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On the Efficiency of a One-Fifth Step Hybrid Block Method for the Numerical Solution of Second-Order Ordinary Differential Equations

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Abstract

In this paper, hybrid methods for solving second-order ordinary differential equations with a one-fourth step length were developed. This was accomplished using interpolation and collocation techniques. The methods were evaluated based on the properties of linear multistep methods and were found to be zero-stable, consistent, and convergent, exhibiting a favorable region of absolute stability. The proposed methods were implemented for second-order ordinary differential initial value problems. The performance of the new methods demonstrated superiority over previously developed methods in the literature, as evidenced by the resolution of five numerical examples. The results were presented in tabular form.

Keywords: Block Method, Hybrid Method, Initial Value Problems, One-Fourth Step, Order Four, Second Order

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Introduction

Mathematical modeling has long played a pivotal role across diverse disciplines such as science, engineering, economics, medicine, psychology, and operations research. These models often serve as tools for understanding complex physical phenomena by expressing relationships through equations involving derivatives of unknown functions with respect to one or more variables—commonly known as Differential Equations (DEs). Differential Equations can be broadly classified into Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs), depending on whether they involve derivatives with respect to a single variable or multiple variables. However, due to the analytical intractability of many DEs, especially nonlinear and higher-order systems, there arises a critical need for robust numerical methods to derive approximate solutions.

Over the decades, several numerical methods have been proposed to address this challenge. Among early developments, Rosser (1967) expanded on Milne's initial proposals to create algorithms suitable for general computation. Building on Rosser's foundation, Lambert (1973) developed a two-step, fourth-order explicit block method. Earlier, Clippinger and Dimsdale's work, as cited in Grabbe (1958), was revisited by Shampine and Watts (1969), who formulated an implicit one-step block method. These foundational contributions catalyzed a growing body of research into more sophisticated block methods.

In line with these efforts, this paper introduces a novel hybrid block method, characterized by a quarter-step length and the inclusion of off-step points. This methodological framework extends the frontiers of existing block methods. Recent advancements include Ibijola (2011), who developed hybrid block methods for ODEs using continuous multi-step collocation techniques. Adeyefa (2013) further refined this approach by employing Chebyshev polynomials as basis functions for second-order ODEs. Fotta et al. (2015) proposed a one-step hybrid block method that utilizes power series and exponential functions for solving first-order initial value problems (IVPs), while Sahi et al. (2016) focused on a continuous fourth-derivative method tailored for third-order boundary value problems (BVPs).

In a similar vein, Adeyeye and Omar (2016) introduced maximal-order block methods for second-order ODEs, emphasizing stability and accuracy. Areo and Rusai (2016) proposed a fourth-order method using a specific step-size parameter ($k = 1/3$) designed specifically for IVPs. Despite these significant advancements, existing methods still exhibit limitations

related to step-size adaptability, error propagation, and solution stability in stiff or multi-dimensional systems.

Hence, this study addresses these gaps by presenting a more flexible and efficient hybrid block approach. Our method aims to improve computational accuracy while reducing the number of function evaluations required per step. Recent developments in the field, such as those by Musa and Omar (2020), who emphasized adaptive collocation techniques for stiff systems, and Chinyoka and Shateyi (2022), who applied hybrid block methods to nonlinear PDEs in thermal systems, underscore the growing relevance and applicability of such techniques in contemporary research. By integrating insights from both classical and modern approaches, the proposed method contributes a timely and innovative solution to the evolving landscape of numerical methods for differential equations.

Methodology

Specification of the Methods

The Derivation of One-fifth step method

In this section, we use Chebyshev polynomial as an approximate solution to be of the form:

$$y(x) = \sum_{i=0}^{p+q-1} a_i T_i(x) \tag{1}$$

where p represents the number of collocation points and q is the number of interpolating points.

The second derivative of equation (11) is gives

$$y''(x) = \sum_{i=0}^{p+q-1} a_i T''_j(x) = f(x, y, y') \tag{2}$$

Equations (1) and (2)are interpolated and collocated at the points $x_{n+p}P = 0, \frac{2}{12}$ and

$q = 0, \frac{1}{12}, \frac{2}{12}, \frac{1}{4}$ to get a system of equations of the form :

$$AX = B \tag{3}$$

where

$$X = [a_0, a_1, a_2, a_3, a_4, a_5]^T, Y = [y_n, y_{n+\frac{1}{12}}, f_n, f_{n+\frac{1}{12}}, f_{n+\frac{2}{12}}, f_{n+\frac{1}{4}}]^T$$

and

$$A = \begin{bmatrix} 0 & 1 & -x_n & x_n^2 & -x_n^3 & x_n^4 - x_n^5 \\ 1 & -\frac{2}{3}x_{n+\frac{2}{12}} & -\frac{1}{9}x^2_{n+\frac{2}{12}} & \frac{22}{27}x^3_{n+\frac{2}{12}} & -\frac{79}{81}x^4_{n+\frac{2}{12}} & \frac{118}{243}x^5_{n+\frac{2}{12}} \\ 0 & 0 & 16 & -96x_n & 320x_n^2 & -800x_n^2 \\ 0 & 0 & 16 & -80x_{n+\frac{1}{12}} & \frac{608}{3}x^2_{n+\frac{1}{12}} & \frac{-9200}{27}x^3_{n+\frac{2}{12}} \\ 0 & 0 & 16 & -64x_{n+\frac{2}{12}} & \frac{320}{3}x^2_{n+\frac{2}{12}} & -\frac{1600}{27}x^3_{n+\frac{2}{12}} \\ 0 & 0 & 16 & -48x_{n+\frac{3}{12}} & 32x^2_{n+\frac{3}{12}} & 80x^3_{n+\frac{3}{12}} \end{bmatrix} \tag{4}$$

By simplifying some notations in equation (4) and solving for a'_j $j = 0,1,2,3,4$ and 5 and substituting the value of a'_js into equation (1) gives a continuous implicit multistep method of the form:

$$y(x) = \alpha_0 y_n + \alpha_{\frac{2}{12}} y_{n+\frac{2}{12}} + h^2 \sum_{j=0}^{\frac{1}{4}} \beta_j f_{n+j} \quad (5)$$

where $j = 0, \frac{1}{12}, \frac{2}{12},$ and$\frac{3}{12}$, α_j, β_j represent continuous coefficients, $y_{n+j} = y(x_n + jh)$ represents numerical solution at point x_{n+j} and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.

Using the transformation $z = \frac{x - x_n}{h}$, the coefficients α_j and β_j are given as:

$$\left.\begin{aligned} \alpha_0(z) &= 1 - 6z \\ \alpha_{\frac{2}{12}}(z) &= 6z \\ \beta_0(z) &= -\frac{7}{270}z + \frac{1}{2}z^2 - \frac{11}{3}z^3 + 12z^4 - \frac{72}{5}z^5 \\ \beta_{\frac{1}{12}}(z) &= -\frac{11}{180}z + 6z^3 - 30z^4 + \frac{216}{5}z^5 \\ \beta_{\frac{2}{12}}(z) &= +\frac{1}{180}z - 3z^3 + 24z^4 - \frac{216}{5}z^5 \\ \beta_{\frac{3}{12}}(z) &= -\frac{1}{540}z + \frac{2}{3}z^3 - 6z^4 + \frac{72}{5}z^5 \end{aligned}\right\} \tag{6}$$

The first derivatives of the equations (6) give

$$\left.\begin{aligned}
\alpha'_0(z) &= -6 \\
\alpha'_{\frac{2}{12}}(z) &= 6 \\
\beta'_0(z) &= z - 11z^2 + 48z^3 - 72z^4 \\
\beta'_{\frac{1}{12}}(z) &= 18z^2 - 120z^3 + 216z^4 \\
\beta'_{\frac{2}{12}}(z) &= -9z^2 + 96z^3 - 216z^4 \\
\beta'_{\frac{3}{12}}(z) &= 2z^2 - 24z^3 + 72z^4
\end{aligned}\right\} \quad (7)$$

By evaluating the first derivative equation (6) together with (7) at points $p = 0, \frac{1}{12}, \frac{2}{12}$ and $\frac{3}{12}$, we obtain

$$\left.\begin{aligned}
y_{n+\frac{1}{12}} &= y_n + \frac{1}{12}hy'_n + h^2\left(\frac{97}{51840}f_n + \frac{19}{8640}f_{n+\frac{1}{12}} - \frac{13}{17280}f_{n+\frac{2}{12}} + \frac{1}{6480}f_{n+\frac{3}{12}}\right) \\
y_{n+\frac{1}{6}} &= y_n + \frac{1}{6}hy'_n + h^2\left(\frac{7}{1620}f_n + \frac{11}{1080}f_{n+\frac{1}{12}} - \frac{1}{1080}f_{n+\frac{2}{12}} + \frac{1}{3240}f_{n+\frac{3}{12}}\right) \\
y_{n+\frac{1}{4}} &= y_n + \frac{3}{12}hy'_n + h^2\left(\frac{13}{1920}f_n + \frac{3}{160}f_{n+\frac{1}{12}} + \frac{3}{640}f_{n+\frac{2}{12}} + \frac{1}{960}f_{n+\frac{3}{12}}\right) \\
y'_{n+\frac{1}{12}} &= y_n + h^2\left(\frac{1}{32}f_n + \frac{19}{288}f_{n+\frac{1}{12}} - \frac{5}{288}f_{n+\frac{2}{12}} + \frac{1}{288}f_{n+\frac{3}{12}}\right) \\
y'_{n+\frac{2}{12}} &= y_n + h^2\left(\frac{1}{36}f_n + \frac{1}{9}f_{n+\frac{1}{12}} - \frac{1}{36}f_{n+\frac{2}{12}}\right) \\
y'_{n+\frac{1}{4}} &= y_n + h^2\left(\frac{1}{32}f_n + \frac{3}{32}f_{n+\frac{1}{12}} + \frac{3}{32}f_{n+\frac{2}{12}} + \frac{1}{32}f_{n+\frac{3}{12}}\right)
\end{aligned}\right\} \quad (8)$$

Analysis of the Methods

The following are the basic principle which can be used to check the validity of the method. The basic principle: Order and Error constant,consistency and zero stability describe the nature of the convergence of the methods.

Definition 1. The linear multi-step method is said to be consistent if it is of oder $p \geq 1$.

Remark: The proposed methods $\frac{1}{4}$HBM are of order $p = 4$ which implies that they are consistent by defination (1)

Using Taylor series expansion on equations (8) we get the order of the new proposed block methods respectively as (4,4,4,4,4,4)(4,4,4,4,4,4) and (4,4,4,4,4,4) with

error constants as

$$\left(\frac{-1}{6144}, \frac{-1}{31104}, \frac{1}{248832}, \frac{-4}{27783}, \frac{-1}{4667544}, \frac{-1}{2058}\right)$$

Zero stability of One-fourth stepLength For Second Order Differential Equations

In order to test for zero stability of the block method (8), we consider the matrix difference equation of the form

$$p^0 = Y_{m+1} = p' y_m + h^2[Q^0 F_{m+1} + Q' F_m + hR'] \tag{9}$$

where in the case,

$$Y_{m+1} = [y_{n+\frac{1}{12}}, \dots y_{n+\frac{1}{4}}]^T, Y_m = [y_{n-\frac{1}{12}}, \dots y_n]^T F_{m+1} = [f_{n+\frac{1}{12}}, \dots f_{n+\frac{1}{4}}]^T, F_m = [f_{n-\frac{1}{12}}, \dots f_n]^T \tag{10}$$

The matrices p^0, p', Q^0, Q' and $R^0$are the coefficients of equation (3.22) which and are defined as follows

$$p^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$p' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Q^0 = \begin{bmatrix} \frac{114}{51840} & \frac{39}{51840} & \frac{8}{51840} \\ \frac{33}{3240} & -\frac{3}{3240} & \frac{1}{3240} \\ \frac{36}{1920} & \frac{9}{1920} & \frac{2}{1920} \end{bmatrix} \tag{12}$$

$$Q' = \begin{bmatrix} 0 & 0 & \frac{97}{51840} \\ 0 & 0 & \frac{14}{3240} \\ 0 & 0 & \frac{13}{1920} \end{bmatrix} R' = \begin{bmatrix} 0 & 0 & \frac{1}{12} \\ 0 & 0 & \frac{2}{12} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{13}$$

A block method is said to be zero stable if the moduli of roots of

$$|[\lambda p^0 - p^1]| = 0$$

are less than or equal to unity and those with unit modulus are simple.

$$|[p^0 - p']| = \left|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} \tag{14}$$

This implies that

Since root has modulus less than or equal to one and is simple the method is zero-stable.

Convergence

The convergence of the proposed method is considered in the light of the basic properties discussed in the fundamental theorem of Dahlquist (Henrichi, 1962) for linear multistep method.

Theorem

(Dahlquist) The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero-stable.

Region of Absolute Stability of the Block Method

Following Ibijola et al (2011), we formulate the stability matrix as follow:

$$M(z) = V + zB(M - zA)U \tag{15}$$

and the stability function

$$p(\eta, z) = det(\eta I - M(z)). \tag{16}$$

then, we represent the block method of the equation (8) in form of

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{97}{51840} & \frac{19}{8640} & -\frac{13}{17280} & \frac{1}{6480} \\ \frac{7}{1620} & \frac{11}{1080} & -\frac{1}{1080} & \frac{1}{3240} \\ \frac{13}{1920} & \frac{3}{160} & \frac{1}{960} & \frac{1}{1500} \end{bmatrix}, B = \begin{bmatrix} \frac{97}{51840} & \frac{19}{8640} & -\frac{13}{17280} & \frac{1}{6480} \\ \frac{7}{1620} & \frac{11}{1080} & -\frac{1}{1080} & \frac{1}{3240} \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & I \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & I \end{bmatrix}$$

$$f(y) = \begin{bmatrix} y_n \\ y_{n+\frac{1}{12}} \\ y_{n+\frac{2}{12}} \\ y_{n+\frac{3}{12}} \end{bmatrix}, Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{12}} \\ y_{n+\frac{3}{12}} \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{12}} y_{n+\frac{3}{12}} \end{bmatrix}, M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$$

Substituting the matrices A,B,V,U,M and I into equation (10) and (11) gives the stability polynomial of the one-fourth step method which are plotted below by the use of matlab

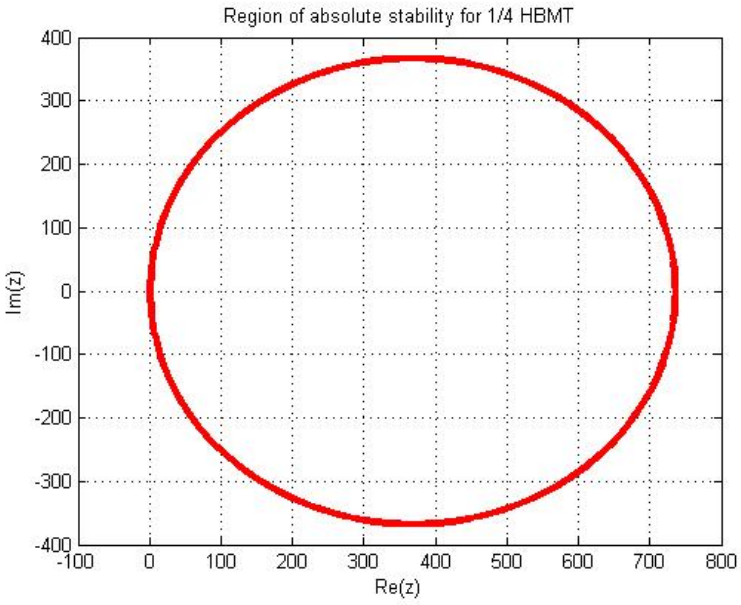


Figure 1: Region of absolute stability of $\frac{1}{4}$ HBM

Numerical Example

In this section, some numerical examples of second order ordinary differential equations are solved. The methods are implemented directly without using any starting value and with use of Matlab and Mapple. The table below shows some notations that were use to present the numerical and graphical results obtained for some test problems by application of the proposed schemes.

Problem 1

Consider a Linear non-homogeneous test problem also solved by Badmus (2014)

$$y'' = 3y' + 8e^{2x}, y(0) = 1, y'(0) = 1, h = 0.05$$

Exact solution:$y(x) = -4e^{2x} + 3e^{3x} + 2$

Problem 2

Consider a specially oscillatory test problem which was also solved by Areo (2016)

$y'' = -\lambda^2 y, y(0) = 1, y'(0) = 2, h = 0.01, \lambda = 2$

Exact solution: $y(x) = cos2x + sin2x$

Problem 3

Consider a singular non-honogeneous test problem also solved by Areo (2016)

$y'' = \frac{2y'}{x} + xe^x - y(1 + \frac{2}{x^2}), y(\frac{\pi}{2}) = 4 - \pi + \frac{1}{4}(e^{\frac{\pi}{2}})(\pi + 2)\ y'(\frac{\pi}{2}) = \frac{\pi}{2}(8 + e^{\frac{\pi}{2}}),$

$h = 0.003125$, Exact solution: $y(x) = 2xcosx + 4xsinx + \frac{1}{2}xe^x$

Problem 4

We consider a linear homogeneous test problem which was also solved by Ramos *et al.*,(2016)

$y'' = y', y(0) = 0, y'(0) = -1, h = 0.1$

Exact solution:$y(x) = 1 - e^x$

Problem 5

A non-linear problem We consider a non-linear non-homogeneous test problem solved by Ramos et al.,(2016)

$y'' = x(y')^2, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1,$

Exact solution: $y(x) = 1 + \frac{1}{2}in(\frac{2+x}{2-x})$

Tabular Presentations

Here we present exact solution, Computed result, errors results for $\frac{1}{4}HBM$ and from the existing method tabularly below

Table: 1 Results for Problem 1

x	Exact	Computed	Error in our method	Error in $Areo(2016)$
0.0100	1.01979867335991	1.019798673359	$5.9952E-14$	$8.4599E-14$
0.0200	1.03918944084761	1.039189440847	$1.5099E-13$	$3.4861E-13$
0.0300	1.05816454641465	1.058164546414	$0.35727E-12$	$7.8870E-13$
0.0400	1.07671640027179	1.076716400271	$0.59486E-12$	$1.4004E-12$

0.0500	1.09483758192485	1.094837581923	$0.92437E-12$	$2.1791E-12$
0.0600	1.11252084314279	1.112520843141	$0.13405E-11$	$3.1208E-12$
0.0700	1.12975911085687	1.129759110855	$0.17863E-11$	$4.2211E-12$
0.0800	1.14654548998987	1.146545489987	$5.4761E-12$	$0.23146E-11$
0.0900	1.16287326621395	1.162873266211	$0.29099E-11$	$6.8801E-12$
0.1000	1.17873590863630	1.178735908632	$0.35638E-11$	$8.4293E-12$

Table: 2 Results for Problem 2

x	Exact	Computed	Error in our method	Error in Areo(2016)
0.0050	1.0051385255	1.005138525	$4.7010E-011$	$1.2349E-09$
0.0100	1.0105582417	1.010558241	$2.05934E-10$	$2.6905E-09$	
0.0150	1.0162654439	1.016265443	$6.8407E-10$	$4.3738E-09$
0.0200	1.0222665428	1.022266541	$8.6773E-10$	$6.2921E-09$
0.0250	1.0285680671	1.028568065	$1.3806E-09$	$8.9697E-09$
0.0300	1.0351766649	1.035176662	$2.0160E-09$	$1.0863E-08$
0.0350	1.0420991060	1.042099103	$2.7352E-08$	$1.6463E-08$
0.0400	1.0493422840	1.049342280	$1.7003E-08$	$4.4563E-08$

Table: 3 Results for Problem 3

x	Exact	Computed	Error in our method	Error in Areo(2016)
0.2000000	-0.2214027581601	-0.2214027581601	$7.66053887E-14$	$5.441E-07$
0.3000000	-0.3498588075760	-0.3498588075759	$0.180411242E-13$	$9.114E-07$
0.4000	-0.4918246976412	-0.4918246976412	$0.345834472E-13$	$1.329E-06$
0.5000	-0.6487212707001	-0.6487212707000	$5.75095527E-13$	$1.447E-06$
0.6000	-0.8221188003905	-0.8221188003904	$8.51541060E-13$	$2.435E-06$
0.7000	-1.0137527074704	-1.0137527074703	$1.21236354E-12$	$3.153E-06$
0.8000	-1.2255409284924	-1.2255409284922	$1.66533454E-12$	$4.965E-06$
0.9000	-1.4596031111569	-1.4596031111567	$2.24487096E-12$	$4.948E-06$
1.0000	-1.7182818284590	-1.7182818284590	2.2555517096E -12	4.365E-06

Table: 5 Results for Problem 5

x	Exact	Computed	Error in our method	Error Ramos et al(2016)
0.1000000	1.0500417292	1.0500417292784	$0.111022302E-14$	$1.18393E-10$
0.2000000	1.1003353477	1.1003353477310	$0.35527E-14$	$2.3749E-10$
0.3000000	1.1511404359	1.1511404359364	$0.13766E-13$	$4.2485E-10$
0.4000	1.2027325540	1.2027325540540	$0.38635E-13$	$6.1628E-10$
0.5000	1.2554128118	1.2554128118829	$0.97033E-13$	$1.0233E-09$
0.6000	1.3095196042	1.3095196042028	$0.21160E-12$	$1.4483E-09$
0.7000	1.3654437542	1.3654437542709	$0.43298E-12$	$2.5449E-09$
0.8000	1.4236489301	1.4236489301927	$0.84021E-12$	$3.7221E-09$
0.9000	1.4847002785	1.4847002785924	$0.15718E-11$	$7.3287E-08$
1.0000	−1.7182818	28.407771366416	$0.15718E-11$	$7.3287E-08$

Discussion of Results

In this study, we present the development of a one-fourth step hybrid block method tailored for the direct solution of second-order initial value problems (IVPs) without the necessity of transforming them into systems of first-order differential equations. The proposed approach leverages both interpolation and collocation techniques to derive a numerically stable and efficient scheme. Unlike traditional methods, which often require reduction to a first-order system—a process that can introduce unnecessary complexity and computational overhead—our method preserves the original structure of the second-order differential equations.

To assess the robustness and versatility of the developed method, five representative numerical examples were carefully selected and examined. These examples encompass a range of differential equations, including linear, nonlinear, and stiff second-order IVPs, thereby providing a comprehensive validation across different classes of problems. The implementation of the one-fourth step method was evaluated in terms of its accuracy, stability, and convergence behavior.

A detailed comparative analysis is presented in the result tables, which juxtapose the numerical outcomes of the proposed method against both the exact solutions and results

from established numerical techniques. Specifically, the first three examples were benchmarked against the method introduced by Areo (2016), while the fourth and fifth examples were compared with the approach proposed by Ramos et al. (2016). Across all test cases, our method demonstrated superior performance, particularly in terms of error minimization and consistency with the exact solutions.

The improved accuracy observed in the stiff and nonlinear cases underscores the capability of the developed method to handle challenging IVPs with high computational efficiency. Furthermore, the results affirm that the proposed one-fourth step hybrid block method not only rivals but outperforms existing methods in several key aspects, thus representing a significant advancement in the numerical solution of second-order differential equations.

This work contributes to the ongoing discourse in numerical analysis by offering a novel, direct approach that eliminates the need for order reduction, expands the applicability of hybrid block methods, and enhances computational accuracy. Future extensions of this research may include the adaptation of the method for boundary value problems and partial differential equations, as well as integration with adaptive step-size control mechanisms for improved dynamic stability.

Conclusion

In this paper, a class of numerical schemes are consturted in which fractions was used as the steplengths for second order ordinary differential equations. The method is consistent and zero stable, therefore it convergences. The method have good region of absolute stability. The derived methods solve second order and third order initial value problems effectively and give good result. The table of result of our method are shown above, we compare our method with Areo (2016) and Rufai (2016) and the new method give a very good result.

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