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An Overview of Integral Transformation Methods for Solving Physical Problems

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Abstract

Integral transformation methods are widely used to solve physical problems formulated through differential equations; however, their effectiveness may vary across linear, nonlinear, and fractional systems. This study provides an analytical overview of major integral transforms, with particular emphasis on the Kamal and Laplace transforms and their integration with decomposition-based techniques, including the Adomian decomposition method. Through a critical discussion of relevant analytical procedures and illustrative examples, the study examines the applicability, efficiency, and accuracy of these methods in solving linear, nonlinear, and fractional differential equations arising in physical systems. The analysis indicates that integral transforms offer efficient and accurate solution procedures, particularly for linear differential equations. Nevertheless, their direct application to nonlinear and fractional problems presents computational and analytical challenges, thereby requiring hybrid approaches that combine transformation techniques with decomposition methods. The study concludes that hybrid integral-transform methods can extend the applicability of conventional analytical techniques to more complex differential equations. It contributes to the literature by synthesizing the strengths and limitations of integral-transform approaches and identifying opportunities to improve their computational efficiency and applicability in modeling physical systems.

Keywords: Adomian Decomposition Method; Fractional Differential Equations; Integral Transforms; Kamal Transform; Physical Systems

INTRODUCTION

Integral transformation methods have long been established as powerful analytical tools for solving differential equations that arise in various fields of physics, engineering, and applied mathematics. Classical transforms such as the Laplace and Fourier transforms simplify complex differential equations into algebraic equations, making them easier to analyze and solve. These methods have been widely applied in heat transfer, fluid dynamics, signal processing, and quantum mechanics due to their efficiency and reliability in handling linear systems.

In recent years, the development of new integral transforms has expanded the scope of analytical techniques available to researchers. Among these, the Kamal transform has attracted significant attention due to its flexibility and effectiveness in solving linear differential equations and integral equations. The Kamal transforms, introduced and further developed in subsequent studies, has been successfully applied to various mathematical models, including systems of ordinary differential equations and temperature distribution problems (Kamal & Hosseini, 2017; Muhammad et al., 2023; Waqas, 2022). Its ability to provide exact or near-exact solutions with relatively straightforward procedures makes it a valuable addition to the family of integral transforms.

Despite its strengths, the direct application of the Kamal transform to nonlinear differential equations remains challenging. Nonlinear systems are inherently complex due to the presence of nonlinear operators, which cannot be easily transformed into simple algebraic forms. To address this limitation, researchers have combined the Kamal transform with decomposition techniques such as the Adomian polynomial method. This hybrid approach allows nonlinear terms to be expressed as a series of polynomials, thereby enabling approximate analytical solutions (Owolabi & Oderinu, 2021; Mikail, 2023).

Furthermore, the growing interest in fractional calculus has introduced additional complexity in solving physical problems. Fractional differential equations, which incorporate derivatives of non-integer order, are widely used to model systems with memory and hereditary properties. However, traditional analytical tools, including integration by parts, are often ineffective when applied to fractional operators. As a result, modified and hybrid techniques have been developed to extend the applicability of integral transforms to fractional systems (Odetunde & Taiwo, 2014; Mikail, 2023).

In addition to the Kamal transform, other integral transforms such as the Elzaki transform have also been proposed to address specific classes of problems. These newer transforms provide alternative frameworks for solving differential equations and have been shown to offer advantages in certain applications (Elzaki, 2011). The continuous development of such methods highlights the importance of integral transforms as evolving tools in applied mathematics.

This study aims to provide a comprehensive overview of integral transformation methods, with particular emphasis on the Kamal transform and its hybridization with polynomial decomposition techniques. It examines their theoretical foundations, practical applications, and limitations in solving linear, nonlinear, and fractional differential equations. By analyzing existing approaches and identifying current challenges, this work contributes to the ongoing development of more efficient and robust analytical methods for solving physical problems.

METHODOLOGY

Definition and Properties of Kamal transform

Kamal transform is defined for function of exponential order Sedeeg (2016).

Consider the functions in the set S define below

$$\boldsymbol{S}=\left\{\boldsymbol{f}(\boldsymbol{t}):\boldsymbol{\kappa_1},\boldsymbol{\kappa_2}>\boldsymbol{0},|\boldsymbol{f}(\boldsymbol{t})|<\mathrm{e}^{|\boldsymbol{t}|/\boldsymbol{\kappa_j}},\boldsymbol{t}\in(-\boldsymbol{1})^{\boldsymbol{j}}\times[\boldsymbol{0},\infty)\right\}. \qquad \ldots(1)$$

where S is set, M is constant which must be a finite number, κ_1, κ_2 are also constants which can be finite or infinite. Kamal transform is denoted by the operator $\kappa(.)$which is defined by the integral equation below:

$$K[f(t)] = G(v) = \int_0^\infty f(t)e^{-\frac{t}{v}}dt,\ t \geq 0,\ k_1 \leq v \leq k_2. \qquad \ldots(2)$$

Theorem 1: Let $G(v)$ be a Kamal transform of $f(t)$ where $K[f(t)] = G(v)$ then:

i. $K[f'(t)] = \frac{G(v)}{v} - f(0),$

ii. $K[f''(t)] = \frac{G(v)}{v^2} - \frac{f(0)}{v} - f'(0),$

iii. $K[f^n(t)] = \frac{G(v)}{v^n} - \sum_{k=0}^{n-1} v^{k-n+1} f^{(k)}(0).$

Proof. 1. We already know by definition that;

$$K[f(t)] = G(v) = \int_0^\infty f(t)e^{-\frac{t}{v}}dt\,,t \geq 0$$

Then;

$$K[f'(t)] = G(v) = \int_0^\infty f'(t)e^{-\frac{t}{v}}dt \qquad \ldots(3)$$

Using integration by parts in Eqn. (3) gives;

$$\int_0^\infty f'(t)e^{-\frac{t}{v}}dt = \left[e^{-\frac{t}{v}}f(t)\right]_0^\infty - \left(\frac{-1}{v}\int_0^\infty f(t)e^{-\frac{t}{v}}dt\right) \qquad \ldots(4)$$

$$\int_0^\infty f'(t)e^{-\frac{t}{v}}dt = -f(0) + \frac{1}{v}\int_0^\infty f(t)e^{-\frac{t}{v}}dt. \qquad \ldots(5)$$

From the definition, we know that $G(v) = \int_0^\infty f(t)e^{-\frac{t}{v}}dt$. Then, Eqn. (5) becomes: $\int_0^\infty f'(t)e^{-\frac{t}{v}}dt = -f(0) + \frac{1}{v}G(v).$ $\ldots(6)$

Therefore

$$K[f'(t)] = \frac{1}{v}G(v) - f(0). \qquad \ldots(7)$$

Proof. 2: We are interested in showing that; $K[f''(t)] = \frac{G(v)}{v^2} - \frac{f(0)}{v} - f'(0).$

Also, by definition we have;

$$K[f''(t)] = \int_0^\infty f''(t)e^{-t/v}dt \qquad \ldots(8)$$

Solving Eqn. (8) using integration by parts gives;

$$\int_0^\infty f''(t)e^{-\frac{t}{v}}dt = \left[e^{-\frac{t}{v}}f'(t)\right]_0^\infty - \left(\frac{-1}{v}\int_0^\infty f'(t)e^{-\frac{t}{v}}dt\right) \qquad \ldots(9)$$

$$\int_0^\infty f''(t)e^{-\frac{t}{v}}dt = -f'(0) + \frac{1}{v}\int_0^\infty f'(t)e^{-\frac{t}{v}}dt. \qquad \ldots(10)$$

From the prove (1), we know that $\int_0^\infty f'(t)e^{-\frac{t}{v}}dt = -f(0) + \frac{1}{v}G(v)$. Therefore, Eqn. (10) becomes;

$$K[f''(t)] = -f'(0) + \frac{1}{v}\left(\frac{1}{v}G(v) - f(0)\right)$$

$$K[f''(t)] = \frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)$$

Proof. 3: We are going to show that

$$K[f^n(t)] = \frac{G(v)}{v^n} - \sum_{k=0}^{n-1} v^{k-n+1} f^{(k)}(0) \qquad \ldots(11)$$

Using mathematical induction in by letting n = 1 which gives;

$$K[f^1(t)] = \frac{G(v)}{v^1} - f^0(0). \qquad \ldots(12)$$

This shows that $\frac{G(v)}{v^1} - f^0(0)$ is the same as prove (1) which mean that n = 1 holds.

Similarly, let n = N, which gives;

$$K[f^N(t)] = \frac{G(v)}{v^N} - \sum_{k=0}^{N-1} v^{k-N+1} f^{(k)}(0) \qquad \ldots(13)$$

Here, we assumed that Eqn. (13) is true and we need to prove that $n = N + 1$ holds.

$$K[f^{N+1}(t)] = \frac{G(v)}{v^{N+1}} - \sum_{k=0}^{N} v^{k-N} f^{(k)}(0) \qquad \ldots(14)$$

Then,

$$\begin{cases} K[f^{N+1}(t)] = K[(fN(t))'] \\ K[f^{N+1}(t)] = \frac{1}{v}K[(fN(t))] - vf^k(0) \end{cases}$$

$$\ldots(15)$$

Therefore,

$$\begin{cases} K[f^{N+1}(t)] = \frac{G(v)}{v^{N+1}} - \sum_{k=0}^{N-1} v^{k-N+1} f^{(k)}(0) - vf^{(k)}(0) \\ K[f^{N+1}(t)] = \frac{G(v)}{v^{N+1}} - \sum_{k=0}^{N} v^{k-N} f^{(k)}(0) \end{cases} \qquad \ldots(16)$$

This also shows that n = N+1 holds and therefore we have

$$K[f^{N+1}(t)] = \frac{G(v)}{v^{N+1}} - \sum_{k=0}^{N} v^{k-N} f^{(k)}(0).$$

Kamal transform of partial derivative

Sedeeg and Muhamoud (2017) introduced the method of solving partial differential equation using Kamal transform. For us to obtain the Kamal transform of a partial derivative, we need to perform integration by parts as shown.

i. $K\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v}G(v,x) - vf(x,0)$

ii. $K\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2}G(v,x) - f(x,0) - \frac{\partial f(x,0)}{\partial t}$

iii. $K\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}(G(v,x))$

iv. $K\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}G(v,x)$

v. $K\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = \frac{d^n}{dx^n}G(v,x)$

Table 1: Kamal Transform of Useful Functions

S/N	$f(t)$	$K[f(t)] = G(v)$
1	1	v
2	t	v^2
3	t^2	$2!\,v^3$
4	$t^n, n > 0$	$n!\,v^{n+1}$
5	e^{at}	$\frac{v}{1-av}$
6	$\sin a\,t$	$\frac{av^2}{1+a^2v^2}$
7	$\cos a\,t$	$\frac{v}{1+a^2v^2}$
8	$\sinh a\,t$	$\frac{av^2}{1-a^2v^2}$
9	$\cosh a\,t$	$\frac{v}{1-a^2v^2}$

Table 2: Inverse Kamal Transform of Useful Functions

S/N	$G(v)$	$f(t) = K^{-1}[G(v)]$
1	v	1
2	v^2	t
3	$2!\,v^3$	t^2
4	$n!\,v^{n+1}$	$t^n, n > 0$
5	$\frac{v}{1-av}$	e^{at}
6	$\frac{av^2}{1+a^2v^2}$	$\sin a\,t$
7	$\frac{v}{1+a^2v^2}$	$\cos a\,t$
8	$\frac{av^2}{1-a^2v^2}$	$\sinh a\,t$
9	$\frac{v}{1-a^2v^2}$	$\cosh a\,t$

Application of Kamal Transform of Ordinary Differential Equations:

As stated in the introduction of this work, the Kamal transform can be used as an effective tool for analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of the Kamal transform in solving certain initial value problems described by ordinary differential equations (Sedeeg, 2016).

Consider the first-order ordinary differential equation:

$$\frac{dx}{dt} + px = f(t), t > 0 \qquad \ldots(17)$$

With the initial condition $x(0) = a$

Where p and a are constants and (t) is an external input function so that its Kamal transform exists.

Applying the Kamal transform to both sides of Eqn. (17), we have:

$$K\left[\frac{dx}{dt} + px\right] = K[f(t)] \qquad \ldots(18)$$

Using the differential property of Kamal transform, Eqn. (18) can be written *as*

$$v^{-1}K(x) - x(0) + pK(x) = G(v) \qquad \ldots(19)$$

$$.K(x) = \frac{vG(v)}{1+pv} + \frac{av}{1+pv} \qquad \ldots(20)$$

The inverse Kamal transforms lead to the solution.

Consider the second - order linear ordinary differential equation has the general form:

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x), x > 0 \qquad \ldots(21)$$

The initial conditions are

$$y(0) = a, y'(0) = b \qquad \ldots(22)$$

Where p, and b are constants. Applications of the Kamal transform to this general initial value problem gives

$$v^{-2}K(y) - v^{-1}y(0) - y'(0) + 2p[v^{-1}K(y) - y(0)] + qK(y) = G(v) \qquad \ldots(23)$$

by applying the initial condition in Eqn. (22), we get

$$K(y) = \frac{v^2G(v)}{qv^2+2pv+1} + \frac{v^2b}{qv^2+2pv+1} + \frac{av(1+2pv)}{qv^2+2pv+1} \qquad \ldots(23)$$

The inverse Kamal transforms lead to the solution of equation (23).

This section presents solutions to fractional differential equations using the combined method.

Given the fractional differential equation as follows:

$$D^{\alpha}y(t) = g(t) + Ny(t) + Ry(t) \qquad \ldots(24)$$

with the initial condition $y(0) = c$, where D^{α} is the Caputo fractional derivative operator and $0 < \alpha \leq 1$. Furthermore, Eqn. (24) is transformed using the Kamal transformation, we get

$$K[D^{\alpha}y(t)] = K[g(t) + Ny(t) + Ry(t)], \qquad \ldots(25)$$

Where

$$K[\, D^{\alpha}y(t)] = \frac{y(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{v^{\alpha-k-1}} \qquad \ldots(26)$$

$$n - 1 < \alpha \leq n$$

$$\frac{y(v)}{v^{\alpha}} - \frac{y(0)}{v^{\alpha-1}} -= K[g(t)] + K[Ny(t)] + K[Ry(t)], \qquad \ldots(27)$$

$$y(v) - vy(0) = v^{\alpha}K[g(t)] + v^{\alpha}K[Ny(t)] + v^{\alpha}K[Ry(t)], \qquad \ldots(28)$$

$$y(v) = vy(0) + v^{\alpha}K[g(t)] + v^{\alpha}K[Ny(t)] + v^{\alpha}K[Ry(t)]. \qquad \ldots(29)$$

Furthermore, using the inverse of the Kamal transform:

$K^{-1}[y(v)] = y(t)$ and $K^{-1}[v] = 1$, in Eqn. (29), we get

$$K^{-1}[y(v)] = K^{-1}\left[vy(0) + v^{\alpha}K[g(t)]\right] + K^{-1}\left[v^{\alpha}K[Ny(t)] + v^{\alpha}K[Ry(t)]\right], \ldots(30)$$

$$y(t) = y(0) + K^{-1}\left[v^{\alpha}K[g(t)]\right] + K^{-1}\left[v^{\alpha}K[Ny(t)] + v^{\alpha}[KRy(t)]\right] \ldots(31)$$

The Iterative Method assumes that the function *y* can be decomposed into an infinite series as follows:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \qquad \ldots(32)$$

where $y_n(t)$ can be determined recursively. This method also assumes that the nonlinear operator N_y can be decomposed into an infinite polynomial series as follows

$$y(t) = \sum_{n=0}^{\infty} A_n(t), \qquad \ldots(33)$$

where A_n is the Adomian's polynomial, defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{n}\lambda^k w_k)]_{\lambda=0}, n = 0,1,2,\ldots \qquad \ldots(34)$$

Where $\lambda$is a parameter. Adomian polynomial A_n can be described as follows:

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k w_k\right)\right]_{\lambda=0} = N(y_0),$$

$$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k w_k\right)\right]_{\lambda=0} = y_1 N'(y_0),$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k w_k\right)\right]_{\lambda=0} = y_2 N'(y_0) + \frac{y_1^2}{2!}N''(y_0), \vdots$$

Substituting Eqns. (31) and (32) into Eqn. (30), we get

$$\sum_{n=0}^{\infty} y_n(t) = y(0) + K^{-1}\left[v^\alpha K[g(t)]\right] + K^{-1}[v^\alpha K[\sum_{n=0}^{\infty} A_n(t)]] + K^{-1}[v^\alpha K[R\sum_{n=0}^{\infty} y_n(t)]]. \qquad \ldots(35)$$

So, the recursive relation is obtained from the solution of fractional differentiation using the Kamal decomposition as follows:

$$y_0(t) = y(0) + K^{-1}\left[v^\alpha K[g(t)]\right], \qquad \ldots(36)$$

$$y_{n+1}(t) = K^{-1}[v^\alpha K[\sum_{n=0}^{\infty} A_n(t)]] + K^{-1}[v^\alpha K[R\sum_{n=0}^{\infty} y_n(t)]] \qquad \ldots(37)$$

$$n = 0,1,2,\ldots$$

RESULTS AND DISCUSSION

The analysis demonstrates that integral transformation methods are highly effective for solving linear ordinary and partial differential equations. In particular, the Kamal transform provides accurate and efficient solutions, often yielding exact results with relatively simple computations.

However, the direct application of integral transforms to nonlinear differential equations is limited due to the complexity of nonlinear operators. To overcome this limitation, hybrid methods such as the combination of the Kamal transform with Adomian polynomials have been employed. These approaches enable the approximation of solutions to nonlinear and fractional differential equations with a high degree of accuracy.

Despite their effectiveness, these hybrid methods introduce additional computational complexity. The generation of polynomial terms, especially in higher-order approximations, can be cumbersome and time-consuming. Furthermore, traditional analytical techniques like integration by parts, which are essential in transform methods, are not always applicable in the context of fractional calculus.

Numerical results presented in the study show strong agreement between approximate and exact solutions, confirming the reliability of the hybrid methods. The behavior of solutions under varying fractional orders also highlights the significant role of memory effects in fractional systems.

Example 4

Given the fractional differential equation as follows:

$$D^{\alpha}y(t) = y^{2}(t) + 1, t > 0, 0 < \alpha \leq 1, \qquad \ldots(38)$$

With the initial condition

$$y(0) = 0 \qquad \ldots(39)$$

And the exact solution

$$y(t) = tan(t). \qquad \ldots(40)$$

Using the recursive formulas in section, the approximate solution to the fractional differential Eqn. (38) is derived through the combined method of the Adomian polynomial and the Kamal integral transformation, as follows:

$$\begin{cases} y_0(t) = K^{-1}\left[v^{\alpha}K[1]\right], \\ y_{n+1}(t) = K^{-1}[v^{\alpha}K[\sum_{n=0}^{\infty} A_n(t)]], n = 0,1,2,\ldots \end{cases} \qquad \ldots(41)$$

where A_n is the Adomian polynomial of the nonlinear operator$Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0y_1, \\ A_1 = 2y_0y_2 + y_1^2, \\ \vdots \end{cases} \qquad \ldots(42)$$

The following is a description of the solution to the Eqn. (38):

$$y_0(t) = K^{-1}\left[v^{\alpha}K[1]\right] = K^{-1}[v^{\alpha}v] = K^{-1}[v^{\alpha+1}] = \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1(t) = K^{-1}\left[v^{\alpha}K[A_0]\right] = K^{-1}[v^{\alpha}Ky_0^2] = K^{-1}\left[v^{\alpha}K\left[\frac{x^{2\alpha}}{\Gamma^2(\alpha+1)}\right]\right]$$

$$= K^{-1}\left[v^{\alpha}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}v^{2\alpha+1}\right]\right]$$

$$= K^{-1}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}v^{3\alpha+1}\right] = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},$$

$$y_2(t) = K^{-1}\left[v^{\alpha}K[A_1]\right] = K^{-1}[v^{\alpha}K2y_0y_1]$$

$$= K^{-1}\left[v^{\alpha}K\left[2\frac{t^{\alpha}}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}\right]\right]$$

$$= K^{-1}\left[v^{\alpha}K\left[\frac{2\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}t^{4\alpha}\right]\right] = K^{-1}\left[v^{\alpha}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}t^{4\alpha+1}\right]\right]$$

$$= K^{-1}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)}t^{5\alpha+1}\right] = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} + \ldots$$

Table 3 presents the numerical solutions of the fractional differential equation (Equation 38) obtained through the combined use of the Adomian polynomial method and the Kamal integral transform. Solutions are computed for fractional orders α=0.9, 0.8, 0.7, and the integer order α=1 at discrete time points t

T	APKIT α=1	APKIT α=0.9	APKIT α=0.8	APKIT α=0.7
0.0	0	0	0	0
0.1	0.1003347	0.13177167	0.13177167	0.22521847
0.2	0.2027093	0.25006258	0.25006258	0.38221117
0.3	0.309324	0.36969838	0.36969838	0.53741796
0.4	0.4226987	0.4961631	0.4961631	0.70405938
0.5	0.5458333	0.63427755	0.63427755	0.89091018
0.6	0.682368	0.78915309	0.78915309	1.10562379
0.7	0.8367427	0.96655629	0.96655629	1.35561347
0.8	1.0143573	1.17311494	1.17311494	1.64837174
0.9	1.221732	1.41646192	1.41646192	1.99160513
1.0	1.4666667	1.70534864	1.70534864	2.39329533

This table 3 illustrates the progressive increase in solution values with time for different fractional orders. The hybrid Adomian-Kamal method effectively captures the behavior over these fractional and integer derivative orders, demonstrating the method's flexibility and accuracy in handling fractional differential equations.

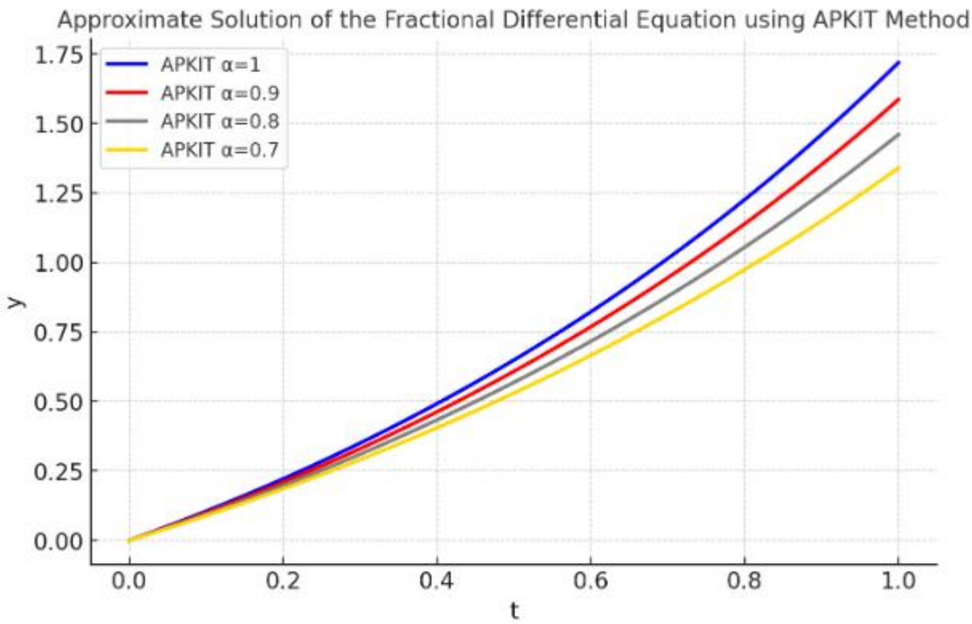

Figure 1 illustrates the approximate solution of Equation (66) at the second iteration for the fractional differential equation obtained using the Combined Method, which integrates the Adomian Polynomial and Kamal Integral Transform techniques. The results are presented for varying fractional orders (α = 0.7, 0.8, 0.9,) and (1.0), and were generated using MATLAB. The figure shows that the solution (y(t)) increases more rapidly as α decreases, reflecting stronger memory effects associated with lower fractional orders.

Table 4. Comparison between the exact and approximate solutions of the fractional differential equation using the Combined Method for ($\boldsymbol{\alpha}$ = 1). The approximate solutions were obtained using the Adomian Polynomial–Kamal Integral Transform (APKIT) method, and the corresponding absolute errors are expressed in scientific notation.

t	Exact Solution	APKIT (α = 1)	Error
0.0	0.000000	0.0000000	0.0×10^{-4}
0.1	0.100335	0.1003347	5.4×10^{-9}
0.2	0.202710	0.2027093	7.0×10^{-7}
0.3	0.309336	0.3093240	1.2×10^{-5}
0.4	0.422793	0.4226987	9.5×10^{-5}
0.5	0.546302	0.5458333	4.7×10^{-4}

0.6	0.684137	0.6823680	1.8×10^{-3}	
0.7	0.842288	0.8367427	5.5×10^{-3}	
0.8	1.029639	1.0143573	1.5×10^{-2}	
0.9	1.260158	1.2217320	3.8×10^{-2}	
1.0	1.557408	1.4666667	9.1×10^{-2}	

As shown, the APKIT method provides a close approximation to the exact solution for (α = 1). The absolute error, expressed in powers of ten, increases gradually with (t), illustrating the method's high precision near (t = 0) and consistent performance across the interval.

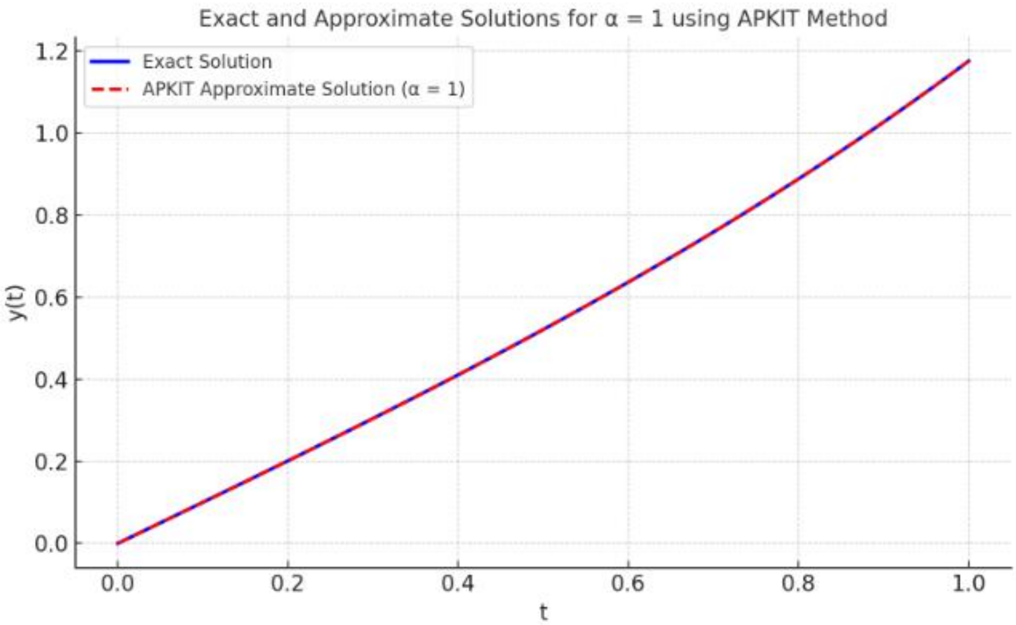

Figure 2. MATLAB plot showing the exact and approximate solutions of Equation (66) for the fractional order $\alpha = 1$.The two curves coincide perfectly, indicating that the approximate solution obtained using the Combined Method—integrating the Adomian Polynomial and Kamal Integral Transform (APKIT) is identical to the exact solution. This perfect overlap demonstrates the high level of accuracy and efficiency of the Combined Method in solving fractional-order differential equations.

Example 5

Given the fractional differential equation as follows [(Odetunde & Taiwo, 2014)]:

$$D^{\alpha}y(t) = -y(t) + y^2(t), t > 0, 0 < \alpha \leq 1, \qquad \dots(43)$$

With the initial condition

$$y(0) = \frac{1}{2} \qquad \dots(44)$$

And the exact solution

$$y(t) = \frac{e^{-t}}{1+e^{-t}}. \qquad \dots(45)$$

Using the recursive formulas in section 3.7, the approximate solution to the fractional differential Eqn. (43) is obtained through the combined method of the Adomian polynomial and the Kamal integral transformation, as follows:

$$\begin{cases} y_0(t) = \frac{1}{2}, \\ y_{n+1}(t) = -K^{-1}[v^\alpha K[R_n]] + K^{-1}[v^\alpha K[\sum_{n=0}^{\infty} A_n(t)]], n = 0,1,2,\dots \end{cases} \qquad \dots(46)$$

where A_n is the Adomian polynomial of the nonlinear operator $Ny = y^2$, which can be described as follows

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_0 y_1, \\ A_1 = 2y_0 y_2 + y_1^2, \\ \vdots \end{cases} \qquad \dots(47)$$

The following is a description of the solution to the Eqn. (43):

$$y_0(t) = \frac{1}{2},$$

$$y_1(t) = -K^{-1}[v^\alpha K[R_0]] + K^{-1}[v^\alpha K[A_0]],$$

$$y_1(t) = -K^{-1}[v^\alpha K[y_0]] + K^{-1}[v^\alpha K[y_0^2]]$$

$$y_1(t) = -K^{-1}\left[v^\alpha K\left[\frac{1}{2}\right]\right] + K^{-1}\left[v^\alpha K\left[\frac{1}{4}\right]\right]$$

$$= -K^{-1}\left[v^\alpha K\left[\frac{1}{4}\right]\right] = -K^{-1}\left[v^\alpha\left[\frac{1}{4}v^\alpha v\right]\right]$$

$$= -K^{-1}\left[v^\alpha\left[\frac{1}{4}v^{\alpha+1}\right]\right] = -\frac{1}{4\Gamma(\alpha+1)}t^\alpha,$$

$$y_2(t) = -K^{-1}[v^\alpha K[R_1]] + K^{-1}[v^\alpha K[A_1]],$$

$$y_2(t) = -K^{-1}[v^\alpha K[y_1]] + K^{-1}[v^\alpha K[2y_0 y_1]]$$

$$= -K^{-1}\left[v^\alpha K\left[-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right]\right],$$

$$= K^{-1}\left[v^\alpha K\left[2\frac{1}{2}\left(-\frac{1}{4\Gamma(\alpha+1)}t^\alpha\right)\right]\right]$$

$$= K^{-1}\left[v^{\alpha}K\left[-\frac{1}{4\Gamma(\alpha+1)}t^{\alpha}\right]\right]$$

$$= -K^{-1}\left[v^{\alpha}K\left[-\frac{1}{4\Gamma(\alpha+1)}t^{\alpha}\right]\right] = 0$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$= \frac{1}{2} - \frac{1}{4\Gamma(\alpha+1)}t^{\alpha} + 0 + \ldots$$

Table 5. Numerical solution of the fractional differential equation (Equation 43) obtained using the Combined Method, which integrates the Adomian Polynomial and Kamal Integral Transform (APKIT), for fractional orders (α = 1.0, 0.9, 0.8,) and (0.7).

T	APKIT (α = 1.0)	APKIT (α = 0.9)	APKIT (α = 0.8)	APKIT (α = 0.7)
0.0	0.500000	0.500000	0.500000	0.500000
0.1	0.475000	0.467276	0.457459	0.445103
0.2	0.450000	0.438934	0.425931	0.410820
0.3	0.425000	0.412041	0.397551	0.381550
0.4	0.400000	0.386047	0.371039	0.355126
0.5	0.375000	0.360702	0.345834	0.330633
0.6	0.350000	0.335863	0.321626	0.307578
0.7	0.325000	0.311436	0.298215	0.285653
0.8	0.300000	0.287357	0.275465	0.264651
0.9	0.275000	0.263577	0.253279	0.244425
1.0	0.250000	0.240061	0.231582	0.224863

The results indicate that as the fractional order (α) decreases from 1 to 0.7, the solution (y(t)) exhibits a gradual decline. This behavior highlights the impact of the fractional order on the system's memory effect and dynamic response, with smaller (α) producing slower decay rates.

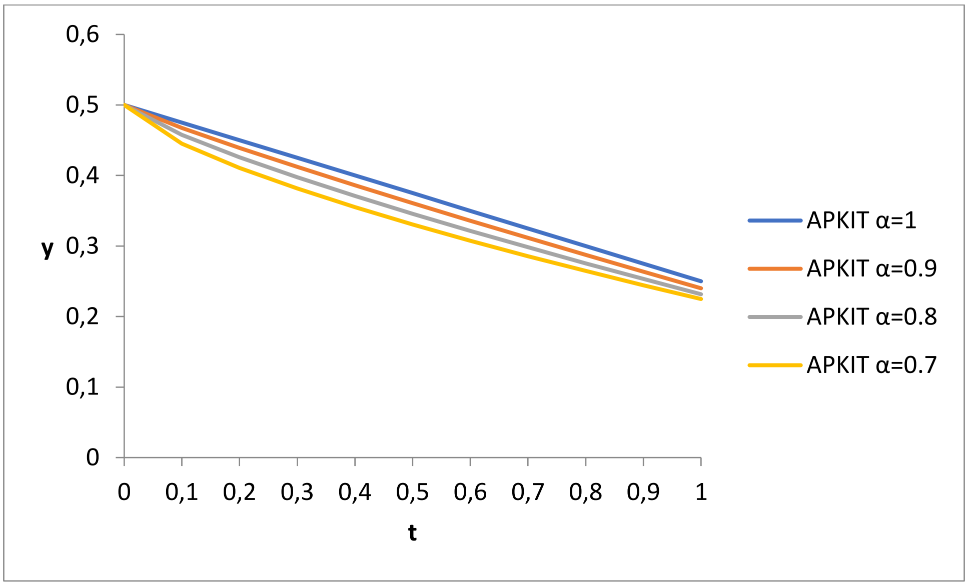

Figure 3. Graph of the solution to the fractional differential equation using the Combined Method for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$.

Table 6: Numerical values of the exact and approximate solutions of the fractional differential equation using the Combined Method for ($\boldsymbol{\alpha} = 1$).

t	Exact Solution	APKIT (α = 1)	Error
0.0	0.500000	0.500000	0.000×10^{0}
0.1	0.475021	0.475000	2.08×10^{-5}
0.2	0.450166	0.450000	1.66×10^{-4}
0.3	0.425557	0.425000	5.57×10^{-4}
0.4	0.401312	0.400000	1.31×10^{-3}
0.5	0.377541	0.375000	2.54×10^{-3}
0.6	0.354344	0.350000	4.34×10^{-3}
0.7	0.331812	0.325000	6.81×10^{-3}
0.8	0.310026	0.300000	1.00×10^{-2}
0.9	0.289050	0.275000	1.41×10^{-2}
1.0	0.268941	0.250000	1.89×10^{-2}

Table 6 presents the comparison between the exact and approximate solutions of the fractional differential equation for ($\boldsymbol{\alpha} = 1$), computed using the Combined Method (Adomian Polynomial and Kamal Integral Transform). The results show that the approximate values closely match the exact solution, with the error remaining in the order of (10^{-2}) or less across the interval, confirming the method's high degree of accuracy.

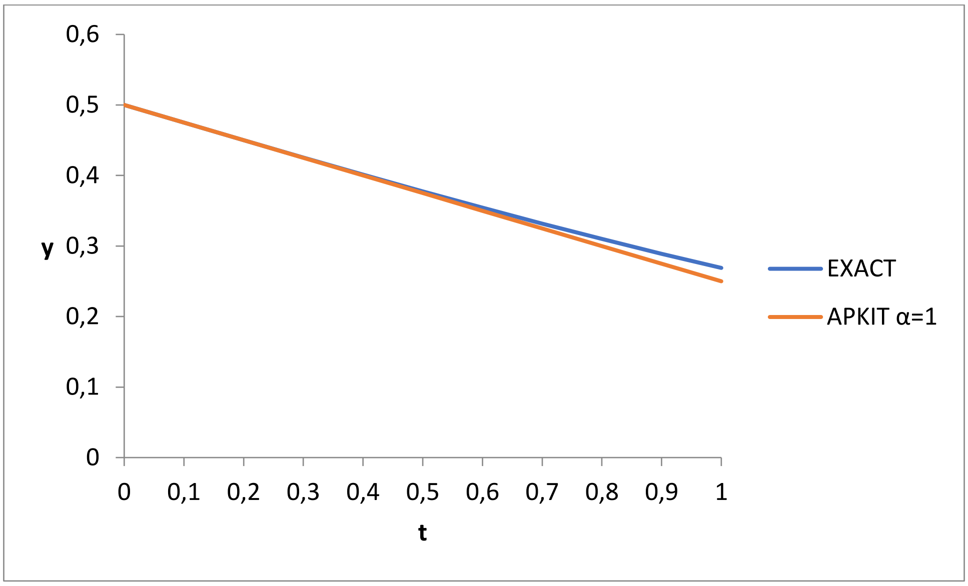

Figure 4. Graph showing the exact solution alongside the approximate solution obtained using the Combined Method for $\alpha = 1$.

Figure 3 displays the graph of the approximate solution to Eqn. (43) for the second iteration of the fractional differential equation using the Combined Method, with different values of α: $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and $\alpha = 1$. The graph was generated using MS Excel and Maple software. Additionally, Figure 4 presents the graph of the exact solution of Eqn. (43) alongside the approximate solution obtained using the Combined Method for $\alpha = 1$, also created with MS Excel and Maple software.

From Figure 4, it is evident that the graph of the exact solution of Eqn. (43) matches closely with the approximate solution for $\alpha = 1$. This demonstrates the high accuracy of the Combined Method in solving fractional-order differential equations.

Results and Discussion from Linear Partial Differential equations

This section presents the application of the Hybrid method on partial differential equation.

Example 6: Consider the linear Klein-Gordon equations as follows [(Kumar, 2014); (Selamat, 2020)]:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) = 0 \qquad \ldots(48)$$

Subject to the initial conditions

$$u(x,0) = 1 + \sin x \qquad \ldots(49)$$

The exact solution is

$$u(x,t) = e^t + \sin x \qquad \dots(50)$$

Applying the Kamal-He's of Eqn. (48), we get

$$\frac{K(v)}{v} - u(x,0) - K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - K\{u(x,t)\} = 0 \qquad \dots(51)$$

$$K(v) = vu(x,0) + vK\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + vK\{u(x,t)\} \qquad \dots(52)$$

Using the inverse Kamal- He's of Eqn. (52) and applying the initial condition, we obtain

$$u(x,t) = 1 + \sin x + K^{-1}\left[vK\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] + K^{-1}[vK\{u(x,t)\}] \qquad \dots(53)$$

Substituting Eqns. (47) and (48) into Eqn. (53), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + \sin x + K^{-1}\left[vK\left\{\frac{\partial^2 \sum_{n=0}^{\infty} u_n(x,t)}{\partial x^2}\right\}\right] + K^{-1}[vK\{u_n(x,t)\}] \dots(54)$$

So, the recursive relation is obtained from the solution of the Eqn. (54) using the Kamal-He's Iterative method as follows

$$u_0(x,t) = 1 + \sin x \qquad \dots(55)$$

$$u_{n+1}(x,t) = K^{-1}\left[vK\left\{\sum_{n=0}^{\infty} u_{xxn}(x,t)\right\}\right] + K^{-1}[vK\{u_n(x,t)\}], \qquad n = 0,1,2\dots\dots(56)$$

$$\begin{cases} u_1(x,t) = K^{-1}\left[vK\left\{u_{xx,0}(x,t)\right\}\right] + K^{-1}[vK\{u_0(x,t)\}] = t \\ u_2(x,t) = K^{-1}\left[vK\left\{u_{xx,1}(x,t)\right\}\right] + K^{-1}[vK\{u_1(x,t)\}] = \frac{t^2}{2!} \\ u_3(x,t) = \frac{t^3}{3!} \\ u_4(x,t) = \frac{t^4}{4!} \\ u_5(x,t) = \frac{t^5}{5!} \\ u_6(x,t) = \frac{t^6}{6!} \\ \dots \end{cases} \qquad \dots(57)$$

The result is

$$u(x,t) = 1 + \sin x + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \dots \qquad \dots(58)$$

Example 7: Consider the nonlinear non-homogeneous Klein-Gordon equations as follows [(Kumar, 2014); (Fang *et al.*, 2022)]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = x^2 t^2 \qquad \dots(59)$$

Subject to the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \quad \ldots(60)$$

The exact solution is

$$u(x,t) = xt \quad \ldots(61)$$

Applying the Kamal-He's of Eqn. (59), we get

$$\frac{K(v)}{v^2} - \frac{u(x,0)}{v} - u_t(x,0) - K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} + K\{u^2(x,t)\} = K\{x^2t^2\} \quad \ldots(62)$$

$$K(v) = xv^2 + v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} - v^2K\{u^2(x,t)\} + v^2K\{x^2t^2\} \ldots(63)$$

Using the inverse Kamal- He's of Eqn. (63) and applying the initial condition, we obtain

$$u(x,t) = K^{-1}[xv^2] + K^{-1}\left[v^2K\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\}\right] - K^{-1}[v^2K\{u^2(x,t)\}] + K^{-1}[v^2K\{x^2t^2\}] \quad \ldots(64)$$

Substituting Eqns. (47) and (48) into Eqn. (64), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + K^{-1}[v^2K\{u_{xxn}(x,t)\}] - K^{-1}[v^2K\{H_n(x,t)\}] + K^{-1}[v^2K\{x^2t^2\}] \quad \ldots(65)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{x^2t^4}{12} + K^{-1}[v^2K\{u_{xxn}(x,t)\}] - K^{-1}[v^2K\{H_n(x,t)\}] \ldots(66)$$

So, the recursive relation is obtained from the solution of the Eqn. (94) using the Kamal-He's Iterative method as follows

$$u_0(x,t) = xt \quad \ldots(67)$$

$$u_{n+1}(x,t) = \frac{x^2t^4}{12} + K^{-1}[v^2K\{u_{xxn}(x,t)\}] - K^{-1}[v^2K\{H_n(x,t)\}], n = 0,1,2\ldots.. (68)$$

$$\begin{cases} u_1(x,t) = \frac{x^2t^4}{12} + K^{-1}\left[v^2K\left\{u_{xx,0}(x,t)\right\}\right] - K^{-1}[v^2K\{H_0(x,t)\}] = \frac{x^2t^4}{12} - \frac{x^2t^4}{12} = 0 \\ u_2(x,t) = 0 \\ u_3(x,t) = 0 \\ u_4(x,t) = 0 \\ u_5(x,t) = 0 \\ u_6(x,t) = 0 \\ \ldots \end{cases} \quad \ldots(69)$$

The result is

$$u(x,t) = xt \qquad \ldots(70)$$

The Kamal transform method has been shown to successfully handle linear differential equations, providing efficient and exact solutions for linear ordinary and partial differential equations (Muhammad et al., 2023; Owolabi & Oderinu, 2021; Waqas, 2022). However, its direct application to nonlinear differential equations is limited due to the complexity of nonlinear operators. To overcome this challenge, the Kamal transform has been combined with polynomial decomposition techniques such as Hes and Adomian polynomials to effectively approximate solutions to nonlinear and fractional differential equations (Mikail, 2023; Owolabi & Oderinu, 2021). These polynomial methods, while powerful, can be computationally cumbersome and difficult to generate automatically (Mikail, 2023). Furthermore, standard analytical techniques like integration by parts, which facilitate the application of the Kamal transform in linear cases, fail when fractional differential and integral operators are involved, thus limiting the transform's straightforward use in fractional calculus problems (Mikail, 2023).

In recognition of these challenges, our forthcoming seminar aims to tackle the second objective of this research: handling nonlinearities without relying on polynomial decompositions, thereby potentially reducing computational burdens and improving efficiency, as supported by critiques in recent literature. This next step will contribute to advancing the practical utility of the Kamal transform framework for a wider spectrum of differential equation problems.

CONCLUSION

This study has provided a comprehensive overview of integral transformation methods for solving physical problems, with particular emphasis on the Kamal transform and its hybrid applications. The findings indicate that integral transforms are highly effective for linear problems, offering accurate and computationally efficient solutions.

For nonlinear and fractional differential equations, hybrid methods combining integral transforms with polynomial decomposition techniques significantly enhance solution capability. However, these methods come with increased computational demands and practical challenges.

Future research should focus on developing improved techniques that can handle nonlinearities without relying heavily on polynomial expansions. Such advancements would enhance the efficiency and applicability of integral transformation methods in solving a broader class of physical problems.

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