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Fibonacci Polynomials and It's Generalization

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Abstract

This article explores the definition, properties, and generalizations of Fibonacci polynomials, providing a comprehensive understanding of their mathematical significance. We have used their Binet's formula and generating function to derive the identities.

Keywords: Generalized Fibonacci Polynomials, Binet's Formula, Generating Function

Introduction

Leonardo Pisano, often known as Fibonacci, was the most gifted Italian mathematician of his middle age and is credited with giving the series its name. He is thought to have been the first mathematician to teach Italians the Hindu-Arabic number system. For this reason, his work "Liber-Abaci" (1202) is well-known. Leonardo presents the well-known "Rabbit Problem" in the Liber Abaci in order to determine the solution to this rabbit dilemma [1].

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Fibonacci polynomials are a fascinating extension of the classical Fibonacci sequence, capturing essential recursive properties within the realm of polynomial algebra. These polynomials hold significant importance in various fields, including number theory, combinatorics, and applied mathematics [2]. This article investigates the definition, properties, and generalizations of Fibonacci polynomials, providing a comprehensive understanding of their mathematical significance.

By adding two arbitrary integers, we will now generalize this recursion algorithm. After that, two numbers will determine the matching sequence. The Fibonacci polynomials are a series of two variable polynomials [3].

Let a and b be real roots of second order equation:

(x-a) (x-b) = x^2 – (a+b) x + ab = 0. Also suppose a+b = p and ab = q then

x^2 – px-q = 0 (1)

These two roots a and b should satisfy equation (1)

a^2- Pa – q = 0 (2)

and b^2- Pb – q = 0 (3)

From equation (2) a^2- Pa – q = 0, the recursion powers a^2= p q + q(1)

$a^3 = (p^2 + q)a$ +qp

$a^4 = p\,(p^2 + 2q)a + q\,(p^2 + q)$

$a^5 = [p^2(p^2 + 2q) + q\,(p^2 + q)]a + qp(p^2 + 2q)$

Now introducing sequences of polynomials of two variables F_n (p, q) :

F_0 (p, q) = 0

F_1 (p, q) =1

F_2 (p, q) =p

F_3 (p, q) = $(p^2 + q)$

F_4 (p, q) = p $(p^2 + 2q)$

F_5 (p, q) = $p^2\,(p^2 + 2q) + q(p^2 + q)$

F_6 (p, q) = $p^3\,(p^2 + 2q) + 2pq(p^2 + q) + q^2p$

F_7 (p, q) = $p^4\ (p^2 + 2q) + 3p^4q(p^2 + q) + 6q^2p^2 + q^3$

Now, by recursion, the nth power is given by

$a^n = F_n(p,q)a$ + q$F_{n-1}(p,q)$ (4)

Similarly, for b,

$b^n = F_n(p,q)a$ + q$F_{n-1}(p,q)$ (5)

is obtained. These both formulas are proved by the method of induction. Subtracting equation (2) from equation (1), we get the Binet formula type of polynomials[9].

$F_n(p,q) = \frac{a^n - b^n}{a-b}$ (6)

Where a, b = $\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$

The equation (3) is the Binet type formula for sequences of Fibonacci polynomials. Now deriving the recursion formula for Fibonacci polynomials:

$F_{n+1}(p,q)$ = A$F_n(p,q)$ + B$F_{n-1}(p,q)$ (7)

To find A and B in equation (7) , we put the Binet formula for Fibonacci polynomials (6) in F_{n+1}, F_n , F_{n-1} . So, we get,

a^{n+1}= Aa^n+ Ba^{n-1} equivalently a^2= Aa+ B

b^{n+1}= Ab^n+ Bb^{n-1} equivalently b^2= Ab+ B

Now, a^2= Aa+ B →pa+q = Aa+B → A = p and B= q

Also, similarly,

b^2= Ab+ B, we get

$F_{n+1}(p,q)$ = p$F_n(p,q)$ + q$F_{n-1}(p,q)$ (8)

In this way, we can get recursion relation for sequences of Fibonacci polynomials. If p and q are arbitrary numbers then we get the sequences of integer numbers.

$F_0(p,q)$ = 0

F_1 (p, q) =1

F_2 (p, q) =p

F_3 (p, q) = $(p^2 + q)$

F_4 (p, q) = p $(p^2 + 2q)$

F_5 (p, q) = p^2 $(p^2 + 2q) + q(p^2 + q)$

F_6 (p, q) = p^3 $(p^2 + 2q) + 2pq(p^2 + q) + q^2p$

F_7 (p, q) = p^4 $(p^2 + 2q) + 3p^4q(p^2 + q) + 6q^2p^2+q^3$ which are called Fibonacci polynomials. But when choosing p = q= 1, then recursion relation will be standard recursion and Fibonacci numbers come [5,6,7]. So, $F_1(1,1) = F_1$.

Generalized Fibonacci Polynomial

Fibonacci polynomials have been generalized in various ways to accommodate broader mathematical contexts. Lucas polynomials are closely related to Fibonacci polynomials, defined by the same recurrence relation but with different initial conditions. Fibonacci polynomials can be generalized using matrix methods. For example, the polynomial sequence can be represented as powers of a companion matrix. Extensions to Fibonacci polynomials include Chebyshev-like or Legendre-like polynomials derived from differential equations or orthogonal conditions [3,4].

Fibonacci polynomial having initial conditions F_0 and F_1 can be generalized to arbitrary initial values G_0 and G_1. So we define generalized Fibonacci polynomials $G_n(p,q)$ by recursion formula

$G_{n+1}(p,q)$ = p $G_{n+1}(p,q)$ + q$G_{n-1}(p,q)$ (9) with initial conditions

$G_0(p,q) = G_0$, $G_1(p,q) = G_1$ (10)

This is simple to show generalized Fibonacci polynomials can be denoted as super-position of Fibonacci sequences:

$G_n(p,q) = G_1\, F_n(p,q)$ + qG_0F (11)

The Binet type formula for generalized Fibonacci polynomials

$G_n(p,q) = \frac{(G_1 - bG_N)a^n - (G_1 - aG_0)b^n}{a-b}$ (12)

where a, b = $\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$

The first few Generalized Fibonacci numbers are:

$G_0(p,q) = G_0$

G_1 (p, q) = G_1

G_2 (p, q) = $G_0 p$ + qG_0

G_3 (p, q) = G_1 $(p^2 + q)$ + qpG_0

G_4 (p, q) = G_1 $(p\ (p^2 + q) + qp)$ + $(q\ (p^2 + q) + G_0$

G_5 (p, q) = G_1 $(p^2\ (p^2 + 2q)$ +q $(p^2 + q)$ + q (p $(p^2 + q) + qp)G_0$

If we take initial conditions G_0 = k and G_1 = l as integer with p = s q = t coefficients then the generalized Fibonacci polynomial numbers. If we take G_0 = 0 and G_1 = p = q=1, then we get the sequences of Fibonacci numbers [6,7,8,9].

Properties of Fibonacci Polynomials

Recurrence Relation: The defining recurrence relation governs the entire structure of Fibonacci polynomials, mirroring the recursive nature of the classical Fibonacci sequence [12,13].

Degree: The degree of is, as each recursion step introduces a higher power.

Orthogonality: Under specific inner product definitions, Fibonacci polynomials can exhibit orthogonality properties, making them valuable in approximation theory.

Roots: The roots of Fibonacci polynomials play a significant role in applications such as signal processing and coding theory. For instance, these roots often exhibit symmetrical properties.

Generating Function: Fibonacci polynomials possess a generating function for analyzing their behavior.

Applications

Fibonacci polynomials and their generalizations have a wide array of applications:

Combinatorics: Counting problems, tiling, and partitioning.

Number Theory: Modular arithmetic and Diophantine equations.

Signal Processing: Filter design and spectral analysis.

Cryptography: Pseudorandom number generation and secure key design.

Conclusion

Fibonacci polynomials capture the beauty of recursive sequences and algebraic structures. Their generalizations, including -Fibonacci polynomials and Lucas polynomials, open new avenues for exploration and application in diverse mathematical and scientific domains. As both a theoretical and practical tool, Fibonacci polynomials continue to inspire mathematical discovery and innovation.

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