Mikailalsys Journal of Mathematics and Statistics

e-ISSN : 3030-816X
p-ISSN : 3030-8399

Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJMS.v3i1.4398

MATRICES OF FIBONACCI NUMBERS

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Oct 21, 2024	Nov 20, 2024	Dec 4, 2024	Dec 19, 2024

Abstract

This paper describes the matrix representation of Fibonacci numbers. The interaction between number theory and linear algebra is emphasized by the study of Fibonacci numbers using matrices. This viewpoint not only makes calculation easier, but it also reveals the sequence's underlying structural characteristics.

Keywords: Fibonacci number and sequence, Pascal triangle, Golden number, Natural Phenomenon

Introduction

The Fibonacci sequence, an elegant numerical pattern where each term is the sum of the two preceding terms, has fascinated mathematicians for centuries. It finds applications in various fields, from biology to computer science. An intriguing and less commonly explored aspect of Fibonacci numbers lies in their relationship with matrices. Through

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matrix algebra, we can derive, analyze, and generalize properties of Fibonacci numbers in a structured and efficient manner.

Since the earliest days of the ancient Greeks, mathematicians have been committed to the explanation of nature. Since mathematics is essentially an examination of all forms, is it more suited to uncover such wrongdoings than wildlife? A closer examination of wildlife reveals several endearing facts about the world's fundamental beauty [1].

Finding patterns in nature is more than simply a pastime; it helps us create mathematical tools that enable us to make estimations. The astounding thing is that mathematical designs are also present in a large number of biological species. It is an illustration of a typical and intriguing sequence seen in nature [2]. In nature, it serves a variety of purposes. It may be found in many biological contexts, such as flower petal patterns and tree branching.

One of the "greatest European mathematicians of the Middle Ages" was Fibonacci. His tomb is situated at the foot of Pisa Cathedral's cemetery tower. He died in 1240, having been born around 1170. Fibonacci grew up among the Moors and received his schooling in North Africa before making a lot of trips along the Mediterranean coast. He then visited traders, helping them with their business and teaching them different mathematical abilities. The "Hindu-Arab" approach was clearly superior than the others, as he quickly proved [1].

The Hindu-Arab system was initially introduced to Europe by Fibonacci. Ten base numerals are employed, each consisting of a zero sign and decimal points: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. In terms of business and finance, it has to do with whole and decimal numbers. Fibonacci numbers are always zero at the beginning and one at the end. There are many amazing natural occurrences that exhibit the Fibonacci sequence [10,11]. Fibonacci numbers were discovered in the 13th century by the Italian mathematician Leonardo Fibonacci [5]. The origins and uses of Fibonacci numbers were thoroughly explained by Bortner and Peterson [6].

Preliminary

A two-by-two matrix is denoted by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, and d represent any real numbers. These numbers are called elements of matrix A.

Two matrices,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

are equal if and only if their corresponding elements are equal. That is,

$$A = B$$

if and only if

$$a = e,\ b = f,\ c = g, \text{ and } d = h.$$

Sum of two matrices A and B is defined to be

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

The zero matrix Z is defined as

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It can be shown as

$$A + Z = Z + A = A$$

and so Z is the additive identity element for the system of two-by-two matrices.

The negative, or additive inverse, of A

$$-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

and the difference of A and B as

$$A - B = A + (-B).$$

It can be verifying that

$$A + (-A) = Z.$$

The product of a scalar s and a matrix A is defined to be

$$sA = s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}.$$

It can be shown that

$(sr)A = s(rA)$.

The product of two matrices A and B is defined to be

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

The identity matrix I is defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$AI = IA = A,$

Matrix multiplication is associative.

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 3 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$$
$$BA = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + 1 \cdot 1 & 3 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}$$

Consequently, $AB \neq BA$. This proves that matrix multiplication is not always commutative.

If $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then we have

$$AB = \begin{pmatrix} 1 \cdot 1 + (-1)1 & 1 \cdot 1 + (-1)1 \\ (-1)1 + 1 \cdot 1 & (-1)1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = Z$$

If the product of two numbers is zero, then at least one of the numbers has to be zero. For matrices this rule does not apply.

If A, B, and C are matrices, it can be shown that matrix multiplication is distributive over matrix addition. That is,

$A(B + C) = AB + AC$ and $(B + C)A = BA + CA$

Matrix multiplication is not always commutative, these two distributive laws do not necessarily say the same thing. The association of each matrix A is a number, called the determinant of matrix A, and denoted by $\det A$, which is defined as

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The matrix A is said to be non-singular if $\det A \neq 0$; otherwise, matrix A is singular.

THEOREM I

The product of the determinants, $\det A$ and $\det B$ is,

$$\det AB = (\det A)(\det B).$$

Proof: we have

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,\ \det B = \begin{vmatrix} c & f \\ g & h \end{vmatrix} = ch - \int\ g,$$

$$\begin{aligned} \det AB &= \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix} \\ &= (ac + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= acef + adeh + bcfg + bdgh - (acef + adfg + bceh + bdgh) \\ &= adeh + bcfg - adfg - bceh \\ &= (ad - bc)(eh - fg) \\ &= (\det A)(\det B). \end{aligned}$$

If a is a nonzero real number, then one can define all integral powers of a as follows:

$$a^0 = 1,\ a^1 = a,\ a^{n+1} = a^n a^1,$$

and $a^{-n} = \frac{1}{a^n}$ for positive integral n.

The definition $a^{n+1} = a^n a^1$ is called an inductive definition. We shall also use an inductive definition for powers of matrices. For any nonzero matrix A,

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,\ A^1 = A,\ \text{and}\ A^{n+1} = A^n A^1$$

for positive integral n.

Let a particular matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ \text{with}\ \det Q = -1,$$

The Fibonacci Sequence

The Fibonacci sequence starts as follows:

While the sequence is defined recursively, matrix representations provide an alternative perspective, offering direct computation methods and insights into its algebraic structure.

Matrix Representation of Fibonacci Numbers

A key insight is that Fibonacci numbers can be encoded using a simple matrix. Consider the matrix:

By observing the powers of, we can relate them to Fibonacci numbers: where represents the -nth Fibonacci number. This formula can be derived using induction and the recurrence relation of the Fibonacci sequence [7,8,9].

Deriving Fibonacci Numbers with Matrices

Using the property of, we can compute Fibonacci numbers directly. For example, to find:

From this matrix, we observe that, matching the standard Fibonacci sequence.

Efficient Computation Using Matrix Exponentiation

Matrix exponentiation offers an efficient way to compute Fibonacci numbers, especially for large indices. Instead of computing Fibonacci numbers recursively or iteratively, we can use fast exponentiation techniques to calculate in time.

The method involves successive squaring:

- Compute powers of by squaring intermediate results.
- Use the binary representation of to determine which powers contribute to,

This approach significantly reduces the computational overhead compared to naive recursion or iteration.

Generalizations and Connections

Matrices not only provide a compact way to express Fibonacci numbers but also lead to fascinating generalizations:

Generalized Fibonacci Sequences: Using different initial conditions or modifying the matrix, we can define analogous sequences with similar recurrence relations.

Binet's Formula: The closed-form expression for Fibonacci numbers, derived from the eigenvalues and eigenvectors of, is: where is the golden ratio.

Applications in Cryptography and Coding: Matrix-based Fibonacci methods have been used in algorithms for pseudorandom number generation, secure communication, and error correction.

THEOREM II

For $n \geq 1$, the nth power of Q is given by

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Proof. We shall use mathematical induction. Clearly, for $n = 1$,

$$Q^1 = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that $Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$; then

$$\begin{aligned} Q^{k+1} = Q^k Q^1 &= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} \\ &= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}. \end{aligned}$$

The proof is complete by mathematical induction.

THEOREM III

$$\det Q^n = (-1)^n,\ n \geq 1$$

Proof.

From Theorem II and Theorem III, we have

$$\det Q^n = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n-1}F_{n+1} - F_n^2 = (-1)^n,\ n \geq 1$$

For a matrix A, the characteristic polynomial is defined to be

$$P(x) = \det(A - xI) = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}\right|$$
$$= \begin{vmatrix} a-x & b \\ c & d-x \end{vmatrix}$$
$$= x^2 - (a+d)x + (ad - bc).$$

The equation $P(x) = 0$, or

$$x^2 - (a+d)x + (ad - bc) = 0$$

is called the characteristic equation of matrix A. The constant term is the determinant of matrix A and the linear term has coefficient $-(a+d)$, which is the negative of $(a+d)$. The sum $(a+d)$ of the elements on the diagonal of matrix A is called the trace of matrix A. The roots of the characteristic equation are called the characteristic roots of matrix A [12].

For Q, the characteristic polynomial is

$$\det(Q - xI) = \left|\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right|$$
$$= \begin{vmatrix} 1-x & 1 \\ 1 & -x \end{vmatrix} = x^2 - x - 1$$

and the characteristic equation is

$$x^2 - x - 1 = 0$$

which we have called the Fibonacci quadratic equation. The characteristic roots of Q are then,

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

The characteristic equation and the characteristic roots of Q''

$$\det(Q^n - xI) = \begin{vmatrix} F_{n+1} - x & F_n \\ F_n & F_{n-1} - x \end{vmatrix}$$
$$= x^2 - (F_{n+1} + F_{n-1})x + (F_{n+1}F_{n-1} - F_n^2)$$

But $F_{n+1} + F_{n-1} = L_n \quad \Gamma_{n+1}F_{n-1} - F_n^2 = (-1)^n[(L_{13})]$. Thus,

$$\det(Q'' - xI) = x^2 - L_n x + (-1)'',$$

and the characteristic equation is

$$x^2 - L_n x + (-1)^n = 0$$

The characteristic roots are given by (using the quadratic formula)

$x = \frac{L_n \pm \sqrt{L_n^2 - 4(-1)^n}}{2}$.

We now suppose from (I_{12}) in Section 10 that $L_n^2 - 4(-1)^n = 5F_n^2$; thus, the roots are $\frac{L_n + \sqrt{5}F_n}{2}$ and $\frac{L_n - \sqrt{5}F_n}{2}$.

THEOREM IV

The characteristic roots of matrix Q^n are the nth powers of the characteristic roots of Q, and the trace of Q^n is L_n.

Now verifying

$Q^2 = Q + I,$

$Q^2 - Q - I = Z.$

Thus, Q may be said to satisfy its characteristic equation

$x^2 - x - 1 = 0$

Conclusion

The study of Fibonacci numbers through matrices highlights the interplay between number theory and linear algebra. This perspective not only simplifies computation but also uncovers deeper structural properties of the sequence. Whether in theoretical research or practical applications, matrices of Fibonacci numbers continue to inspire exploration and discovery in mathematics and beyond.

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