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Common Fixed Point under Commutative Contraction Mappings in Digital Metric Spaces

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Abstract

The contraction mapping principle is a fundamental tool for establishing solutions to linear and nonlinear systems of ordinary differential equations. It guarantees that a contraction self-mapping on a complete metric space has a unique fixed point that can be approximated through an iterative sequence generated from an arbitrary initial point. This study examines the existence, uniqueness, and related analytical properties of fixed points for contraction mappings in digital metric spaces, including the uniform continuity of such mappings. Using a theoretical and analytical approach, existing results on dual-commutative contraction mappings were reviewed to determine the conditions under which unique fixed points exist in digital metric spaces. These results were subsequently extended from dual-commutative to triple-commutative contraction mappings. The analysis establishes that the proposed triple-commutative mappings satisfy the stated contraction conditions required for the existence and uniqueness of fixed points within the digital metric framework. Illustrative examples are provided to demonstrate the applicability of the established results. This extension broadens the theoretical framework of fixed-point analysis in digital metric spaces and provides a foundation for further investigations of higher-order commutative contraction mappings.

Keywords: Cauchy Convergence; Contraction Mapping; Digital Metric Space; Fixed Point; Lipschitz Constant

Introduction

The principle of contractions mapping play significant roles in studying systems of linear and non - linear differential equations and integro- equations such as Fredholm and Volterra - integral equations.

The principle was found by Banach and the concepts of abstract metric spaces which give rise to the fundamental concepts and back ground for the principle of Banach contraction mapping in a complete metric space were also found at the same time(Nawab and Iram, 2019).

This principle is a self-continuous mapping defined in a metric space (X, d) stated that, for some real number $L > 0$ such that if $L \in (0,1)$ for $x, y \in X$, we can define a distance function as $d(T(x), T(y)) \leq Ld(x, y)$, where L is a Lipschitz constant of the map T, which means every contraction map is a Lipschitz continuous function and also uniformly continuous.

The Banach contraction principle states that every self-continuous map defined on a metric space is complete if there exists a unique point and that for every element in a metric space, the iterated function sequence of such element converges to a fixed-point (Asha et al., 2016).

Recently, several researchers have provided prove for the uniqueness and existence of a fixed-point by considering unalike forms of continuous maps and techniques. (Ozgur et al., 2020) studied commutative and compatible type mapping under the context of digital metric spaces and provided some fixed-point consequences through the property of commutative mappings. Also, some common fixed-point results of compatible mappings and their variants in the framework of digital metric context were established. (Deepak, 2018) study the close maps in a classical metric space and in its digital context. The author added a common fixed-point result of two corresponding friendly mappings in digital metric spaces. (Sang, 2016) proposed a Banach contraction mapping result for digital images, and also extends the concepts of contraction map and to image processing and restricted to only single type contraction mapping. (Rao at al., 2019) provided some results of dual-mappings and quadruple self-mappings by adjusting distances of sub-compatible functions in five variables and difference functions with generality of only contractive mapping condition.

There are numerous differences between classical metric space and digital metric space, which is shown in the **Table 1** below;

Aspect	Classical metric space	Digital metric space
1. Nature of the space	Continuous	Discrete
2. Underline set	Any set, often $\mathbb{R}^n$	Subset of $\mathbb{Z}^n$
3. Distance values	Real numbers ($\mathbb{R}$)	Integers ($\mathbb{Z}$)
4. Continuity	Topologically continuous	Digitally continuous
5. Adjacency	Not required	Essential
6. Cardinality	Denumerable	Countably finite
7. Main focus	Analysis, topology and geometry	Analysis, Digital topology and imaging

Adjacency is nothing but how we formalize the idea that two digital points in a digital metric space $X \subset \mathbb{Z}_n$ (usually pixels or voxels) are neighbors. The digital metric space is written as (X, d, μ), where X is a set of digital elements or points and $X \subset \mathbb{Z}_n$, d is a metric derive from the space l_p, and μ is called the adjacency relation.

Two elements $x, y \in X$ are called adjacency if they are distinct, and they satisfy the specific closeness rule determine by the adjacency μ. Mathematically, if $d(x, y) = 1$, we say x and y are adjacency, simply μ −adjacency, if μ is the adjacency relation.

Examples:

(i) 4 −adjacency in $\mathbb{Z}_2$ implies $x \sim y, iff\ d_1(x, y) = 1$, this means two points are 4 −adjacency, if they share an edge.

(ii) 8 −adjacency in $\mathbb{Z}_2$, by the metric l_∞, implies $x \sim y, iff\ d_\infty(x, y) = 1$, this means two points are 8 −adjacency, if they share either an edge or a corner.

Definition 1(Adagonye and Ayuba, 2023): The function f is (x, x) continuous, whenever two points are $x - adjacent$ in the codomain, that is, their images are either equal or $x - adjacent$ in the codomain.

Definition 2(Kalaiarasia, 2022): An ordered pair (X, k) with $k -$ adjacency and X as a finite subset of $\mathbb{Z}_n$ for some positive integers is called a digital image.

Definition 3(Kalaiarasia, 2022): give that $(X, k) \subset \mathbb{Z}_n$ and d be a metric function on $\mathbb{Z}_n$ such that $d: X \times X \to \mathbb{Z}_n$ is define as

$d(t_1, t_2) = (\sum_{i=1}^{i=n} (t_{1\,i}, t_{2i})^2)^{\frac{1}{2}}$.

Then, the following properties holds for $t_1, t_2, t_3 \in X$, the following are satisfied

(i) $d(t_1 \quad, t_2) \geq 0, d(t_1, t_2) = 0\ iff\ x_1 = x_2$, (Positive definiteness property)

(ii) $d(t_1, t_2) = d(t_2, t_1)$, (Symmetry property)

(iii) $d(t_1, t_2) \leq d(t_1, t_3) + d(t_3, t_2)$ (Triangle Inequality)

Then, pair (X, k) with the metric function d define as (X, d, k) is known as digital metric space.

Definition 4(Md. Abdul et al., 2021) A self-mapping $T: X \to X$ on a set X with a point $x \in X$ is a fixed- point on the map T , if $T(x) = x$.

Definition 5(Asha et al, 2016): Let (M, d, k) be a complete digital metric space and T_1, T_2 be two maps defined on X . Then T_1, T_2 are said to be commutative if $T_1 \circ T_2(x) = T_2 \circ T_1(x)$.

Definition 6(Kareem et al., 2021): Suppose (X, d) is a metric space andthere is a sequence (x_n) in it. Then, the sequence (x_n) is Cauchy sequence if for every $\epsilon > 0$, there is an integer $N > 0$ so that $d(x_a, x_b) < \epsilon, \forall a, b \in N$.

Example 1.1: Let $X = \mathbb{Z}_n$ with two adjacencies (i.e when the neighbors differ by 1), which define by $f(\mu) = \mu + 1$.

If μ and $\mu + 1$ are adjacent, then $f(\mu) = \mu + 1$ and $f(\mu + 1) = \mu + 2$, which are also adjacency, this shows that the function f is $(2,2)$ continuous.

Example 1.2: If we consider the metric space $(\mathbb{R}, d)$ where d is the Euclidean Metric. Then the function $f: \mathbb{R} \to \mathbb{R}$, where $f(t) = \frac{t}{m} + n$ is a contraction if $m > 1$. In the case, we can find a fixed point, by recalling $f(t) = t$, from the definition of fixed point, then, by letting $t = \frac{t}{m} + n$, we solve for t to get $t = \frac{mn}{m-1}$.

Example 1.3: If we create another contraction that is similar to the first example in a two dimensional Euclidean space by letting $f: \mathbb{R}^2 \to \mathbb{R}^2$, where $f(p, q) = (\frac{p}{m} + n, \frac{q}{r} + n)$. Is a contraction if $m, r > 1$. From the definition of fixed point, $f(p, q) = (p, q)$, solving this, we get $p = \frac{mn}{m-1}$, and $q = \frac{rn}{r-1}$.

Example 1.4: Let the map $T: X \to X$ be define by $Tn = n^2$, the fixed point of the map T can be determine by letting $$Tn = n^2 \qquad 1$$

and from definition 1, we have $Tn = n$ 2

compare 1 with 2, we have $n^2 = n, \Longrightarrow n^2 - n = 0 \Longrightarrow n(n-1) = 0$, which shows that $n = 0$ or $n = 1$.

Thus, the fixed points of T are 0 and 1.(Kareem et al., 2021)

Example 1.5: Translation map has no fixed point.

Illustration: Let $Tn = n + k$, from definition 1, we have $Tn = n$, by comparing these two equations, we have $n + k = n + 0 \Longrightarrow k = 0$ [by left cancellation law].

Since $Tn = n + k$ is a translation map, so $k \neq 0$. Therefore, the translation map $Tn = n + k$ has no fixed point(Md. Abdul et al., 2021).

Example 1.6: Given that $T: X \to X$ be define by $d(a, b) = |b - a|$, where

$Ta = \begin{cases} a - \frac{1}{2}e^a, for\ a \leq 0 \\ -\frac{1}{2} + \frac{1}{2}a,\ for\ a \geq 0 \end{cases}$, examine whether or not the map T has fixed point.

Solution:Given that $Ta = \begin{cases} a - \frac{1}{2}e^a, for\ a \leq 0 \\ -\frac{1}{2} + \frac{1}{2}a,\ for\ a \geq 0 \end{cases}$, so that $Tb = \begin{cases} b - \frac{1}{2}e^b, for\ b \leq 0 \\ -\frac{1}{2} + \frac{1}{2}b,\ for\ b \geq 0 \end{cases}$

For $a, b \leq 0$, $d(Ta, Tb) \leq |b - \frac{1}{2}e^b - (a - \frac{1}{2}e^a)| \leq |(b - a) - \frac{1}{2}(e^b - e^a)| \leq |b - a|$

For $a, b \geq 0$, $d(Ta, Tb) \leq |-\frac{1}{2} + \frac{1}{2}b - (-\frac{1}{2} + \frac{1}{2}a)| \leq |\frac{1}{2}(b - a)| \leq |b - a|$.

Thus, the contraction map T satisfies $d(Ta, Tb) \leq |b - a|$.

But, from definition of fixed point, we have $Ta = a$ and $Tb = b$.

Now,

for $a \leq 0, Ta = a = a - \frac{1}{2}e^a \Rightarrow a = a - \frac{1}{2}e^a \Rightarrow -\frac{1}{2}e^a = 0 \Rightarrow e^a = 0 \Rightarrow lne^a = lno$

$\Rightarrow a = e^0 = 1$, this is contrary to the fact that $a \leq 0$.

For $a \geq 0$, $Ta = a = -\frac{1}{2} + \frac{1}{2}a \Rightarrow a = -\frac{1}{2} + \frac{1}{2}a \Rightarrow \frac{a}{2} = -\frac{1}{2} \Rightarrow a = -1$,this is also contradicting the fact that $a \geq 0$. Thus, the map T define above satisfies the condition of Banach contraction principle but has no fixed point.

Thus, by fixed point, $Tb = b$,

now for $b \le 0$, $Tb = b = b - \frac{1}{2}e^b \Rightarrow b = b - \frac{1}{2}e^b \Rightarrow -\frac{1}{2}e^b = 0 \Rightarrow e^b = 0 \Rightarrow b = 1$, which is contradicting the fact that $b \le 0$.

Finally, for $b \ge 0$, $Tb = b = -\frac{1}{2} + \frac{1}{2}b \Rightarrow y = -\frac{1}{2} + \frac{1}{2}b \Rightarrow \frac{1}{2}b = -\frac{1}{2} \Rightarrow b = -1$, which contradict the fact that $b \ge 0$. Thus, $Tb = \begin{cases} b - \frac{1}{2}e^b, for\ b \le 0 \\ -\frac{1}{2} + \frac{1}{2}b,\ for\ b \ge 0 \end{cases}$ is an example that satisfies condition of Banach contraction principle but has no fixed point.

METHODOLOGY

Here, the method of (Adagonye et al., 2025), (Saresh et al 2024), Kannan and Reich contraction maps as stated in **Theorem 3.2** and **Theorem 3.3** below were used to achieve our results with the help of the following notions;

Let $f(x,t)$ be a real valued function. This function is called a Lipschitz condition in a region R in a space (x,t), whenever we have positive constant say $L < 1$ so that

$$\| f(x,t) - f(y,t) \| \le L|x - t| \qquad 2.1$$

From equation 3.1, L is a Lipschitz constant and when $(x,t) \in \mathbb{R}$, $(x,t) \in \mathbb{R}^2$(Christiana, 2023).

Suppose the space(X,d,μ) is a digital metric space and the map $T: (X,d,\mu) \to (X,d,\mu)$ a self – digital map. If there a positive number $L < 1$ so that $\forall x,y \in X$,

$$d(T(x),T(y) \le Ld(x,y) \qquad 2.2$$

$d(Tx,Ty) \le Ld(x,y), L > 0 \Longrightarrow$ Lipchitz condition.

$d(Tx,Ty \le Ld(x,y), L < 1 \Longrightarrow$ Contraction constant, which is also Lipchitz constant.

If $L = 1$ such that $d(Tx,Ty) \le d(x,y)$, then the map T is called non-expansive map or an isometry. Then T is known as digital contraction map which possess a unique fixed-point a in a metric space$X$say, and for every$b$ in a metric space X, $T^n = a, n \to \infty$, showing that any contraction mapping on a digital metric space must possess a unique fixed-point.

Lemma 2.1(Asha et al, 2016): If $M \subseteq \mathbb{Z}_n$ and (M,d,μ) is a digital metric space, there is a sequence $\{a_n\} \in M$ so that $d(a_{n+1},a_n) < d(a_n,a_{n-1})$, for all$n \in \mathbb{N}$

Theorem 2.1(Adeyemi, 2021): Given that X is a metric space and $T: X \to X$ be a contraction mapping, then there exists $L < 1$ so that $d(T(x),T(y) \le L[d(x,Tx) +$

$d(y, Ty)]$, $\forall x, y \in X$.Then the map T possess a unique fixed-point $\alpha \in X$ and for every $x \in X$, the sequences $\{x_n\}$ iterates $\{T^n x\}$ by $x_{n+1} = T^n x$.

In a contraction conditions for digital metric space established by (Choonkil et al., 2019), the authors worked on fixed-point theorems of different type contraction conditions for digital metric space and the following corollary were proposed:

Corollary 2.1(Choonkil et al., 2019): Give that T is a contraction mapping on a complete metric space M and K be a contraction constant with a fixed-point m_o. Then, for every $m_o \in M$, with T-iterates $\{m_n\}$, the following estimate holds:

(1) $d(m_n, m_o) \leq \frac{K^n}{1-K} d(m_o, T(m_o))$,

(2) $d(m_n, m_o) \leq K d(m_{n-1}, m_o)$, and

(3) $d(m_n, m_o) \leq \frac{K}{1-K} d(m_{n-1}, m_n)$

Theorem 2.2(Choonkil et al., 2019): Given that (M, d, μ) is a digital metric space with a self – contraction map H, then the relation:

$$d\big(H(u), H(v)\big) \leq a_1 d(u, Hu) + a_2 d(v, Hv) + a_3 d(u, v), \forall\, u, v \in H \qquad 2.3$$

and $a_1, a_2, a_3 \geq 0$, and $\sum_{i=1}^{3} a_i < 1$., then the self – contraction map H has a unique fixed-point in M.

Corollary 2.2(Asha et al, 2016): given that H and N are contraction mappings that maps space (M, d, μ)into itself. IfH is continuous and the inclusion$N(a) \subseteq H(a)$, there exists $L \in (0,1)$ and integer $k > 0$ such that $d\big(N_k(a), N_k(b)\big) \leq Ld(H(a), H(b))$. Then H and N have a common point.

Theorem 2.3(Asha et al., 2016): Given that H is a continuous self-map in a complete digital metric space (M, d, μ) , then the map H has a unique point in M if and only if there exist an $L \in (0,1)$such that a mapping $N: M \longrightarrow M$commutes with H and $N(a) \subseteq H(a)$satisfies

$$d\big(N_k(a), N_k(b)\big) \leq Ld(H(a), H(b)) \qquad 2.4$$

Therefore, for every $a, b \in M$, the mappings H and N have a common fixed-point that is unique if equation (2.4) above holds.

Theorem 2.4(Kannan, 1968): Let $S, T: X \longrightarrow X$ be a self – contraction mappings. Then, there is$K \in (0, \frac{1}{2}]$such that that $d(Sa, Sb) \leq K[d(a, Sa) + d(b, Sb)]$, $\forall a, b \in X$, and

$$d(Ta, Tb) \leq K[d(a, Ta) + d(b, Tb)], \forall\, a, b \in X.$$

Theorem 2.5(Reich, 1971): Let $T: X \longrightarrow X$ be a self – contraction mapping in a digital metric space (M, ρ, μ). $\forall x_1, x_2 \in X$, $a_1, a_2, a_3 \geq 0$, and $\sum_{i=1}^{3} a_i < 1$. Then,

$$\rho(Tx_1, Tx_2) \leq a_1\rho(x_1, Tx_1) + a_2\rho(x_2, Tx_2) + a_3\rho(x_1, x_2).$$

Theorem 2.6(Aarti, 2021): Suppose (M, ρ, μ) is a digital metric space and A, B are self – digital contraction maps. If $A(M) \subset B(M)$, and φ is a continuous real valued function such that $\varphi(k) < k$, for $n > 0, \forall m, n \in M$, 2.5

And

$$\rho(Bm, Bn) \leq \varphi(\rho(Am, An)). \qquad 2.6$$

Then, A and B possess a unique fixed point in a digital metric space (M, ρ, μ).

Theorem 2.7(Aarti, 2021): Suppose (N, d, k) is a digital metric space and A, B, P, Q are $\alpha - \psi - \varphi$ type self – digital contraction mappings of digital metric space. If the pairs (P, A) and (Q, B) are weakly compatible. Then, for all $m, n \in N$

$$\alpha(m, n)\psi\big(d(Pm, Qn)\big) \leq \psi\big(N(m, n)\big) - \alpha(N(m, n)) \qquad 2.7$$

and

$$N(m, n) = max\left\{d(Am, Bn), d(Bn, Am), d(Am, Pm), d(Qn, Pn), d(Pm, Bn), \left(\frac{2d(Am, Pm)}{1 + d(Qn, Bn)}\right)\right\}.$$ Then, contraction maps $A, B, P \; and \; Q$ have a common fixed point in a digital metric space (N, d, μ).

Example 3.1: Suppose $I_z, S, T: M \to M$ is given by $Sa = \frac{2}{5}a^2 + \frac{1}{5}$, $Ta = \frac{3}{7}a^2 + \frac{2}{7}$, and $\rho(a, b) = |b - a|$, where I is closed and bounded interval $[0,1]$. Establish the uniqueness of fixed point in contraction maps S and T.

Solution:

Recall the inequality 2.6, in the **Theorem 2.6**, $\rho(Ta, Tb) \leq \varphi(\rho(Sa, Sb))$

$$\Rightarrow (\frac{3}{7}b^2 + \frac{2}{7} - \frac{3}{7}a^2 - \frac{2}{7}) \leq \varphi(\frac{2}{5}b^2 + \frac{1}{5} - \frac{2}{5}a^2 - \frac{1}{5})$$

$$\Rightarrow \frac{3}{7}(b^2 - a^2) \leq \varphi(\frac{2}{5}(b^2 - a^2))$$

$\Rightarrow \frac{3}{7}(b^2 - a^2) < (\frac{2}{5}(b^2 - a^2))$. This shows that the two self – contraction maps S and T have a common fixed point in M.

Example3.2: Suppose $M = \mathbb{N}$, set of Natural numbers and $\rho(a, b) = |b - a|$, $(M, d, 4)$ is a digital metric space with 4 −adjacency and the contraction mappings A, B, S, and T have common fixed point defined by $Aa = a + 2$, $Bb = b + 2$, $Sa = a - 2$, $Tb = b - 2$.

Solution: By inequality (2.7), in the **Theorem 2.7** above, we have

$$\alpha(a, b)\psi\big(\rho(Aa, Bb)\big) \leq \psi\big(M(a, b)\big) - \alpha(M(a, b)) \qquad 2.7$$

Where

$M(a, b) =$

$max\left\{\rho(Sa, Tb), \rho(Bb, Sa), \rho(Sa, Aa), \rho(Bb, Tb), \rho(Aa, Tb), \left(\frac{2\rho(Sa,Aa)}{1+\rho(Bb,Tb)}\right)\right\}$.

Therefore,

$$M(a, b) = max\Big\{|b - 2 - a + 2|, |a - 2 - b + 2|, |a + 2 - a + 2|, |b - 2 - b + 2|, |b - 2 - a - 2|, \left(\frac{2|a + 2 - a + 2|}{1 + |b - 2 - b + 2|}\right)\Big\}$$

$$M(a, b) = max\{|b - a|, |a - b|, |4|, 0, |b - a - 4|, 8\} = |b - a|$$

Therefore, the inequality (2.7) hold and self-contraction maps $A, B, S,$ and T have a common fixed point in a digital metric space $(M, d, 4)$(Aarti, 2021).

RESULTS

In this section, we are solidly rely on the method of (Kannan, 1968) and (Reich, 1971) for contraction of two - commutative maps studied in(Adagonye and Ayuba, 2023) to provide some analytical notions of commutative contractions such as existence and uniqueness of fixed points, Cauchy convergency of sequence of function arise and the uniformly continuity of contraction mappings associated with digital metric space.

Theorem 3.1: Given that$f$ and $g$are commutativeself-contraction maps on a digital metric space (M, d, k),then the following relations holds:

$$d(f(m), f(n)) \leq L\{d(m, f(m))+d(n,f(n)),\text{and}$$

$d(g(m), g(n) \leq L\{d(m\,, g(m)) + \mathrm{d}(n, g(n))\}$,where$f(m) \subset g(m)$,for every $m, n \in M$ and $L < 1$, thenmaps f and g have a unique fixed point in M.

Proof: Suppose $d(f(m), f(n)) \leq L\{d(m\,,f(n))+d(y,f(n))$

and $d(g(m), g(n) \leq L\{d(m\,, g(m)) + \mathrm{d}(n, g(n))\}$.

Since$f(m) \subset g(m) \Rightarrow d(f(m), f(n)) \leq d(g(m), g(n))$.Then

$$d(f(m), f(n)) \leq L\ \{d(m\,,f(m)) +(n,f(n))\} \leq d(g(m), g(n))$$

$$\leq L\{d(m\,, g(m)) + \mathrm{d}(n, g(n)) \Longrightarrow\ d(f(m), f(n)) \leq d(g(m), g(n)).$$

Let $m_o \in M$and take the iteration of the sequence of the function$f(m)$ as $f(m_{s+1}) = g(m_s)$ and $f(f(m_s)) = f(m_{s+1}), \forall s \in N$ such that

$$f(m_1) = g(m_o)$$

$$f(m_2) = g(m_1)$$

$$.f(m_{s+1}) = g(m_s)$$

By **Theorem 2.2**, we get

$$\begin{aligned} d(f(m_1), f(m_2)) &\leq L\{d(f(m_o), f(m_1)) + d(f(m_1), f(m_2))\} \\ &\leq Ld(f(m_o), f(m_1)) + Ld(f(m_1), f(m_2)) \end{aligned}$$

$$\Longrightarrow d(f(m_1), f(m_2)) - Ld(f(m_1), f(m_2)) \leq Ld(f(m_o), f(m_1))$$

$$(1 - L)\ d(f(m_1), f(m_2)) \leq Ld(f(m_o), f(m_1))$$

$$\Longrightarrow\ d(f(m_1), f(m_2)) \leq (\tfrac{L}{1-L})\ d(f(m_o), f(m_1)),$$

$$d(f(m_2), f(m_3)) \leq \left(\frac{L}{1-L}\right)^2 d(f(m_o), f(m_1)),$$

$$d(f(m_3), f(m_4)) \leq \left(\frac{L}{1-L}\right)^3 d(f(m_o), f(m_1)),$$

$$d(f(m_s), f(m_{s+1})) \leq \left(\tfrac{L}{1-L}\right)^s d(f(m_o), f(m_1)) \qquad 3.1.1$$

$$d(f(m_{s+1}), f(m_{s+2})) \leq \left(\tfrac{L}{1-L}\right)^{s+1} d(f(m_o), f(m_1)) \qquad 3.1.2$$

Let$\frac{L}{1-L}$ be τ such that $(\frac{L}{1-L})^2 = \tau^2$, $\left(\frac{L}{1-L}\right)^3 = \tau^3$, and $\left(\frac{L}{1-L}\right)^{s+1} = \tau^{s+1}$

By these, we obtain the following from inequalities (3.1.1)and (3.1.2)above

$d\big(f(m_s), f(m_{s+1})\big) \leq \tau^s d\big(f(m_o), f(m_1)\big)$3.1.3

and

$d\big(f(m_{s+1}), f(m_{s+2})\big) \leq (\tau)^{s+1} d\big(f(m_o), f(m_1)\big)$3.1.4

Then by**corollary 2.1**,we get from the inequalities (3.1.3)and (3.1.4) above

$$\begin{aligned} d\big(f(m_s), f(m_{s+r})\big) \\ &\leq d\big(f(m_s), f(m_{s+r})\big) + d\big(f(m_{s+1}), f(m_{s+2})\big) + \cdots \\ &+ d\big(f(m_{s+r-1}), f(m_{s+r})\big) \end{aligned}$$

$\Longrightarrow d\big(f(m_s), f(m_{s+r})\big) \leq \tau^s d(f(m_s), f(m_{s+1})) + (\tau)^{s+1} d\big(f(m_{s+1}), f(m_{s+2})\big) + \cdots + (\tau)^{s+r-1} d\big(f(m_{s+r-1}), f(m_{s+r})\big) \leq (\tau^s + (\tau)^{s+1} \ldots + (\tau)^{s+r-1}) d\big(f(m_{s+r-1}), f(m_{s+r})\big) \leq \tau^s (1 + \tau + \tau^2 + \cdots) d\big(f(m_o), f(m_1)\big)$.

Let$(1 + \tau + \tau^2 + \cdots)$ be an infinite exponential series such that its sum to infinity is

$S_\infty = \frac{x_1}{1-r}$,here, $x_1 = a = 1$which is the first term of exponential sequence, common ratio(r) =τ. So, sum to infinity, $S_\infty = \frac{1}{1-\tau}$.

Therefore,$d\big(f(m_s), f(m_{s+r})\big) \leq \tau^s (\frac{1}{1-\tau}) d\big(f(m_o), f(m_1)\big)$.

Recall that $\lim_{n\to\infty} r^s = 0, iff\ r < 1$, by this, $\lim_{s\to\infty} \tau^s = 0$, $\Longrightarrow$ $\tau^s (\frac{1}{1-\tau}) d\big(f(m_o), f(m_1)\big) \to 0$ as $s \to \infty$. This shows that the sequence $f(x_n)$ is a digital Cauchy sequence which converges in digital metric space (M, d, k).

Since the sequence $f(m_s)$ is a digital Cauchy, that means it converges to a point z in a digital metric space (M, d, k) and the sequence $g(m_s) = f(m_{s+1})$ also converges to a limit point z in the digital metric space (M, d, k). Since (m, m) −continuity of f implies (m, m) −continuity of g. Since we are considering sequence of function, this implies that (m, m) − uniformly continuity of $f$implies (m, m) −uniformly continuity of g, since the function of any sequence is uniformly continuous, the function $f(m_s)$ is uniformly continuous. Therefore, the sequence$\{g(f(m_s))\}$ converges to $g(z)$.

Since both functions f and g are commutative, then

$f\big(g(m_s)\big) = g(f(m_s))$3.1.5

Since the sequence $g(f(m_s))$ converges to $g(z)$, this implies that the sequence $f\big(g(m_s)\big)$ converges to $f(z)$.

From (i) above, we see that

$f(z) = g(z)$ 3.1.6

Due to the uniqueness of the limit, we have from (3.1.6) that

$f(f(z)) = f(g(z))$3.1.7

By **corollary 2.2** and **Theorem 2.2**, we have $d(g(z), g(g(z))) \leq \tau d(f(z), f(f(z))) \leq d(g(z), g(g(z)))$

Let $g(z) = f(f(z))$3.1.8

By comparing (3.1.7) and (3.1.8), we get $g(z) = f\big(g(z)\big) = g(g(z))$, from this we conclude that $g(z)$ is a common fixed point of the maps f and g.

Next, we need to provide prove for the uniqueness of the fixed points of the maps f and g, by letting$m, n \in M$, and m and n are distinct such that $f(m) = m = g(m)$ and $f(n) = n = g(n)$ such that $d(a, b) \leq \tau d(g(a), g(b)) \leq \tau d(f(a), f(b)) \leq \tau d(a, b)$, then, we have

$$d(m, n) - \tau d(m, n) \leq \tau d(m, n) - \tau d(m, n),$$

$(1 - \tau) d(m, n) \leq 0$ 3.1.9

But from property of metric space, $d(m, n) \nless 0$, it implies that

$(1 - \tau) < 0$, then, divide both sides of (3.1.9) by $(1 - \tau)$, we get

$d(m, n) \geq 0$ 3.1.10

From (3.1.10) above, we get $m = n$, which shows the uniqueness of two fixed points m and n.

Theorem 3.2:given that$f$ and g are self-commutative maps on a digital metric space(M, d, k).Then ,the followingrelation are satisfied:

$d(f(m), f(n)) \leq a_1 d(f(m), f(f(n))) + a_2 d\big(f(n), ff(n)\big) +$
$a_3 d(f(m), f(n))$, and

$d(g(m), g(n)) \leq a_1 d(g(m), g(g(n))) + a_2 d\big(g(n), g(g(n)\big) +$
$a_3 d(g(m), g(n))$,were $f(m) \subset g(n), \forall\, m, n \in M$, and $\sum_{i=1}^{3} a_i < 1$. Then f and g have a unique fixed point in$M$.

Proof:Let $f(m_0) \in M$, and the sequence

$f(m_{s+1}) = f(f(m_s))$,$f(m_s) = f(f(m_{s-1}))$, since $f(m) \subset g(m)$,

$f(m) \leq g(m)$, by **Theorem 2.3**, we get

$d(f(m), f(n)) \leq d(g(m), g(n))$. Therefore,

$d(f(m), f(n)) \leq a_1 d(f(m),f(f(m))) + a_2 d\big(f(n), ff(n)\big) +$
$a_3 d(f(m), f(n) \leq$

$d(g(m), g(n)) \leq a_1 d(g(m),g(g(n))) + a_2 d\big(g(n), g(g(n)\big) +$
$a_3 d(g(m), g(n))$,

and

$d(f(m), f(n)) \leq d(g(m), g(n)) \leq a_1 d(f(m), f(f(m)))+a_2 d\big(f(n), ff(n)\big) +$
$a_3 d(f(m), f(n))$ 3.2.1

Let $m = m_1$, and $n = m_2$ in the equation (*) above, we get

$d(f(m_1), f(m_2)) \leq a_1 d(f(m_1), f(f(m_1))) + a_2 d\big(f(m_2), ff(m_2)\big) +$
$a_3 d(f(m_1), f(m_2))$ 3.2.2

Consider the following sequences

$f\big(f(m_s)\big) = f(m_{s+1})$ 3.2.3

and $f(m_s) = ff(m_{s-1})$ 3.2.4

from equation (3.2.3), equation (3.2.2)becomes

$d(f(m_1), f(m_2)) \leq a_1 d(f(m_1), f(m_2)) + a_2 d\big(f(m_2), f(m_3)\big) +$
$a_3 d(f(m_1), f(m_2)$3.2.5

from equation (3.2.4), equation (3.2.5)becomes

$d(f(m_1), f(m_2)) \leq a_1 d(f(m_o), f(m_1)) + a_2 d\big(f(m_1), f(m_2)\big) +$
$a_3 d(f(m_o), f(m_1))$

$\Rightarrow\ d\big(f(m_1), f(m_2)\big) - a_2 d\big(f(m_1), f(m_2)\big) \leq a_1 d(f(m_o), f(m_1)) +$
$a_3 d(f(m_o), f(m_1))$

$$\Rightarrow (1 - a_2) d\big(f(m_1), f(m_2)\big) \leq (a_1 + a_3) d(f(m_o), f(m_1))$$

Divide through by $(1 - a_2)$, we get

$$d\big(f(m_1), f(m_2)\big) \leq \frac{a_1 + a_3}{1 - a_2} d(f(m_o), f(m_1))$$

Similarly, $d\big(f(m_2), f(m_3)\big) \leq (\frac{a_1 + a_3}{1 - a_2})^2 d(f(m_o), f(m_1))$.

$$d\big(f(m_s), f(m_{s+1})\big) \leq (\frac{a_1 + a_3}{1 - a_2})^s d(f(m_o), f(m_1)) \qquad 3.2.6$$

and

$$d\big(f(m_{s+1}), f(m_{s+2})\big) \leq (\frac{a_1 + a_3}{1 - a_2})^{s+1} d(f(m_o), f(m_1)) \qquad 3.2.7$$

Let $m = \frac{a_1 + a_3}{1 - a_2}$ in equations(3.2.6) and (3.2.7) above, we have

$$d\big(f(m_n), f(m_{n+1})\big) \leq u^s d(f(m_o), f(m_1)) \qquad 3.2.8$$

and

$$d\big(f(m_{n+1}), f(m_{n+2})\big) \leq u^{s+1} d(f(m_o), f(m_1)) \qquad 3.2.9$$

Then,by **Theorem 2.2** above, we get

$$\begin{aligned} d\big(f(m_s), f(m_{s+r})\big) \\ &\leq\ d\big(f(m_s), f(m_{s+1})\big) +\ d\big(f(m_{s+1}), f(m_{s+2})\big) + \cdots \\ &+ d\big(f(m_{s+r-1}), f(m_{s+r})\big) \\ &\leq (u^s + u^{s+1} + \cdots + u^{s+r-1}) d(f(m_o), f(m_1)) \\ &\leq u^s (1 + u + u^s + \cdots) d(f(m_o), f(m_1)) \end{aligned}$$

Note that $(1 + u + u^s + \cdots)$ is an infinite geometric series which its sum to infinity is written as $S_\infty = \frac{a}{1-r}$, where $a = 1, r = u$, showing that $S_\infty = \frac{1}{1-u}$, therefore, we have

$d\big(f(m_s), f(m_{s+r})\big) \leq \frac{u^s}{(1-u)}\, d(f(m_o), f(m_1))$, recall that $\lim_{s\to\infty} r^s = 0, iff\ r < 1$, this implies that

$$\frac{u^s}{(1-u)}\, d\big(f(m_o), f(m_1)\big) = 0$$

So, the sequence $\{f(m_n)\}$is a Cauchy sequence in a digital metric space (M, d, k). Since the digital metric space(M, d, k) is complete , then the sequence must converge to a limit point$l$,say, in the digital metric space (M, d, k) and hence , the sequence $g(m) = f(m_{n+1})$ must also converges to a limit point l, say ,in a digital metric space(M, d, k).Note that since $(m, m)-$ uniformly continuity of the commutative map f implies(m, m) $-$uniformly continuity of the commutative map g, then the sequence $\{g(f(m_n))\}$ converges to $g(l)$.Since both maps f and g arecommutative on adigital metric space (X, d, k) and $g(f(m_n))$=$f(g(m_n))$,therefore the sequence $\{g(f(m_n))\}$converges to$\omega(L)$, which shows that $\omega(l) = g(l)$, which is the uniqueness of thelimit of the digital Cauchy sequence arise in the digital contraction mappings.Now,since the uniqueness of the limits exists, then we can have $f(g(l))$=$g(f(l))$,which is uniqueness of digital Cauchy sequence arise in commutative digital contraction maps. Next,according to **Theorem2.3**,we can have $d\Big(g(l), g\big(g(l)\big)\Big) \leq md\Big(f(l), f\big(f(l)\big)\Big) \leq d\Big(g(l), g\big(g(l)\big)\Big)$by the fact that $g(l) = f(f(l))$, $g(l) = g\big(g(l)\big) = f(g(l))$.Thus the point$g(l)$ is a common fixed point of commutative maps f and g.Finally, we shall prove the uniqueness of fixed point of commutative maps$f$ and g by letting a and b be fixed points of commutative maps f and $g$respectively, where$a \neq b, \forall a, b \in M$. From the definition of fixed point, it means for commutative maps$f$ and g, $f(a) = a = g(a)$and$g(b) = b = f(b)$. Then, we have

$$d(a, b) \leq d(g(a), g(b)) \leq d(f(a), f(b)) \leq d(a, b)$$

$$\Longrightarrow\ d(g(a), g(b)) \leq d(f(a), f(b)) \leq 0$$

$\Longrightarrow d(g(b), g(a)) \geq d(f(b), f(a)) \geq 0$, by positive definite property of metric space.

Let either $d(g(b), g(a)) \geq 0$ or $d(f(b), f(a)) \geq 0$,if $d\big(g(b), g(a)\big) = 0$, then $g(b) =\ g(a)$, showing that $b = a$. Hence, $b = a$ shows the uniqueness of the fixed points of the commutative digital contraction maps f and g.

In the next result, we are going to consider **Theorem 2.4** and**Theorem2.5** called The Kannan and Reich contraction mappings respectively to extend the **Theorem 3.1** and **Theorem 3.2** (dual – contraction mappings) to the case of Triple - contraction mappingsto establish the existence and uniqueness as well as uniform continuity of digital contraction mappings in a digital metric space by using Sandwich Theorem to relate the three contraction mappings involved.

Theorem3.3: Given that$(M, d, k)$is a digital metric space and f, $g$are two commutativeself-maps in(M, d, k).If there exist an arbitrary maps h such that $f(m) \subset h(m) \subset g(m)$ and the following relations holds;

$$d(f(m), f(n)) \leq L\{d(m, f(m)) + d(n, f(n))\}, d(h(m), h(n)) \leq L\{d(m, h(m)) + d(n, h(n))\},$$ and

$d(g(m), g(n)) \leq L\{d(m, g(m)) + d(n, g(m))\}$, for every $m, n \in X$ and $L < 1$. Then, f, g, and h have a unique fixed point in M.

Proof: From the inclusion relation $f(m) \subset h(m) \subset g(m)$, we have

$$f(m) \leq h(m) \leq g(m) \qquad 3.3.1$$

From (3.3.1) above, we can have the following relations

$$f(m) \leq h(m), \qquad 3.3.2$$

$$h(m) \leq g(m) \qquad 3.3.3$$

$$\Longrightarrow f(m) \leq g(m) \qquad 3.3.4$$

In the relation, let's consider equation (3.3.4), by **Theorem 3.1**, we get

$$d(f(m), f(m)) \leq \{d(m, f(m)) + d(m, f(m))\},$$ and

$$d(g(m), g(n)) \leq L\{d(m, g(m)) + d(n, g(n))\}$$

Since $f(m) \leq g(m)$, by (3.3.4), we get

$$d(f(m), f(n)) \leq d(g(m), g(n))$$

Now, if $m_o \in X$ and we consider the sequence $f(m_{s+1}) = g(m_s)$ for all $s \in \mathbb{N}$, and$f(f(m_s)) = f(m_{s+1})$ such that

$$f(m_1) = g(m_o)$$

$$f(m_2) = g(m_o)$$

$$f(m_{s+1}) = g(m_s)$$

By **Theorem 3.1**above, we have the following;

$$d\big(f(m_1), f(m_2)\big) \leq Ld\big(f(m_o), f(m_1)\big) + Ld(f(m_1), f(m_2))$$

Then, $d\big(f(m_1), f(m_2)\big) \leq \frac{L}{1-L}\, d\big(f(m_o), f(m_1)\big)$.

By continuing this iteration this way as in shown in **Theorem 3.1**, we get

$$d\big(f(m_s), f(m_{s+1})\big) \leq \left(\frac{L}{1-L}\right)^s d\big(f(m_o), f(m_1)\big) \qquad 3.3.5$$

and

$$d\big(f(m_{s+1}), f(m_{s+2})\big) \leq \left(\frac{L}{1-L}\right)^{s+1} d\big(f(m_o), f(m_1)\big) \qquad 3.3.6$$

We can let $\frac{L}{1-L} = \tau$, as in **Theorem 3.1**, so that

$$d\big(f(m_s), f(m_{s+1})\big) \leq \tau^n d\big(f(m_o), f(m_1)\big)$$

and

$d\big(f(m_{s+1}), f(m_{s+2})\big) \leq \tau^{s+1} d\big(f(m_o), f(m_1)\big)$.

If we continue this iteration in such a way that

$$\begin{aligned} d\big(f(m_s), f(m_{s+r})\big) \\ &\leq d\big(f(m_s), f(m_{s+1})\big) + d\big(f(m_{s+1}), f(m_{s+2})\big) + \cdots \\ &+ d\big(f(m_{s+r-1}), f(m_{s+r})\big) \end{aligned}$$

$$\Longrightarrow\ d\big(f(m_s), f(m_{s+r})\big) \leq \tau^s(1 + \tau + \tau^2 + \cdots)d\big(f(m_o), f(m_1)\big)$$

$$\Longrightarrow\ d\big(f(m_s), f(m_{s+r})\big) \leq \frac{\tau^s}{1-\tau} d\big(f(m_o), f(m_1)\big)$$

Since $\lim_{s\to\infty} \tau^s = 0, iff \tau < 1 \Longrightarrow \frac{\tau^s}{1-\tau} d\big(f(m_o), f(m_1)\big) = 0, as\ s \to \infty$

This shows that the sequence $f(m_s)$ is a digital Cauchy sequence.

Recall the relation $f(m) \leq h(m) \leq g(m)$, from **Theorem 3.1**, we have known that both sequences $f(m_s)$ and $g(m_s)$ are digital Cauchy sequence, and since Cauchy sequence converges, then both sequences $f(m_s)$ and $g(m_s)$ converges.

Now recall the Sandwich Theorem which states that; if $x_n, y_n, z_n$are sequences such that$m_s \leq n_s \leq q_s$. If $\lim_{s\to\infty} m_s = \lim_{s\to\infty} q_s = L$, then $\lim_{s\to\infty} n_s = L$, this means, if m_s and q_s converges to a limit, then the sequencen_s converges to the same limit.

For this, the sequence $h(m_s)$ formed from the inclusion relation is Cauchy and hence, converges.

Therefore, both sequences$f(m_s), h(m_s), g(m_s)$ converges and hence. have a unique limit point.

Assume that $\lim_{n\to\infty} f(m_s) = \lim_{n\to\infty} g(m_s) = w$, by Sandwich Theorem, $\lim_{s\to\infty} h(m_s) = w$, which shows the uniqueness of the limit. From the Theorem1, since the limit is the fixed point, then the limit is unique, as we can see by letting $a, b, c \in M$ such that $f(a) = a = h(a) = a = g(a), f(b) = b = h(b) = b = g(b)$, and

$f(c) = c = h(c) = c = g(c)$, such that

$$d(a,b) \leq \tau d(f(a), f(b)) \leq \tau d(h(a), h(b)) \leq \tau d(g(a), g(b)) \leq \tau d(a,b) \quad 3.3.7$$

$$\Rightarrow \quad d(a,b) \leq \tau d(f(a), f(b)) \leq \tau d(g(a), g(b)) \leq \tau d(a,b)$$

$$\Rightarrow \quad d(a,b) - \tau d(a,b) \leq \tau d(a,b) - \tau d(a,b) = 0 \quad ,$$

So, $(1-\tau)d(a,b) \leq 0$, if we let $(1-\tau) < 0$, as in Theorem1, we see that $d(a,b) \geq 0$

By definiteness property, if $d(a,b) = 0$, then $a = b$ 3.3.8

Also, if we replace b by c in equation (vii) above, we have $d(a,c) = 0$, then

$$a = c \quad 3.3.9$$

By equations (3.3.8) and (3.3.9), we see that$a = b = c$, which shows the uniqueness of the fixed points.

Next, in order to justify our results, we make used of **Theorem 2.6** and **Theorem 2.7** stated in the methodologyabove to provides the following examples.

Example 3.1: Suppose $S,T: X \to X$are commutative contraction maps defined by $Sx = \frac{x}{2} + \frac{1}{2}$, $Tx = \frac{x}{4} + \frac{3}{4}$, and $d(x,y) = |y - x|$, where I is closed and bounded interval $[0,1]$. Establish the uniqueness of fixed point in contraction maps S and T.

Solution:

Recall the inequality B, in the **Theorem 2.6**, $d(Tx,Ty) \leq \varphi(d(Sx,Sy))$

$$\Rightarrow \frac{y}{4} + \frac{3}{4} - (\frac{x}{4} + \frac{3}{4}) \leq \varphi\left(\frac{y}{2} + \frac{1}{2} - (\frac{x}{2} + \frac{1}{2})\right)$$

$$\frac{y}{4} + \frac{3}{4} - \frac{x}{4} - \frac{3}{4} \leq \varphi\left(\frac{y}{2} + \frac{1}{2} - \frac{x}{2} - \frac{1}{2}\right)$$

$$\frac{y}{4} - \frac{x}{4} \leq \varphi\left(\frac{y}{2} - \frac{x}{2}\right)$$

$$\Rightarrow \frac{1}{4}(y - x) \leq \varphi\frac{1}{2}(y - x)$$

$$\Rightarrow (y - x) \leq \varphi(y - x).$$

Example 3.2: Suppose $S,T: X \to X$are commutative contraction maps defined by $Sx = \frac{x}{2} + \frac{3}{2}$, $Tx = 2x - 3$, and $d(x,y) = |y - x|$. Establish the uniqueness of fixed point in contraction maps S and T.

Solution: As we have seen in **example 3.1**above, Recall the inequality B, in the **Theorem 2.6**, $d(Tx,Ty) \leq \varphi(d(Sx,Sy))$, such that $Sx = \frac{x}{2} + \frac{3}{2}, Sy = \frac{y}{2} + \frac{3}{2}$, and $Tx = 2x - 3, Ty = 2y - 3$.

Then, By $d(Tx,Ty) \leq \varphi(d(Sx,Sy))$,

$$2y - 3 - (2x - 3) \leq \varphi\left(\frac{y}{2} + \frac{3}{2} - (\frac{x}{2} + \frac{3}{2})\right)$$

$$2y - 3 - 2x + 3 \leq \varphi\left(\frac{y}{2} + \frac{3}{2} - \frac{x}{2} - \frac{3}{2}\right)$$

$$2y - 2x \leq \varphi\left(\frac{y}{2} - \frac{x}{2}\right)$$

$$2(y - x) \leq \varphi\frac{1}{2}(y - x)$$

$$\Rightarrow (y - x) \leq \varphi(y - x)$$

CONCLUSION

The contraction conditions for digital metric spaceswere proposed by (Adagonye and Ayuba, 2023). In this work, the dual commutative map for Kannan and Reich contractions principle for digital metric spaces were studied, the commutative nature of the couple maps with respect to digital contraction maps and the uniquenessof fixed point of the mappings, its digital Cauchy convergency and uniform continuity were established. The uniqueness and common fixed was established by consider two different points such as a, b and, a, b, c in case of Triple – mappings for both contraction maps f and g as shown in Theorem 3.1, 3.2 and 3.3 above. If this these points are distinct, the result will totally contradict the nature of Lipschitz constant, but we have shown that they are equal. Therefore, $a = b$, $a = b = c$is unique and common fixed point.

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