. It provides a baseline from which deviations can be studied when the economy is subject to shocks.

Next, log-linearizing around the steady state approximates the nonlinear equations, simplifying the problem. This involves taking the natural logarithm of the variables and then linearizing the equations by taking first-order Taylor expansions around the steady, which we will explain in the next section. The resulting system of linear equations is much easier to solve and analyze than the original nonlinear system. Log-linearization transforms the model into a form where the variables represent percentage deviations from their steady-state values. (Valdivia, 2015)

After that, we solve for the recursive law of motion, which describes how the state variables evolve over time in response to shocks. This involves finding a solution to the system of linearized equations. Techniques such as the method of undetermined coefficients or the Blanchard-Kahn method are commonly used to solve this system. The solution typically takes the form of a state-space representation, where the current state of the economy depends on its past state and the realization of shocks.

Finally, we compute impulse responses and moments, analyzing the effects of shocks and providing statistical summaries. With the recursive law of motion in hand, we can compute impulse response functions (IRFs) and moments. IRFs show how the endogenous variables respond over time to a one-time shock to an exogenous variable. This helps in understanding the dynamic properties of the model and the propagation of shocks through the economy. Additionally, we compute moments such as means, variances, and autocorrelations of the variables to provide statistical summaries of the model's behavior.

Figure 2: Algorithm for solving DSGE Models



Source: the authors.

2- How Log-Linearization Works in DSGE Models:

DSGE models, widely used in macroeconomics, rely on log-linearization to simplify their analysis and enable tractable solutions. This method involves taking the logarithms of variables and then linearizing these logs around a steady-state path where all real variables grow at the same rate (King et al., 1988). The steady-state path is critical because the economy's stochastic fluctuations tend to center around these values, making the approximation valid (Cooley & Prescott, 1995)—log-linearization results in a set of linear equations representing the deviations of log variables from their steady-state values.

This approach offers several advantages. First, it significantly simplifies the model's equations, avoiding the need for extensive derivative calculations and making the analysis more manageable (DeJong & Dave, 2011). Second, it allows for using standard linear techniques to solve the model, such as the Kalman filter and the spectral decomposition method (Hamilton, 1994). Third, log-linearization has been shown to provide a reasonable approximation of the true solution, particularly in the neighborhood of the steady-state (Schmitt-Grohé & Uribe, 2004).

However, it is essential to note that log-linearization is an approximation technique, and its accuracy can be affected by the model's structure and the size of the shocks. Sometimes, the approximation may need to be sufficiently accurate, particularly in large shocks or highly nonlinear relationships between variables (Sims, 2002).

Despite its limitations, log-linearization remains a powerful tool for analyzing DSGE models and has been instrumental in advancing our understanding of macroeconomic dynamics.

Log-linearization in DSGE models is employed as follows. Let X_t be a strictly positive variable, X its steady state and $x_t = log(X_t) - log(X)$ the logarithmic deviation. First notice that, for x near zero, $log(1 + x) = x$

$$x_t = log(X_t) - log(X) = log\left(\frac{X_t}{X}\right) = log\left(\frac{X + X_t - X}{X}\right) = log(1 + \%\,change)$$
$$= \%\,change$$

The basic idea of the log-linearization approach lies in the fact that any variable can be expressed as:

$$X_t = X\,\frac{X_t}{X} = Xe^{x_t}$$

Note that:

$$e^{x_t} = e^{log\left(\frac{X_t}{X}\right)} = \frac{X_t}{X}$$

The key concept is that the first-order Taylor approximation of the exponential function e^x may be expressed as:

$$e^{x_t} \approx 1 + x_t$$

So, we can write variables as

$$X_t \approx X.(1 + x_t)$$

Log-linearization: useful trick 1 (Sun, 2009)

Log-linearizing the term $X_{1t}^{\alpha_1} X_{2t}^{\alpha_2} \cdots X_{nt}^{\alpha_n}$, we have:

$$X_{1t}^{\alpha_1} X_{2t}^{\alpha_2} \cdots X_{nt}^{\alpha_n} \approx X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}(1 + \alpha_1 x_{1t} + \alpha_2 x_{2t} + \cdots + \alpha_n x_{nt})$$

Example:

Production function:

$$AK_t^{\alpha} L_t^{\beta} \approx AK^{\alpha} L^{\beta}(1 + \alpha\,k_t + \beta\,l_t)$$

Real wage:

$$\frac{W_t}{P_t} = W_t P_t^{-1} \approx \frac{W}{P}(1 + w_t - p_t)$$

Log-linearization: useful trick 2: Product of two variables

The second trick is applicable when dealing with variables that are being multiplied together, as in the following example:

$$X_t Y_t \approx XY(1 + x_t)(1 + y_t) \approx XY(1 + x_t + y_t)$$

In other words, terms such as $x_t y_t = 0$ are set because we are examining little deviations from the steady-state condition, and when these small deviations are multiplied together, the resulting term is near to zero.

Log-linearization: useful trick 3: Adding two variables

$$X_t + Y_t \approx X(1 + x_t) + Y(1 + y_t)$$

Some examples:

In this section, we demonstrate how the log-linearization technique simplifies functions in DSGE modeling, specifically focusing on production and utility functions. We will dive into the practical applications of this technique with popular functions like CES, Cobb-Douglas, and LES, as well as the economic identity, revealing how they behave within the context of the DSGE model. Each example will demonstrate the process of log-linearization and its advantages for analysis and computation, particularly for non-linear economic relationships.

Log-Linearization Example 1: ***The national accounting identity of a closed economy without government***

Start with

$$\mathbf{Y_t = C_t + I_t}$$

Re-write it as

$$\mathbf{Y^* e^{y_t} = C^* e^{c_t} + I^* e^{i_t}}$$

Using the first-order approximation, this becomes

$$\mathbf{Y^*(1 + y_t) = C^*(1 + c_t) + I^*(1 + i_t)}$$

Note, though, that the steady-state terms must obey identities so

$$\mathbf{Y^* = C^* + I^*}$$

Canceling these terms on both sides, we get

$$\mathbf{Y^* y_t = C^* c_t + I^* i_t}$$

which we will write as

$$\mathbf{y_t = \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t}$$

Log-Linearization Example 2: ***The production function***

Now consider

$$\mathbf{Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}}$$

This can be re-written in terms of steady-states and log-deviations as

$$\boldsymbol{Y^* e^{\{y_t\}} = \left(A^* e^{\{a_t\}}\right)(K^*)^{\{\alpha\}} e^{\{\alpha k_{\{t-1\}}\}} (N^*)^{\{1-\alpha\}} e^{\{(1-\alpha) n_t\}}}$$

Again, use the fact the steady-state values obey identities so that

$$\mathbf{Y^* = A^*(K^*)^{\alpha}(N^*)^{1-\alpha}}$$

So canceling gives

$$\mathbf{e^{\{y_t\}} = e^{\{a_t\}} e^{\{\alpha\, k_{\{t-1\}}\}} e^{\{(1-\alpha) n_t\}}}$$

Using first-order Taylor approximations, this becomes

$$\mathbf{(1 + y_t) = (1 + a_t)\left(1 + \alpha k_{\{t-1\}}\right)(1 + (1 - \alpha) n_t)}$$

Ignoring cross-products of the log-deviations, this simplifies to

$$\mathbf{y_t = a_t + \alpha k_{\{t-1\}} + (1 - \alpha) n_t}$$

This is the log-linearized version of the production function. It expresses the percentage deviation of output ($\mathbf{Y_t}$) from its steady state ($\boldsymbol{Y^*}$) as a function of the percentage deviations of capital ($\mathbf{K_{t-1}^{\alpha}}$), technology ($\mathbf{A_t}$), and labor ($\boldsymbol{N_t}$) from their respective steady states. This linear approximation is much easier to handle in analytical or numerical analysis, especially within a DSGE model framework. (Madeira, 2013)

Log-Linearization Example 3: ***The production function (CES)***

Let's start with the CES (Constant Elasticity of Substitution) functional form

$$Y_t = A_t\left[\alpha K_t^{\rho}+(1-\alpha)N_t^{\rho}\right]^{\frac{1}{\rho}}, \quad \text{(IV. 3.1)}$$

where $\varepsilon_s = -1/(1+\rho)$, is the elasticity of substitution and $0 < \alpha \le 1$.

Subsequently, we use the fact that steady-state values obey identities, so that

$$Y^* = A^*[\alpha(K^*)^{\rho}+(1-\alpha)(N^*)^{\rho}]^{\frac{1}{\rho}}, \quad \text{(IV. 3.2)}$$

$$\begin{cases} y_t = \log Y_t - \log Y^* = \log \dfrac{Y_t}{Y^*} \\ a_t = \log A_t - \log A^* = \log \dfrac{A_t}{A^*} \\ k_t = \log K_t - \log K^* = \log \dfrac{K_t}{K^*} \\ n_t = \log N_t - \log N^* = \log \dfrac{N_t}{N^*} \end{cases} \quad \text{(IV. 3.3)}$$

The next step is to find y_t as a function of a_t, k_t and n_t.

We have

$$\log Y_t = \log\left[A_t\left[\alpha K_t^{\rho}+(1-\alpha)N_t^{\rho}\right]^{\frac{1}{\rho}}\right]$$
$$= \log A_t + \frac{1}{\rho}\log\left[\alpha K_t^{\rho}+(1-\alpha)N_t^{\rho}\right] \quad \text{(IV. 3.4)}$$

The first-order Taylor expansion of this function around the point $Y_t = Y^*$, is

$$\log Y_t \approx \log Y^* + \left.\frac{\partial \log Y_t}{\partial \log A_t}\right|_{A^*}(\log A_t - \log A^*) + \left.\frac{\partial \log Y_t}{\partial \log K_t}\right|_{K^*}(\log K_t - \log K^*) + \left.\frac{\partial \log Y_t}{\partial \log N_t}\right|_{A^*}(\log N_t - \log N^*), \quad \text{(IV. 3.5)}$$

By (IV. 3.3), we can re-written (IV. 3.5), as

$$y_t \approx a_t \frac{\partial \log Y_t}{\partial \log A_t}\bigg|_{(A^*,K^*,N^*)} + k_t \frac{\partial \log Y_t}{\partial \log K_t}\bigg|_{(A^*,K^*,N^*)} + n_t \frac{\partial \log Y_t}{\partial \log N_t}\bigg|_{(A^*,K^*,N^*)} \qquad (\text{IV}.3.6)$$

It's easy to find that

$$\frac{\partial \log Y_t}{\partial \log A_t}\bigg|_{(A^*,K^*,N^*)} = 1, \qquad (\text{IV}.3.7)$$

We then calculate the other two partial derivatives, $\frac{\partial \log Y_t}{\partial \log K_t}$ and $\frac{\partial \log Y_t}{\partial \log N_t}$

$$\frac{\partial \log Y_t}{\partial \log K_t}\bigg|_{(A^*,K^*,N^*)} = \frac{\partial Y_t}{\partial K_t} \cdot \frac{K_t}{Y_t}\bigg|_{(A^*,K^*,N^*)},$$

$$\frac{\partial Y_t}{\partial K_t} = A_t \frac{1}{\rho}\left[\alpha K_t^{\rho} + (1-\alpha) N_t^{\rho}\right]^{\frac{1}{\rho}-1} \cdot \alpha\rho\left(K_t^{\rho-1}\right),$$

$$\frac{\partial Y_t}{\partial K_t} \cdot \frac{K_t}{Y_t} = \left(A_t\left[\alpha K_t^{\rho} + (1-\alpha) N_t^{\rho}\right]^{\frac{1}{\rho}} \cdot \alpha\left(K_t^{\rho-1}\right)\right) \cdot \left(\frac{K_t}{A_t\left[\alpha K_t^{\rho} + (1-\alpha) N_t^{\rho}\right]^{\frac{1}{\rho}}}\right)$$

$$= \frac{\alpha K_t^{\rho}}{\alpha K_t^{\rho} + (1-\alpha) N_t^{\rho}}.$$

Hence,

$$\frac{\partial \log Y_t}{\partial \log K_t}\bigg|_{(A^*,K^*,N^*)} = \frac{\partial Y_t}{\partial K_t} \cdot \frac{K_t}{Y_t}\bigg|_{(A^*,K^*,N^*)} = \frac{\alpha (K^*)^{\rho}}{\alpha (K^*)^{\rho} + (1-\alpha)(N^*)^{\rho}} \qquad (\text{IV}.3.8)$$

Similarly, we find that

$$\frac{\partial \log Y_t}{\partial \log N_t}\bigg|_{(A^*,K^*,N^*)} = \frac{\partial Y_t}{\partial N_t} \cdot \frac{N_t}{Y_t}\bigg|_{(A^*,K^*,N^*)} = \frac{(1-\alpha)(N^*)^{\rho}}{\alpha (K^*)^{\rho} + (1-\alpha)(N^*)^{\rho}} \qquad (\text{IV}.3.9)$$

Substituting $(\text{IV}.3.7)$, $(\text{IV}.3.8)$ and $(\text{IV}.3.9)$ in $(\text{IV}.3.6)$, we obtain

$$y_t \approx a_t + \frac{\alpha(K^*)^\rho}{\alpha(K^*)^\rho + (1-\alpha)(N^*)^\rho} k_t + \frac{(1-\alpha)(N^*)^\rho}{\alpha(K^*)^\rho + (1-\alpha)(N^*)^\rho} n_t \qquad \text{(IV. 3.10)}$$

Let's rewrite our formula as follows

$$y_t \approx a_t + \theta_K k_t + \theta_L n_t \qquad \text{(IV. 3.11)}$$

Where:

$$\theta_K = \frac{\alpha(K^*)^\rho}{\alpha(K^*)^\rho + (1-\alpha)(N^*)^\rho}, \quad \theta_L = \frac{(1-\alpha)(N^*)^\rho}{\alpha(K^*)^\rho + (1-\alpha)(N^*)^\rho}. \qquad \text{(IV. 3.12)}$$

The coefficients θ_K and θ_L represent the relative shares of capital and labor contributions (respectively) to production in the CES function.

We call them shares because $\theta_K + \theta_L = 1$, and the adjective relative is used because these parameters are the coefficients of the variables k_t and n_t, which represent the relative variations of the two factors.

Log-Linearization Example 4: *The New Classical model*

Let's take the first model described in the first section

$$\frac{1}{C_t} = \beta E_t\left\{\frac{1}{C_{t+1}}(1 + R_{t+1} - \delta)\right\} \qquad \text{(IV. 4.1)}$$

$$H_t^\eta = \frac{W_t}{C_t} \qquad \text{(IV. 4.2)}$$

$$Y_t = C_t + X_t + G_t \qquad \text{(IV. 4.3)}$$

$$Y_t = K_t^\alpha (Z_t H_t)^{1-\alpha} \qquad \text{(IV. 4.4)}$$

$$W_t = (1-\alpha)\frac{Y_t}{H_t} \qquad \text{(IV. 4.5)}$$

$$R_t = \alpha \frac{Y_t}{K_t} \qquad \text{(IV. 4.6)}$$

$$K_{t+1} = (1-\delta)K_t + X_t \qquad \text{(IV. 4.7)}$$

The log-linearization of this model, also described in the same section

$$c_t = E_t(c_{t+1}) - (1 - \beta + \beta\delta)\, E_t(r_{t+1}) \qquad (\text{IV}.4.1')$$

$$\eta h_t = w_t - c_t \qquad (\text{IV}.4.2')$$

$$\phi_1 x_t = y_t - \phi_2 c_t - \phi_3 g_t \qquad (\text{IV}.4.3')$$

$$y_t = (1 - \alpha)(z_t + h_t) + \alpha k_t \qquad (\text{IV}.4.4')$$

$$w_t = y_t - h_t \qquad (\text{IV}.4.5')$$

$$r_t = y_t - k_t \qquad (\text{IV}.4.6')$$

$$k_{t+1} = \delta x_t + (1 - \delta) k_t \qquad (\text{IV}.4.7')$$

$$z_{t+1} = \rho_z z_t + \xi_{t+1} \qquad (\text{IV}.4.8')$$

$$g_{t+1} = \rho_g g_t + \varepsilon_{t+1} \qquad (\text{IV}.4.9')$$

The aim is to demonstrate this log-linearization, i.e. the transition from the model $\big((\text{IV}.4.1) - (\text{IV}.4.7)\big)$ to its log-linearized version $\big((\text{IV}.4.1') - (\text{IV}.4.9')\big)$.

Let $(C^*, R^*, H^*, W^*, X^*, Y^*, G^*, K^*, Z^*)$ be the values of the variables in the stationary state (SS).

We start with the log-linearization of (1)

$$\frac{1}{C_t} = \beta\, E_t\left\{\frac{1}{C_{t+1}}(1 + R_{t+1} - \delta)\right\} \qquad (\text{IV}.4.1)$$

In the steady state, it is

$$\frac{1}{C^*} = \beta\left[\frac{1}{C^*}(1 + R^* - \delta)\right] \quad \text{that implies} \quad 1 = \beta[(1 + R^* - \delta)]$$

Then:

$$R^* = \frac{1}{\beta} - 1 + \delta$$
$$= \frac{1 - \beta + \beta\delta}{\beta} \qquad (\text{IV}.4.8)$$

We can rewrite $(\text{IV}.4.1)$ as

$$(C_t)^{-1} = \beta\, E_t\{(C_{t+1})^{-1}(1 + R_{t+1} - \delta)\} = \beta E_t\{(C_{t+1})^{-1}R_{t+1} + (C_{t+1})^{-1}(1 - \delta)\} \quad (\text{IV}.4.9)$$

The Taylor development of the three variables (C_t, C_{t+1}, R_{t+1}), therefore, allows us to find their values as a function of their logarithmic deviations (c_t, c_{t+1}, r_{t+1}),

$$\begin{cases} (C_t)^{-1} = \dfrac{1 - c_t}{C^*} \\ (C_{t+1})^{-1} = \dfrac{1 - c_{t+1}}{C^*} \\ (C_{t+1})^{-1}R_{t+1} = \dfrac{R^*}{C^*}(1 - c_{t+1} + r_{t+1}) \end{cases} \quad (\text{IV}.4.10)$$

Thus, $(\text{IV}.4.1)$ becames

$$\frac{1 - c_t}{C^*} = \beta E_t \left\{ \frac{R^*}{C^*}(1 - c_{t+1} + r_{t+1}) + \frac{(1 - c_{t+1})}{C^*}(1 - \delta) \right\}$$

Which implies

$$1 - c_t = \beta E_t\{R^*(1 - c_{t+1} + r_{t+1}) + (1 - c_{t+1})(1 - \delta)\} = \beta E_t\{-c_{t+1}(1 + R^* - \delta) + R^*(1 + r_{t+1}) + (1 - \delta)\}$$

$$-c_t = -\beta(1 + R^* - \delta)E_t(c_{t+1}) + R^*\beta E_t(r_{t+1}) + \beta(1 + R^* - \delta) - 1$$

From $(\text{IV}.4.8)$, we have $1 = \beta(1 + R^* - \delta)$ and $R^*\beta = 1 - \beta + \beta\delta$, hence

$$c_t = E_t(c_{t+1}) - (1 - \beta + \beta\delta)E_t(r_{t+1}) \quad Q.E.D$$

For equation $(\text{IV}.4.2)$

$$H_t^{\eta} = \frac{W_t}{C_t} \quad (\text{IV}.4.2)$$

In the steady state

$$(H^*)^\eta = \frac{W^*}{C^*} \quad \text{(IV. 4.11)}$$

Hence

$$\left(\frac{H_t}{H^*}\right)^\eta = \frac{\left(\frac{W_t}{C_t}\right)}{\left(\frac{W^*}{C^*}\right)}$$

$$\log\left(\frac{H_t}{H^*}\right)^\eta = \log\left(\frac{\left(\frac{W_t}{C_t}\right)}{\left(\frac{W^*}{C^*}\right)}\right)$$

which implies

$$\begin{aligned}\eta \log\left(\frac{H_t}{H^*}\right) &= \log(W_t) - \log(C_t) - (\log(W^*) - \log(C^*)) \\ &= \log(W_t) - \log(C_t) - \log(W^*) + \log(C^*) \\ &= [\log(W_t) - \log(W^*)] \\ &- [\log(C_t) - \log(C^*)]. \quad \text{(IV. 4.12)}\end{aligned}$$

Let

$$\begin{cases} h_t = \log\left(\frac{H_t}{H^*}\right) = \log(H_t) - \log(H^*) \\ w_t = \log\left(\frac{W_t}{W^*}\right) = \log(W_t) - \log(W^*) \\ c_t = \log\left(\frac{C_t}{C^*}\right) = \log(C_t) - \log(C^*) \end{cases} \quad \text{(IV. 4.13)}$$

Substituting $(\text{IV. }4.13)$ in $(\text{IV. }4.12)$ gives $(\text{IV. }4.2')$.

We'll now log-linearize equation $(\text{IV. }4.3)$.

$$Y_t = C_t + X_t + G_t \quad \text{(IV. 4.3)}$$

In the steady state

$$Y^* = C^* + X^* + G^* \quad \text{(IV. 4.14)}$$

We take the logarithmic deviations of these variables

$$\begin{cases} y_t = \log\left(\frac{Y_t}{Y^*}\right) \\ c_t = \log\left(\frac{C_t}{C^*}\right) \\ x_t = \log\left(\frac{X_t}{X^*}\right) \\ g_t = \log\left(\frac{G_t}{G^*}\right) \end{cases} \quad \text{which implies} \quad \begin{cases} Y_t = Y^* e^{y_t} \\ C_t = C^* e^{c_t} \\ X_t = X^* e^{x_t} \\ G_t = G^* e^{g_t} \end{cases} \qquad (\mathrm{IV}.4.15)$$

Using the first-order Taylor expansion of the exponential function $(e^x \approx 1 + x)$, we obtain

$$\begin{cases} Y_t \approx Y^*(1 + y_t) \\ C_t \approx C^*(1 + c_t) \\ X_t \approx X^*(1 + x_t) \\ G_t \approx G^*(1 + g_t) \end{cases} \qquad (\mathrm{IV}.4.16)$$

We have

$$\begin{aligned} y_t = \log\left(\frac{Y_t}{Y^*}\right) &= \log\left(\frac{C_t}{Y^*} + \frac{X_t}{Y^*} + \frac{G_t}{Y^*}\right) \\ &\approx \log\left(\frac{C^*(1 + c_t)}{Y^*} + \frac{X^*(1 + x_t)}{Y^*} + \frac{G^*(1 + g_t)}{Y^*}\right) \\ &\approx \log\left[\left(\frac{C^*}{Y^*} + \frac{X^*}{Y^*} + \frac{G^*}{Y^*}\right) \right. \\ &\quad \left. + \left(\frac{C^*}{Y^*} c_t + \frac{X^*}{Y^*} x_t + \frac{G^*}{Y^*} g_t\right)\right] \end{aligned}$$

From $(\mathrm{IV}.4.14)$, we have

$$\frac{C^*}{Y^*} + \frac{X^*}{Y^*} + \frac{G^*}{Y^*} = 1$$

Thus

$$y_t \approx \log\left[1 + \left(\frac{C^*}{Y^*} c_t + \frac{X^*}{Y^*} x_t + \frac{G^*}{Y^*} g_t\right)\right]$$

Introducing the linear approximation of the function $\log(1 + x) \approx x$, we obtain

$$y_t \approx \frac{X^*}{Y^*}x_t + \frac{C^*}{Y^*}c_t + \frac{G^*}{Y^*}g_t$$

It can also be written as

$$y_t = \phi_1 x_t + \phi_2 c_t + \phi_3 g_t \qquad (\text{IV}.4.16)$$

With $\phi_1 = X^*/Y^*$, $\phi_2 = C^*/Y^*$, $\phi_3 = G^*/Y^*$, and therefore $\phi_1 + \phi_2 + \phi_3 = 1$

a simple rearrangement of $(\text{IV}.4.16)$ yields $(\text{IV}.4.3')$.

The coefficients (ϕ_1, ϕ_2, ϕ_3) represent the respective shares of investment, consumption and government spending in aggregate output in the steady state.

We'll move on to the log-linearization of the following model equation

$$W_t = (1-\alpha)\frac{Y_t}{H_t} \qquad (\text{IV}.4.5)$$

In the steady state

$$W^* = (1-\alpha)\frac{Y^*}{H^*} \qquad (\text{IV}.4.17)$$

let's go back to the formulas for logarithmic deviations already described above

$$\begin{cases} y_t = \log\left(\frac{Y_t}{Y^*}\right) \\ w_t = \log\left(\frac{W_t}{W^*}\right) \\ h_t = \log\left(\frac{H_t}{H^*}\right) \end{cases}$$

we've also

$$w_t = \log\left(\frac{W_t}{W^*}\right) = \log\left(\frac{(1-\alpha)\frac{Y_t}{H_t}}{(1-\alpha)\frac{Y^*}{H^*}}\right) = \log\left(\frac{\frac{Y_t}{H_t}}{\frac{Y^*}{H^*}}\right).$$

We use the same trick as in $(\text{IV}.4.12)$ to obtain $(\text{IV}.4.5')$.

For $(\text{IV}.4.6)$ use the same procedure as for $(\text{IV}.4.5)$ to obtain $(\text{IV}.4.6')$ as a log-linearization of $(\text{IV}.4.6)$.

We now turn to equation $(\text{IV}.4.7)$, we have

$$K_{t+1} = (1-\delta)K_t + X_t \qquad (\text{IV}.4.7)$$

In the steady state, we have

$$K^* = (1-\delta)K^* + X^* \quad \text{which implies} \quad X^* = \delta K^* \qquad (\text{IV}.4.18)$$

The logarithmic deviations of the variables are

$$\begin{cases} k_{t+1} = \log(K_{t+1}) - \log(K^*) \\ k_t = \log(K_t) - \log(K^*) \\ x_t = \log(X_t) - \log(X^*) \end{cases} \quad \text{which implies} \quad \begin{cases} K_{t+1} = K^* e^{k_{t+1}} \approx K^*(1+k_{t+1}) \\ K_t = K^* e^{k_t} \approx K^*(1+k_t) \\ X_t = X^* e^{x_t} \approx X^*(1+x_t) \end{cases}$$

Hence

$$K^*(1+k_{t+1}) = (1-\delta)K^*(1+k_t) + X^*(1+x_t)$$

$$= (1-\delta)K^*(1+k_t) + \delta K^*(1+x_t)$$

The elimination of K^*, and a simple rearrangement of this last equation leads us to $(\text{IV}.4.7')$.

For the last two equations of our log-linearized model, they are derived from

$$\begin{cases} \log Z_{t+1} = (1-\rho_z)\log Z^* + \rho_z \log Z_t + \xi_{t+1} \\ \log G_{t+1} = \left(1-\rho_g\right)\log G^* + \rho_g \log G_t + \varepsilon_{t+1} \end{cases} \quad \text{where} \quad \begin{cases} \xi_{t+1} \sim i.i.d.\mathcal{N}(0,\sigma_z^2) \\ \varepsilon_{t+1} \sim i.i.d.\mathcal{N}\left(0,\sigma_g^2\right) \end{cases}$$

Log-Linearization Example 5: *New Keynesian model*

We now detail the log-linearization of the second model described in section 1.

$$\frac{1}{Y_t} = \beta\, E_t \left\{ \frac{1}{Y_{t+1}} \frac{R_t}{\Pi_{t+1}} \right\} \quad (\text{IV}.5.1)$$

$$(\pi_t - \pi) + \frac{1}{\phi} = \phi \left(\frac{Y_t}{Z_t} \right) + \beta E_t (\Pi_{t+1} - \Pi) \quad (\text{IV}.5.2)$$

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi} \right)^{\frac{1}{\beta}} U_t \quad (\text{IV}.5.3)$$

Based on these two formulas $X_t = (Y_t / Z_t)$, $G_t = (Z_{t+1} / Z_t)$, we can reproduce our model in the following form

$$1 = \beta E_t \left(\left(\frac{X_t}{X_{t+1}} \right) \left(\frac{1}{G_t} \right) \left(\frac{R_t}{\Pi_{t+1}} \right) \right) \quad (\text{IV}.5.1')$$

$$(\Pi_t - \Pi) + \frac{1}{\phi} = \phi X_t + \beta E_t (\Pi_{t+1} - \Pi) \quad (\text{IV}.5.2')$$

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi} \right)^{\frac{1}{\beta}} U_t \quad (\text{IV}.5.3')$$

In the steady state, the variables $\left((X_t, X_{t+1}), (G_t, G_{t+1}), R_t, (\Pi_t, \Pi_{t+1}), (U_t, U_{t+1})\right)$ represent (X, G, R, Π, U)

The model is completed by adding equations describing the evolution of the state variables G_t and U_t.

$$\log(G_{t+1}) = (1 - \rho_g) \log(G) + \rho_g \log(G_t) + \varepsilon_{t+1} \quad (\text{IV}.5.4')$$

$$\log(U_{t+1}) = (1 - \rho_u) \log(U) + \rho_u \log(U_t) + \xi_{t+1} \quad (\text{IV}.5.5')$$

The log-linearized model is given by the following series of equations

$$x_t = E_t(x_{t+1}) - (r_t - E_t \pi_{t+1} - g_t) \quad (\text{IV}.5.1'')$$

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa x_t \quad (\text{IV}.5.2'')$$

$$r_t = \frac{1}{\beta}\pi_t + \log U + u_t \qquad (\text{IV}.5.3'')$$

$$g_{t+1} = \rho_g g_t + \varepsilon_{t+1} \qquad (\text{IV}.5.4'')$$

$$u_{t+1} = \rho_u u_t + \xi_{t+1} \qquad (\text{IV}.5.5'')$$

Our work then consists of showing that the log-linearization of equations $\big((\text{IV}.5.1') - (\text{IV}.5.5')\big)$ are $\big((\text{IV}.5.1'') - (\text{IV}.5.5'')\big)$. We start with the first equation

$$1 = \beta E_t\left(\left(\frac{X_t}{X_{t+1}}\right)\left(\frac{1}{G_t}\right)\left(\frac{R_t}{\Pi_{t+1}}\right)\right). \qquad (\text{IV}.5.1')$$

In the steady state, we have

$$1 = \beta\left(\frac{X}{X}\right)\left(\frac{1}{G}\right)\left(\frac{R}{\Pi}\right) \quad \text{which implies} \quad \frac{1}{\beta} = \left(\frac{R}{G\Pi}\right).$$

On the other hand, $(\text{IV}.5.1')$ can be written as

$$1 = \beta E_t\left[\left(\frac{X}{X}\right)\left(\frac{1}{G}\right)\left(\frac{R}{\Pi}\right)(1 + x_t - x_{t+1} - g_t + r_t - \pi_{t+1})\right]$$

hence

$$1 = \beta\left(\frac{R}{G\Pi}\right)E_t(1 + x_t - x_{t+1} - g_t + r_t - \pi_{t+1})$$

And by the fact $1/\beta = (R/G\Pi)$, we obtain

$$1 = 1 + x_t + E_t(x_{t+1}) - g_t + r_t - E_t(\pi_{t+1})$$

this latter equation implies $(\text{IV}.5.1'')$.

Let's turn to the second equation

$$(\Pi_t - \Pi) + \frac{1}{\phi} = \phi X_t + \beta E_t(\Pi_{t+1} - \Pi) \qquad (\text{IV}.5.2')$$

In the steady state

$$(\Pi - \Pi) + \frac{1}{\phi} = \phi X + (\Pi - \Pi) \quad \Rightarrow \quad \phi X - \frac{1}{\phi} = 0$$

The variables according to their logarithmic deviations are given by:

$$\begin{cases} X_t = Xe^{x_t} \approx X(1 + x_t) \\ \Pi_t = \Pi e^{\pi_t} \approx \Pi(1 + \pi_t) \\ \Pi_{t+1} = \Pi e^{\pi_{t+1}} \approx \Pi(1 + \pi_{t+1}) \end{cases}$$

Equation $(\mathrm{IV}.5.2')$ becames

$$(\Pi(1 + \pi_t) - \Pi) + \frac{1}{\phi} = \phi X(1 + x_t) + \beta E_t(\Pi(1 + \pi_{t+1}) - \Pi)$$

That implies

$$\Pi\pi_t = \phi X + \phi X x_t + \beta \Pi E_t(\pi_{t+1}) - \frac{1}{\phi} \quad \Longrightarrow \quad \pi_t = \beta E_t(\pi_{t+1}) + \frac{\phi X}{\Pi} x_t$$

This equation is the same as $(\mathrm{IV}.5.2'')$, whose $\kappa = \phi X / \Pi$.

Let's take equation $(\mathrm{IV}.5.3')$

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi}\right)^{\frac{1}{\beta}} U_t \qquad (\mathrm{IV}.5.3')$$

The logarithmic deviations of the variables are

$$\begin{cases} \log \frac{R_t}{R} = r_t \\ \log \frac{\Pi_t}{\Pi} = \pi_t \\ \log U_t = \log U - u_t \end{cases}$$

Thus, using $(\mathrm{IV}.5.3')$ and these logarithmic deviations, we obtain

$$r_t = \log \frac{R_t}{R} = \log \left(\frac{\Pi_t}{\Pi}\right)^{\frac{1}{\beta}} U_t = \frac{1}{\beta}\pi_t + \log U + u_t \qquad Q.E.D.$$

For these two equations

$$\log(G_{t+1}) = \left(1 - \rho_g\right) \log(G) + \rho_g \log(G_t) + \varepsilon_{t+1} \qquad (\mathrm{IV}.5.4')$$

$$\log(U_{t+1}) = (1 - \rho_u) \log(U) + \rho_u \log(U_t) + \xi_{t+1} \qquad (\mathrm{IV}.5.5')$$

We can obtain $(\mathrm{IV}.5.4'')$ and $(\mathrm{IV}.5.5'')$, by using the logarithmic deviations of $Z_t, Z_{t+1}, G_t, G_{t+1}$.

$$\begin{cases} g_t = \log(G_t) - \log(G) \\ g_{t+1} = \log(G_{t+1}) - \log(G) \end{cases} \qquad \begin{cases} u_t = \log(U_t) - \log(U) \\ u_{t+1} = \log(U_{t+1}) - \log(U) \end{cases}$$

$(\mathrm{IV}.5.4')$ and $(\mathrm{IV}.5.5')$can be rewritten as

$$\begin{cases} \log(G_{t+1}) = \log(G) - \rho_g \log(G) + \rho_g \log(G_t) + \varepsilon_{t+1} \\ \log(U_{t+1}) = \log(U) - \rho_u \log(U) + \rho_u \log(U_t) + \xi_{t+1} \end{cases}$$

Which implies

$$\begin{cases} \underbrace{\log(G_{t+1}) - \log(G)}_{g_{t+1}} = \rho_g \underbrace{(\log(G_t) - \log(G))}_{g_t} + \varepsilon_{t+1} \\ \underbrace{\log(U_{t+1}) - \log(U)}_{u_{t+1}} = \rho_u \underbrace{(\log(U_t) - \log(U))}_{u_t} + \xi_{t+1} \end{cases} \text{where} \begin{cases} \varepsilon_{t+1} \sim i.i.d.\mathcal{N}(0, \sigma_g^2) \\ \xi_{t+1} \sim i.i.d.\mathcal{N}(0, \sigma_u^2) \end{cases}$$

So, these last equations are $(\mathrm{IV}.5.4'')$ and $(\mathrm{IV}.5.5'')$

Log-Linearization Example 6: *Marginal cost function*

In this example, we'll study the log-linearization of the marginal cost function, which takes the following form:

$$MC_t = \frac{W_t}{Z_t H_t} \qquad (\mathrm{IV}.6.1)$$

Marginal cost MC_t is the cost of producing an additional unit, a function of wages W_t and productivity $Z_t H_t$.

First, we define the logarithmic deviations of the variables,

$$\begin{cases} mc_t = \log(MC_t) - \log(MC^*) \\ w_t = \log(W_t) - \log(W^*) \\ z_t = \log(Z_t) - \log(Z^*) \\ h_t = \log(H_t) - \log(H^*) \end{cases} \text{therefore} \begin{cases} MC_t \approx MC^*(1 + mc_t) \\ \dfrac{W_t}{Z_t H_t} \approx \dfrac{W^*}{Z^* H^*}(1 + w_t - z_t - h_t) \end{cases} \qquad (\mathrm{IV}.6.2)$$

In steady state, we have

$$MC^* = \frac{W^*}{Z^* H^*} \qquad (\mathrm{IV}.6.3)$$

A substitution of $(\mathrm{IV}.6.2)$ into $(\mathrm{IV}.6.1)$ gives us

$$MC^*(1 + mc_t) = \frac{W^*}{Z^* H^*}(1 + w_t - z_t - h_t) \qquad (\mathrm{IV}.6.4)$$

and by $(\mathrm{IV}.6.3)$, we rewrite $(\mathrm{IV}.6.4)$ to obtain the log-linearized form of $(\mathrm{IV}.6.1)$

$$mc_t = w_t - z_t - h_t \quad \text{(IV. 6.5)}$$

Log-linearization is useful for understanding price and wage dynamics.

Log-Linearization Example 7: ***Monetary demand function***

Consider the following monetary demand function:

$$M_t = P_t C_t \left(\frac{1}{IR_t}\right)^{\epsilon} \quad \text{(IV. 7.1)}$$

This function describes the demand for money M_t as a function of the price level P_t, consumption C_t, and the nominal interest rate IR_t.

Given that, in the steady state, we have

$$M^* = P^* C^* \left(\frac{1}{IR^*}\right)^{\epsilon}. \quad \text{(IV. 7.2)}$$

The logarithmic deviations of the variables are given by,

$$\begin{cases} M_t = M^* e^{m_t} \\ P_t = P^* e^{p_t} \\ C_t = C^* e^{c_t} \\ IR_t = IR^* e^{ir_t} \end{cases} \quad \text{(IV. 7.3)}$$

which implies

$$\begin{cases} M_t \approx M^*(1 + m_t) \\ P_t C_t \left(\frac{1}{IR_t}\right)^{\epsilon} = P^* C^* \left(\frac{1}{IR^*}\right)^{\epsilon} (1 + p_t + c_t - \epsilon ir_t) \end{cases} \quad \text{(IV. 7.4)}$$

then, (IV. 7.1) becames,

$$M^*(1 + m_t) = P^* C^* \left(\frac{1}{IR^*}\right)^{\epsilon} (1 + p_t + c_t - \epsilon ir_t) \quad \text{(IV. 7.5)}$$

and from (IV. 7.2), we have

$$m_t = p_t + c_t - \epsilon i r_t. \quad (\text{IV}.7.6)$$

The interest of log-linearization is often used on this function, in order to study the relationship between monetary policy and the real economy.

Log-Linearization Example 8: The investment demand function

We assume that investment is given by the following function

$$I_t = \left(\frac{K_{t+1} - (1-\delta)K_t}{\eta}\right) \quad (\text{IV}.8.1)$$

This function describes the link between investment I_t, capital stock in time t is K_t,and capital stock in time $t+1$ is K_{t+1}, and capital depreciation δ, and parameter η represents the investment cost adjustment.

We define the steady state by,

$$I^* = \left(\frac{K^* - (1-\delta)K^*}{\eta}\right) \text{ that implies } I^* = \frac{\delta K^*}{\eta} \quad (\text{IV}.8.2)$$

The logarithmic deviations are given by

$$\begin{cases} I_t = I^* e^{i_t} \approx I^*(1 + i_t) \\ K_t = K^* e^{k_t} \approx K^*(1 + k_t) \\ K_{t+1} = K^* e^{k_{t+1}} \approx K^*(1 + k_{t+1}) \end{cases} \quad (\text{IV}.8.3)$$

Using $(\text{IV}.8.3)$, we can rewrite $(\text{IV}.8.1)$ like

$$I^*(1 + i_t) = \frac{K^*}{\eta}[(1 + k_{t+1}) - (1-\delta)(1 + k_t)] \quad (\text{IV}.8.4)$$

According to $(\text{IV}.8.2)$ we know that $I^*/\delta = K^*/\eta$, therefore

$$(1 + i_t) = \frac{1}{\delta}[1 + k_{t+1} - (1 - \delta)k_t - (1 - \delta)]. \quad (\text{IV}.8.5)$$

Then

$$i_t = \frac{1}{\delta}[k_{t+1} - (1 - \delta)k_t]. \quad (\text{IV}.8.6)$$

Log-linearization allows us to explore the dynamics of investment in the face of economic shocks.

Conclusion

This paper has demonstrated the power and practicality of log-linearization as a technique for simplifying complex nonlinear equations within DSGE modeling. By leveraging Taylor approximation centered around the steady state, log-linearization offers a powerful tool for transforming nonlinear functions into more manageable, log-linear representations. This transformation allows for easier analysis, computation, and solution finding within the context of dynamic macroeconomic models.

While log-linearization provides a significant advantage in simplifying DSGE models, it's important to acknowledge its limitations. The accuracy of the log-linear approximation is dependent on the proximity of the model's variables to the steady state. Furthermore, the technique relies on the assumption of a stable steady state, which might not always be present in real-world scenarios.

Future research could explore the application of log-linearization in more complex DSGE models, including those with heterogeneous agents or richer dynamics. Investigating the potential impact of non-linearity on model outcomes, particularly when considering larger deviations from the steady state, would also be a valuable area for further exploration.

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