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Some Entropies Derivation for Entropy Transformed Exponential Distribution with Application to Health Data

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Abstract

This study aims to estimate and comparatively evaluate the performance of four entropy measures—Havrda–Charvat, Kapur, Verma, and Mathai–Haubold—in modeling newborn weight. A quantitative approach was adopted through analytical derivations and Monte Carlo simulation techniques. The performance of each entropy measure was assessed across varying sample sizes using bias, mean squared error (MSE), and root mean squared error (RMSE) as evaluation criteria. The findings indicate that the Havrda–Charvat entropy measure demonstrates superior accuracy, consistency, and convergence toward the true entropy values, thereby exhibiting robust performance under the entropy-transformed exponential distribution (ETED). These results contribute to the theoretical development of entropy-based modeling by extending current understanding of estimator performance within ETED and providing comparative evidence on the suitability of alternative entropy measures for newborn weight modeling.

Keywords: Entropy Measures; Entropy-Transformed Exponential Distribution; Havrda–Charvat Entropy; Monte Carlo Simulation; Newborn Weight

INTRODUCTION

Uncertainty and variability are inherent characteristics of real-life phenomena, particularly in health, engineering, and survival-related data. Researchers have increasingly relied on entropy-based measures to quantify uncertainty in probabilistic systems, with applications spanning reliability engineering, survival analysis, epidemiology, and biomedical sciences (Naz Sindhu, 2024). In health sciences, entropy has proven especially useful for modeling complex biological processes such as neonatal outcomes, disease progression, and physiological signals, where classical distributional assumptions often fail to capture underlying variability (Tekle *et al.*, 2023).

In response to these challenges, statisticians have proposed flexible probability distributions capable of better capturing uncertainty and tail behavior in lifetime and health-related data. One such development is the one-parameter entropy-transformed exponential distribution (ETED), which extends the classical exponential distribution through entropy transformation to improve flexibility and model adequacy (Mathew *et al.*, 2024). According to Mathew *et al.* (2024), the entropy transformation framework enhances the ability of traditional distributions to represent complex stochastic behavior commonly observed in survival and reliability data. Experts in information theory further argue that entropy-based transformations provide a theoretically grounded mechanism for embedding uncertainty directly into distributional structures, thereby improving inferential performance (Hossam *et al.*, 2022).

Previous empirical and theoretical studies demonstrate substantial interest in deriving and estimating entropy measures for newly proposed or modified lifetime distributions. For instance, Baji´c *et al.* (2021) investigated multiple entropy measures, including sample entropy and permutation entropy, in analyzing cardiovascular signals related to COVID-19. Similarly, Kumar *et al.* (2024) examined Shannon entropy estimation for the Maxwell lifetime model under progressive censoring schemes using both classical and Bayesian approaches. Tsallis entropy was derived out by Jamal *et al.* (2021) for a proposed power Ailamujia distribution. Other studies have derived residual entropy, Rényi entropy, Tsallis entropy, Verma entropy, and Mathai–Haubold entropy for distributions such as the Lomax, Fréchet, Weibull, finite-range, and exponentiated transformed Gumbel distributions (Ijaz *et al.*, 2019; Adubisi *et al.*, 2023; Sindhu *et al.*, 2021; Athira & Jeevanand, 2021; Kayal *et al.*, 2015). These studies collectively underscore the relevance of entropy measures in characterizing uncertainty and information content in lifetime models.

Despite these contributions, a critical gap remains in the literature. While many studies derive one or two entropy measures for proposed distributions, only a limited number conduct comprehensive simulation studies to assess estimator performance, convergence behavior, and robustness across varying sample sizes. Moreover, real-life health data applications particularly, neonatal data are often absent or few, limiting the practical interpretability of the derived entropy measures (Visnovcova *et al.*, 2022).

Notably, no known study has systematically examined and compared multiple entropy measures under the ETED distribution using both simulation techniques and real neonatal health data. This gap is significant, given that theoretical derivations alone are insufficient to establish estimator reliability without empirical validation. This study advances entropy modeling through the comparative evaluation of four entropy measures Havrda–Charvat, Kapur, Verma, and Mathai–Haubold.

METHODS

This section presents the derivation of the information generating function and four entropy measures namely: (H-CEM, KEM, VEM, and M-HEM) for the entropy transformed exponential distribution (ETED).

The IGF Derivation

The IGF for a ETED is derived as follows

$$E_v(\sigma) = \int_0^{\infty} [f(z,\sigma)]^v \, dz \tag{1}$$

where, f (z, σ) is the PDF of the ETED and v is a constant parameter.

The PDF of the ETED is given as

$$f(z,\sigma) = \sigma^2 z e^{-\sigma z} \tag{2}$$

Therefore, equation (1) can be derived as follows

$$E_v(\sigma) = \int_0^{\infty} \left(\sigma^2 z e^{-\sigma z}\right)^v dz = \int_0^{\infty} \left(\sigma^{2v} z^v e^{-\sigma z v}\right) dz = \sigma^{2v} \int_0^{\infty} z^v e^{-\sigma z v} dz \tag{3}$$

Let $y=\sigma z v$, $z=\dfrac{y}{\sigma v}$, $\dfrac{dy}{dz}=\sigma v$ and $dz=\dfrac{dy}{\sigma v}$. Substituting these expressions into equation (3) gives

$$E_v(\sigma)=\sigma^{2v}\int_0^\infty\left(\frac{y}{\sigma v}\right)^v e^{-y}\frac{dy}{\sigma v}$$

$$=\sigma^{2v}\int_0^\infty y^v(\sigma v)^{-v}(\sigma v)^{-1}e^{-y}dy$$

$$=\sigma^{2v}\int_0^\infty y^v(\sigma v)^{-v-1}e^{-y}dy$$

$$=\sigma^{2v}(\sigma v)^{-(v+1)}\int_0^\infty y^v e^{-y}dy$$

$$E_v(\sigma)=\frac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}} \qquad (4)$$

Measures of Entropy for an ETED

In this section, four entropy measures are derived for ETED. These entropy measures include: H-CEM, KEM, VEM, and M-HEM.

Havrda-Charvat Entropy Measure (H-CEM)

The H-CEM is defined as:

$$HC_v=\frac{1}{2^{1-v}-1}\left[\left(E_v(\sigma)\right)^{\frac{1}{v}}-1\right],\ v>0,\ v\neq 1. \qquad (5)$$

Substituting equation (4) into equation (5), the H-CEM for a ETED is obtained as follows

$$HC_v=\frac{1}{2^{1-v}-1}\left[\left(\frac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}}\right)^{1/v}-1\right] \qquad (6)$$

Kapur Entropy Meassure (KEM)

The KEM is defined as:

$$K_{\lambda,v}=\frac{1}{\lambda-v}\log\left[\frac{E_v(\sigma)}{E_\lambda(\sigma)}\right],\ v,\lambda>0,\ v,\lambda\neq 1,\ v\neq\lambda. \qquad (7)$$

Substituting equation (4) into equation (7), the KEM for a ETED is obtained as follows

$$K_{\lambda v} = \frac{1}{\lambda - v}\log\left[\frac{\left(\dfrac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}}\right)}{\left(\dfrac{\sigma^{2\lambda}\Gamma(\lambda+1)}{(\sigma\lambda)^{\lambda+1}}\right)}\right] \tag{8}$$

$$K_{\lambda v} = \log\left(\frac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}} X \frac{(\sigma\lambda)^{\lambda-1}}{\sigma^{2\lambda}\Gamma(\lambda-1)}\right) \tag{9}$$

Verma Entropy Measure (VEM)

The VEM is defined as:

$$V_{v,\lambda} = \frac{1}{v-\lambda}\log\left[E(\sigma)\right],\ v-1<\lambda<v,\ v<1,\ v\neq 1. \tag{10}$$

Substituting equation (4) into equation (19), the VEM for a ETED is obtained as follows

$$V_{v,\lambda} = \frac{1}{v-\lambda}\log\left(\frac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}}\right) \tag{11}$$

Mathai-Houbold Entropy Measure (M-HEM)

The M-HEM is defined as:

$$MH_v = \frac{1}{1-v}\left[E(\sigma)\right],\ v>0,\ v\neq 1. \tag{12}$$

Substituting equation (4) into equation (12), the M-HEM for a ETED is obtained as follows

$$MH_v = \frac{1}{1-v}\left(\frac{\sigma^{2v}\Gamma(v+1)}{(\sigma v)^{v+1}}\right) \tag{13}$$

RESULTS AND DISCUSSION

Numerical Simulation for Entropy Measure of an ETED

This section presents the stability evaluation result for ETED, to examine the stability and numerical behavior of the entropies estimate the simulations study were conducted across several sample size by using performance matrix such as the bias, MSE, and RSME. Graphical plots of MSE versus sample size and RMSE versus sample size are represented.

TABLE 1: Simulated Havrda- Charvat Entropy Measure (H-CEM) Estimates

Sample Size	Entropy	True Entropy	Estimated Entropy	Bias	MSE	RMSE
20	H-CEM	0.25	0.23697	-0.01303	0.04066	0.20165
50	H-CEM	0.25	0.25816	0.00816	0.01439	0.11996
100	H-CEM	0.25	0.25636	0.00636	0.00802	0.08958
200	H-CEM	0.25	0.26679	0.01679	0.00392	0.06258
500	H-CEM	0.25	0.26507	0.01507	0.00178	0.04218
750	H-CEM	0.25	0.26566	0.01566	0.00127	0.03569
1000	H-CEM	0.25	0.26797	0.01797	0.00109	0.03296
1500	H-CEM	0.25	0.26690	0.01690	0.00079	0.02820

Table 1 shows the simulation result for H-CEM, it can be observed that the estimated entropy values converge steadily towards the true entropy value as the sample size grows. From the Table 1, the most improved accuracy occurs where the sample size is low that is, the first three sample size in the table. This implies that moderate sample size may provide an accurate entropy estimate which is valuable in estimating entropy in birth weight data, where sample size may be limited. The steady decrease in MSE and RMSE indicate improved and consistent estimators which give a better accuracy with large sample data.

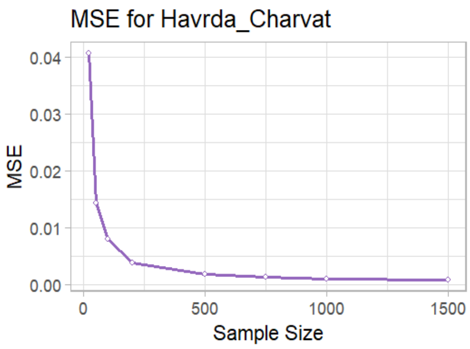

Figure 1a: MSE plot for H-CEM

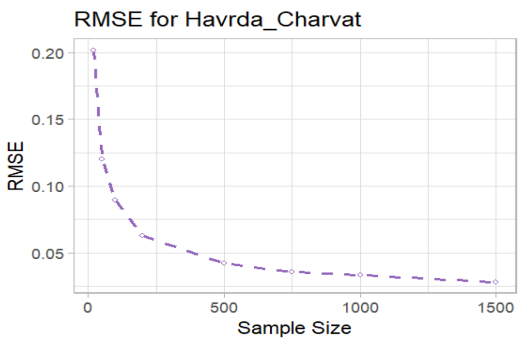

Figure 1b: RMSE for H-CEM

The simulation results for the KEM presented in Table 2 below. The findings reveal that KEM consistently overestimates the true entropy across all sample sizes, exhibiting a high level of bias that remains unaffected by changes in sample size. Although the estimated entropy values appear stable, the absence of noticeable improvement in both MSE and RMSE as the sample size increases suggests persistent overestimation

TABLE 2: Simulated Kapur Entropy Measure (KEM) Estimates

Sample Size	Entropy	True Entropy	Estimated Entropy	Bias	MSE	RMSE
20	KEM	0.25	0.40266	0.15266	0.02564	0.16011
50	KEM	0.25	0.40482	0.15482	0.02485	0.15764
100	KEM	0.25	0.40357	0.15357	0.02407	0.15513
200	KEM	0.25	0.40564	0.15564	0.02445	0.15636
500	KEM	0.25	0.40494	0.15494	0.02410	0.15525
750	KEM	0.25	0.40502	0.15502	0.02410	0.15523
1000	KEM	0.25	0.40557	0.15557	0.02425	0.15572
1500	KEM	0.25	0.40527	0.15527	0.02414	0.15537

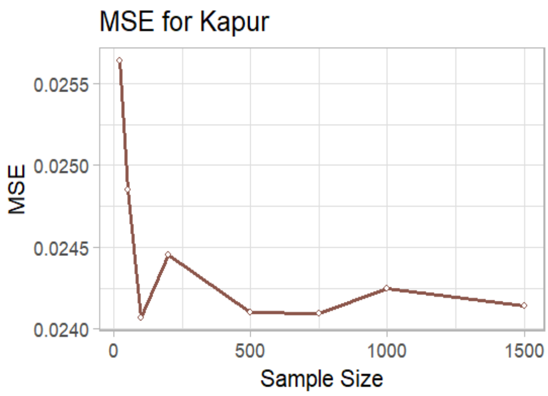

Figure 2a: MSE Plot for KEM

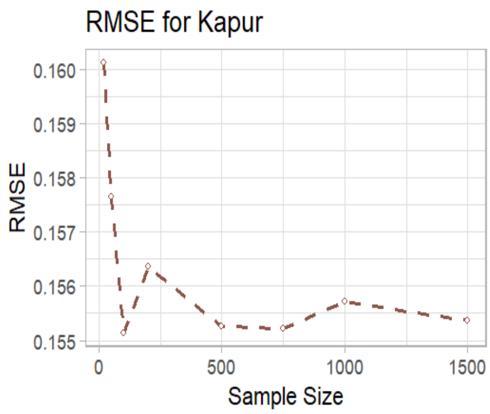

Figure 2b: RMSE Plot for KEM

TABLE 3: Simulated M-HEM Estimates

Sample Size	Entropy	True Entropy	Estimated Entropy	Bias	MSE	RMSE
20	M-HEM	0.25	-0.7872	-1.0372	1.10919	1.05318
50	M-HEM	0.25	-0.76208	-1.01208	1.03535	1.01752
100	M-HEM	0.25	-0.76206	-1.01206	1.03045	1.01511
200	M-HEM	0.25	-0.75191	-1.00191	1.00657	1.00328
500	M-HEM	0.25	-0.75288	-1.00288	1.00694	1.00346
750	M-HEM	0.25	-0.75224	-1.00224	1.00527	1.00263
1000	M-HEM	0.25	-0.75017	-1.00017	1.00092	1.00046
1500	M-HEM	0.25	-0.75104	-1.00104	1.00246	1.00123

The simulation results for the M-HEM, as shown in Table 3, the estimated entropy consistently produces large negative values, across all the sample size. These highlight a significant deviation from the true entropy values. Such consistently large negative estimates, along with a mean square error (MSE) and root mean square error (RMSE) of 1, highlight the poor performance of this estimator and its incompatibility with the ETED in this context.

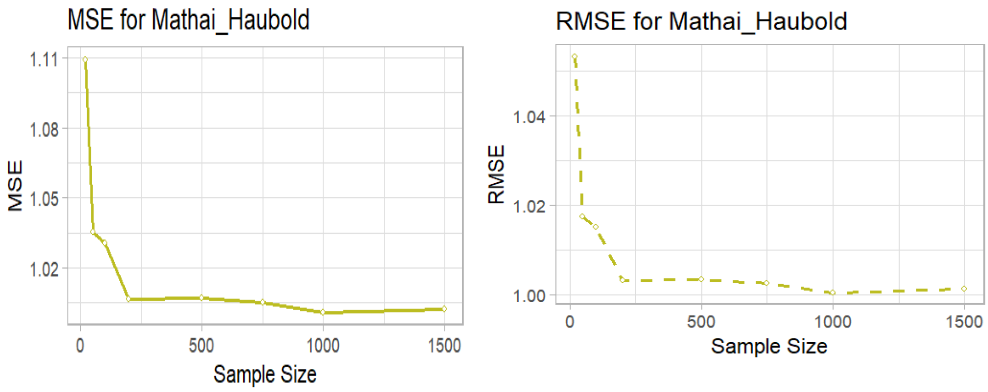

Figure 3a: MSE Plot for M-HEM   Figure 3b: RMSE Plot for M-HEM

Table 4: Simulated Verma Entropy Measure (VEM) Estimates

Sample Size	Entropy	True Entropy	Estimated Entropy	Bias	MSE	RMSE
20	VEM	0.25	-0.17669	-0.42669	0.20472	0.45246
50	VEM	0.25	-0.18742	-0.43742	0.19971	0.44689
100	VEM	0.25	-0.18464	-0.43464	0.19353	0.43993
200	VEM	0.25	-0.1917	-0.4417	0.19725	0.44413
500	VEM	0.25	-0.18992	-0.43992	0.19444	0.44096
750	VEM	0.25	-0.19025	-0.44025	0.19443	0.44094
1000	VEM	0.25	-0.19197	-0.44197	0.19579	0.44248
1500	VEM	0.25	-0.19109	-0.44109	0.19487	0.44144

Table 4 presents the simulation results for VEM, this uncertainty measure persistently produces negative entropy estimates. These estimates indicate deviance from the true entropy value. The persistently high MSE and RMSE values, along with the lack of responsiveness to larger sample sizes, suggest that VEM is poorly suited for the ETED.

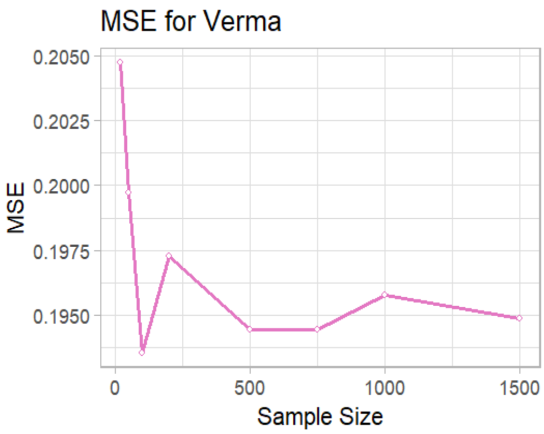

Figure 4a: MSE Plot for VEM

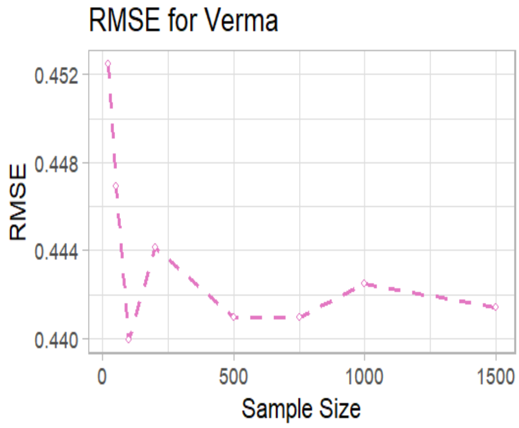

Figure 4b: RMSE Plot for VEM

Simulation Result Discussion

The performance evaluation of the EM based on the simulation study revealed that the H-CEM demonstrated superior accuracy, consistency and efficiency. However, EM like KE consistently overestimated the true entropy value and were associated with high MSE and RMSE, this behavior suggests lack of accuracy, consistency, and efficiency. Similarly, M-HEM and VEM highlighted poor performance by underestimated the true entropy, often producing negative values and maintaining high MSE and RMSE even at larger sample sizes which suggest instability and unsuitability under this context.

Application to Real Life health Data Estimation

Table 5 presents the H-CEM, KEM, M-HEM and VEM for new born weights estimate across increasing values of the parameter *c*. These measures assess the overall uncertainty in the distribution of birth weights with the estimated value of θ given as 0.71988.

Table 5: Estimated Entropies for New Born Weights at Delivery

c-Value	H-CEM	KEM	VEM	M-HEM
1.1	2.34116	1.87581	-1.87581	8.28962
1.3	1.83165	1.77956	-5.47851	1.92730
1.5	1.53209	1.71173	-8.93317	0.81859
1.7	1.33803	1.66146	-12.27947	0.41842
1.9	1.20383	1.62274	-15.54313	0.2348 2
2.1	1.10660	1.59204	-18.74172	0.13953
2.3	1.03371	1.56710	-21.88778	0.08620
2.5	0.97762	1.54646	-24.99055	0.05478
2.7	0.93359	1.52909	-28.05705	0.03557
2.9	0.89847	1.51427	-31.09272	0.02349

Comparative Analysis of the Real Data Estimation Result for ETED

The comparative evaluation of new born weights estimation results using standard entropy measures, as presented in Table 5, reveals a consistent trend among entropy estimators. Measures such as H-CEM and KEM, shows a decreasing pattern in estimates values as the parameter c increases. This smooth and interpretable trend across increasing values of c indicates the potential of the H-CEM and KEM entropy measures as reliable estimators for new born weight. In contrast, VEM consistently yields negative values across all c values, suggesting instability in this context. Additionally, M-HEM begin with relatively high estimates of 8.28962 at c (1.1) but sharply drop to approximately 0.024 by the end of the range c (2.9). This rapid decline, indicates instability and questions the reliability of these estimators for estimating new born weights.

Discussion of Results

The application of entropy for new born weight data shows patterns of variability and uncertainty. H-CEM which emerges with superior performance in the stability evaluation also delivered interpretable and stable results in modeling new born weights. From the estimated standard entropies for new born weight measure like H-CEM and KEM estimate values reduces as c increases. These reductions in the estimated value as c value grows indicate that the uncertainty within birth weight distribution decline. At the highest value of parameter c, the uncertainty estimators that behaved well possess an interpretable estimate value, which demonstrated that, birth weights are becoming less uncertain (easier to predict as c value grows).

CONCLUSION

In conclusion, this study successfully estimated and examine the stability of four entropy measures for the ETED, both through simulation using mean squared error (MSE) and root mean squared error (RMSE) and further estimate this measure using real-life data on new born weight at delivery. These estimations provided valuable insights into the degree of uncertainty, variability, and distributional spread of the data evaluated across varying values of the parameter c (1.1 to 2.9). The simulation results revealed that H-CEM demonstrated high levels of accuracy, efficiency and reliability compared to others. The findings for real-life datasets on new born weights data validate the simulation finding. In contrast, measures such as M-HEM, and VEM demonstrate instability due to their lack of accuracy, efficiency and consistency under the ETED in both contexts.

REFERENCES

Adubisi, O. D., David, I. J., Adubisi, C. E., & Oboh, S. (2023). The extended odd Fréchet inverted exponential distribution: Theoretical properties and application. *Journal of Statistical Modelling and Analytics, 5*(1), 1–24. https://doi.org/10.22452/josma.vol5no1.1

Ahmad, M. A. M. (2019). An estimation of the entropy for a Fréchet distribution based on generalized hybrid censored samples. *International Journal of Sciences: Basic and Applied Research, 48*(7), 226–236. https://gssrr.org/JournalOfBasicAndApplied/article/view/10647

Alomani, G., & Kayid, M. (2023). Further properties of Tsallis entropy and its application. *Entropy, 25*(2), Article 199. https://doi.org/10.3390/e25020199

Athira, N. R., & Jeevanand, E. S. (2021). Estimation of residual entropy function of finite range distribution using record values. *International Journal of Statistics and Applied Mathematics, 6*(6), 1–5. https://www.mathsjournal.com/pdf/2021/vol6issue6/PartA/6-5-29-325.pdf

Bajić, D., Đajić, V., & Milovanović, B. (2021). Entropy analysis of COVID-19 cardiovascular signals. *Entropy, 23*(1), Article 87. https://doi.org/10.3390/e23010087

Hossam, E., Abdulrahman, A., Gemeay, A. M., Alshammari, N., Alshawarbeh, E., & Mashaqbah, N. K. (2022). A novel extension of Gumbel distribution: Statistical inference with COVID-19 application. *Alexandria Engineering Journal, 61*(11), 8823–8842. https://doi.org/10.1016/j.aej.2022.01.071

Ijaz, M., Asim, S. M., & Alamgir. (2019). Lomax exponential distribution with an application to real-life data. *PLOS ONE, 14*(12), Article e0225827. https://doi.org/10.1371/journal.pone.0225827

Jamal, F., Chesneau, C., Aidi, K., & Ali, A. (2021). Theory and application of the power Ailamujia distribution. *Journal of Mathematical Modeling, 9*(3), 391–413. https://doi.org/10.22124/jmm.2020.17547.1512

Kayal, S., Kumar, S., & Vellaisamy, P. (2015). Estimating the Rényi entropy of several exponential populations. *Brazilian Journal of Probability and Statistics, 29*(1), 94–111. https://doi.org/10.1214/13-BJPS230

Kumar, K., Kumar, I., & Ng, H. K. T. (2024). On estimation of Shannon's entropy of Maxwell distribution based on progressively first-failure censored data. *Stats, 7*(1), 138–159. https://doi.org/10.3390/stats7010009

Mahmoud, M. R., Ahmad, M. A. M., & Mohamed, B. S. K. (2021). Estimating the entropy and residual entropy of a Lomax distribution under generalized Type-II hybrid censoring. *Mathematics and Statistics, 9*(5), 780–791. https://doi.org/10.13189/ms.2021.090518

Maya, R., Abdul-Sathar, E. I., & Rajesh, G. (2014). Non-parametric estimation of the generalized past entropy function with censored dependent data. *Statistics & Probability Letters, 90*, 129–135. https://doi.org/10.1016/j.spl.2014.03.012

Sindhu, T. N., Shafiq, A., & Al-Mdallal, Q. M. (2021). Exponentiated transformation of Gumbel Type-II distribution for modeling COVID-19 data. *Alexandria Engineering Journal, 60*(1), 671–689. https://doi.org/10.1016/j.aej.2020.09.060

Sindhu, T. N., Shafiq, A., Lone, S. A., Al-Mdallal, Q. M., & Abushal, T. A. (2024). Distributional properties of the entropy transformed Weibull distribution and applications to various scientific fields. *Scientific Reports, 14*, Article 31827. https://doi.org/10.1038/s41598-024-83132-w

Stephen, M., John, D. I., & Mutah, Y. (2024). A study on one-parameter entropy-transformed exponential distribution and its application. *Journal of Reliability and Statistical Studies, 17*(1), 17–44. https://doi.org/10.13052/jrss0974-8024.1712

Tekle, G., Roozegar, R., & Ahmad, Z. (2023). A new Type 1 alpha power family of distributions and modeling data with correlation, overdispersion, and zero-inflation in the health data sets. *Journal of Probability and Statistics, 2023*, Article 6611108. https://doi.org/10.1155/2023/6611108

Visnovcova, Z., Kozar, M., Kuderava, Z., Zibolen, M., Ferencova, N., & Tonhajzerova, I. (2022). Entropy analysis of neonatal electrodermal activity during the first three days after birth. *Entropy, 24*(3), Article 422. https://doi.org/10.3390/e24030422