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Inference and Simulation Study for the Exponentiated Novel α-Power Gumbel Model

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Abstract

This study introduces and investigates a new flexible lifetime model, termed the Exponentiated Novel α-Power Gumbel (ENAPG) distribution, by applying the exponentiation technique to the recently proposed novel α-power Gumbel model. The proposed distribution extends the classical Gumbel family through the inclusion of an additional shape parameter, thereby enhancing its flexibility for modeling right-skewed and heavy-tailed data. To establish its theoretical usefulness, the study derives key statistical properties of the ENAPG distribution, including the survival and hazard rate functions, quantile function, moments, moment-generating function, Rényi and Tsallis entropies, and order statistics. Parameter estimation is carried out using the maximum likelihood estimation approach, with the resulting nonlinear likelihood equations solved numerically through iterative optimization routines. A comprehensive Monte Carlo simulation is further conducted to assess the finite-sample performance of the estimators across different sample sizes using bias, mean square error, root mean square error, and mean relative error criteria. The results indicate that the maximum likelihood estimators exhibit consistency and improved efficiency as sample size increases. Overall, the ENAPG distribution provides

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a robust and flexible alternative to existing Gumbel-type models and offers potential applications in reliability analysis, survival studies, and extreme-value modeling.

Keywords: Exponentiated Distribution; α-Power Transformation; Gumbel Distribution; Maximum Likelihood Estimation; Reliability Analysis

INTRODUCTION

In applied statistical modeling, the development of generalized continuous probability distributions remains an active and important area of research due to the limitations of classical models in capturing various data features such as heavy tails, skewness, and varied hazard rate behaviors. Standard models often lack sufficient flexibility to provide adequate fits for diverse real-world data, particularly in reliability, survival analysis, and extreme value applications. Current research has therefore focused on generating more flexible families of distributions using structural transformation techniques and additional shape parameters to better accommodate diverse empirical phenomena (ElSherpieny & Almetwally, 2022; Hurairah and Almazaqi, 2024; Uuwadi et al., 2024; Klakattawi, 2025). One widely adopted strategy for enhancing flexibility is the exponentiation method, in which an additional shape parameter is introduced to extend the tail behavior and curvature properties of a distribution. Such exponentiated families have been shown to improve goodness-of-fit and adaptability for real data across numerous contexts (Chen et al., 2024; Chipepa et al., 2025). Recent work on exponentiated transformation methods has underscored both theoretical and practical strengths of these techniques, including the derivation of quantile functions, hazard rate shapes, and entropic measures that are not possible with simpler baseline models (George & George, 2025; Alshawarbeh et al., 2025).

A particularly important special case arises in the context of Gumbel-type distributions, which are central to extreme value theory and are used for modeling maxima, minima, and other tail phenomena in hydrology, engineering, and risk assessment. While the classical Gumbel distribution has a long history of application, it exhibits limited flexibility in accounting for various tail behaviors, motivating the creation of extended Gumbel families (Anghel, 2024). Recent research has introduced several modified Gumbel variants, including exponentiated Gumbel-Lomax model, that add shape parameters to accommodate skewed and heavy-tailed data more effectively than existing two-parameter forms (Uuwadi et al., 2024). Despite these advances, there remains scope for constructing more adaptable Gumbel-based models that are both analytically tractable and capable of representing a wider spectrum of hazard rate configurations. The proposed Exponentiated Novel α-Power Gumbel (ENAPG) distribution, introduced in this paper, builds upon these contemporary research developments by applying an exponentiation technique to a base α-power transformed Gumbel model. The resulting distribution integrates additional shape parameters to extend tail flexibility and offers a robust platform for modeling lifetime and extreme-value data.

The remainder of the paper is organized as follows. Section 2 presents the formulation of the ENAPG distribution and its fundamental properties. Section 3 discusses parameter estimation via maximum likelihood. Section 4 reports the results of the simulation study. Section 5 provides concluding remarks and potential directions for future research.

METHODS

Exponentiated novel alpha-power Gumbel model

The exponentiated family of distributions introducted by Gupta et al. (1998) with the cumulative distribution function (CDF) specified as

$$F(x;\theta,\varsigma) = G(x;\varsigma)^{\theta}, \tag{1}$$

with corresponding probability density function (PDF) specified as

$$f(x;\theta,\varsigma) = \theta g(x;\varsigma)G(x;\varsigma)^{\theta-1}. \tag{2}$$

where $\theta > 0$ is the scale parameter and ς denote the vector parameter of the baseline model with CDF and PDF denoted with $G(x;\varsigma)$ and $g(x;\varsigma)$, respectively. The baseline model utilized in this study is the novel alpha-power Gumbel (NAPG) model introduced by Hossam et al. (2022) with CDF given as

$$G(x) = \alpha^{-e^{-\eta x}}, \quad x \in \mathbb{R}, \tag{3}$$

and the corresponding PDF given as

$$g(x) = \eta \log(\alpha) e^{-\eta x} \alpha^{-e^{-\eta x}}. \tag{4}$$

where $\alpha >, \alpha \neq 1$ is the shape parameter and $\eta > 0$ is the scale parameter..

Given that X is a non-negative random variable following the ENAPG distribution with parameters (η, α, θ), the CDF and PDF derived by inserting Equations (3) and (4) into Equations (1) and (2) are respectively, specified as

$$\mathrm{F}(\mathrm{x}; \eta, \alpha, \theta) = \left(\alpha^{-e^{-\eta x}}\right)^{\theta}, \quad \alpha >, \alpha \neq 1, \eta > 0, \mathrm{x} \in \mathbb{R}, \tag{5}$$

$$\mathrm{f}(\mathrm{x}; \eta, \alpha, \theta) = \theta\eta \log(\alpha) e^{-\eta x} \left(\alpha^{-e^{-\eta x}}\right)^{\theta} \tag{6}$$

Figure 1 presents the PDF and CDF plots for the ENAPG model. As graphically depicted, the ENAPG is unimodal and right-skewed. More so, the CDF plot graphically shows that the ENAPG is a valid probability distribution within its domain.

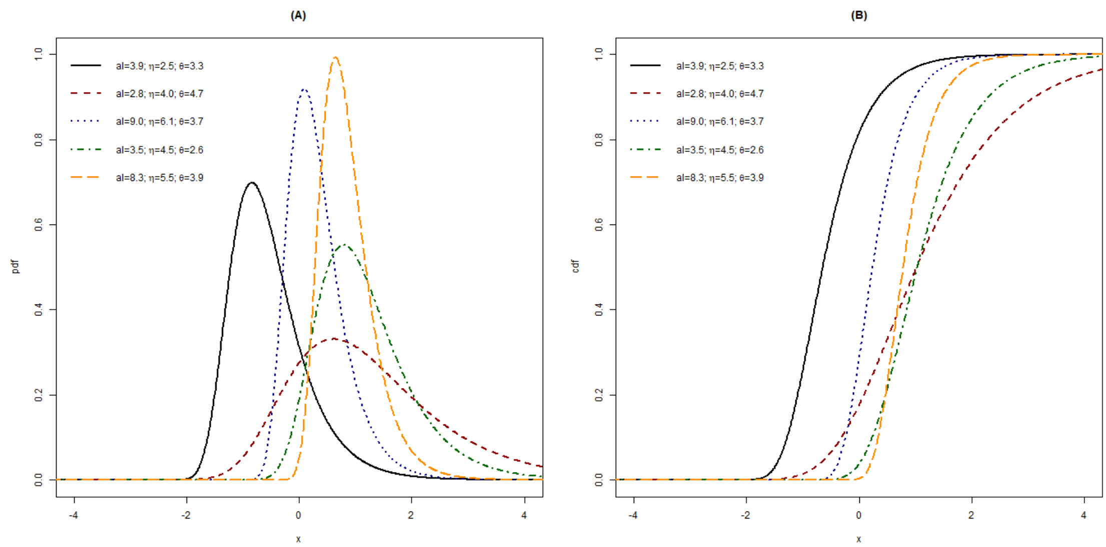

Figure 1: The density (A) and distribution (B) functions plots of the ENAPG model with selected parameter value..

Properties of the ENAPG model

Here, the statistical properties of the ENAPG model are presented.

Reliability Analysis

The survival function $\mathrm{s}(\mathrm{x}; \eta, \alpha, \theta)$ and hazard rate function $\mathrm{h}(\mathrm{x}; \eta, \alpha, \theta)$ of the ENAPG model are specified as, respectively.

$$\mathrm{s}(\mathrm{x}; \eta, \alpha, \theta) = 1 - \left(\alpha^{-e^{-\eta x}}\right)^{\theta} \tag{7}$$

and

$$\mathrm{h}(\mathrm{x}; \eta, \alpha, \theta) = \frac{\theta\eta \log(\alpha) e^{-\eta x} \left(\alpha^{-e^{-\eta x}}\right)^{\theta}}{1 - \left(\alpha^{-e^{-\eta x}}\right)^{\theta}} \tag{8}$$

Also, the cumulative hazard rate $\mathrm{H}(\mathrm{x};\eta,\alpha,\theta)$, reversed hazard rate $\mathrm{R}(\mathrm{x};\eta,\alpha,\theta)$ and odds $\mathrm{O}(\mathrm{x};\eta,\alpha,\theta)$ functions are respectively, specified as

$$\mathrm{H}(\mathrm{x};\eta,\alpha,\theta) = -\log[1-\left(\alpha^{-e^{-\eta x}}\right)^{\theta}] \quad (9)$$

$$\mathrm{R}(\mathrm{x};\eta,\alpha,\theta) = \theta\eta\log(\alpha)e^{-\eta x} \quad (10)$$

and

$$\mathrm{O}(\mathrm{x};\eta,\alpha,\theta) = \frac{\left(\alpha^{-e^{-\eta x}}\right)^{\theta}}{1-\left(\alpha^{-e^{-\eta x}}\right)^{\theta}} \quad (11)$$

The plots of the survival and hazard rate functions are presented in Figure (2). The plots depict an increasing hazard rate function.

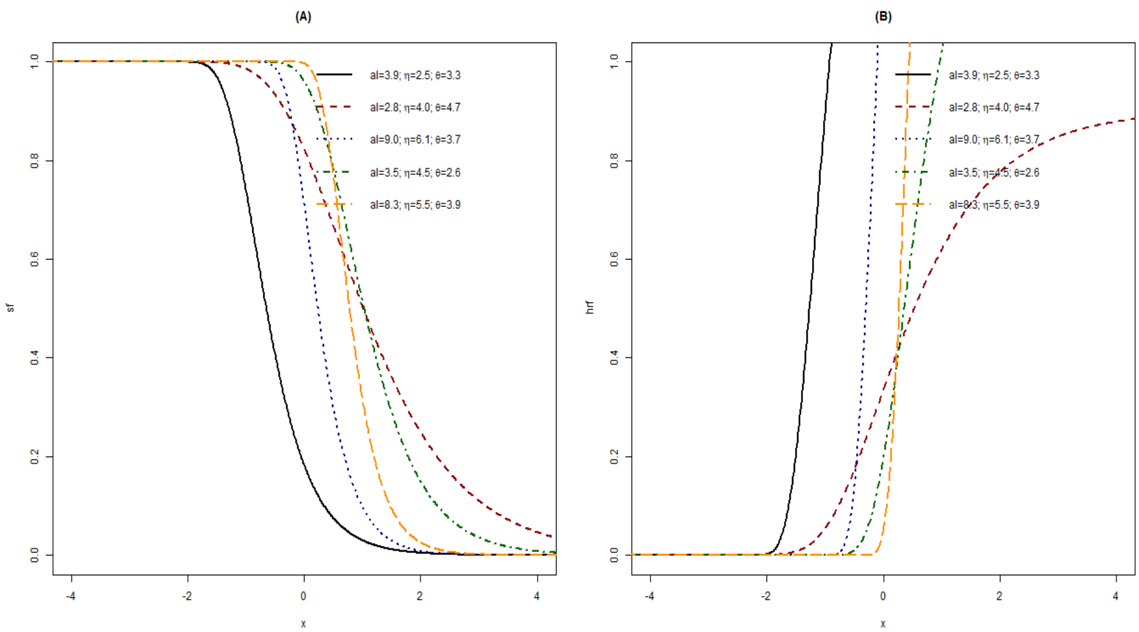

Figure 2: The survival (A) and hazard rate (B) functions plots of the ENAPG model with selected value

Quantile Function

The quantile function, say $\mathrm{Q(u)} = F^{-1}(u)$ of the ENAPG model is obtained by inverting Equation (5). The quantile function is specified as

$$\mathrm{Q(U)} = \frac{1}{\eta}[-\log(\frac{-\log u^{\frac{1}{\theta}}}{\log\alpha})] \quad (12)$$

where $u \sim Uniform(0.1)$. The quantile function is very effective for generating random variables from any continuous probability distribution. Hence, the numerical values of the first four moments, coefficient of variation (CV), standard deviation (SD) and dispersion index (DI) in Table 1 are simulated using the ENAPG quantile function with four selected parameters settings:

$$A: (\eta = 1.5, \alpha = 1.5, \theta = 1.5), B: (\eta = 2.0, \alpha = 2.5, \theta = 2.0),$$

$C: (\eta = 1.8, \alpha = 1.2, \theta = 1.2)$, $D: (\eta = 1.3, \alpha = 1.3, \theta = 2.2)$.

These settings yield varied statistical profiles, reflecting the influence of parameter variations on distribution characteristics. C exhibits the highest maximum value (2.3481) and elevated dispersion metrics (CV = 1.6212, DI = 1.3101), indicating a heavier upper tail and greater sensitivity to extreme values. A and D show higher coefficients of variation (1.7538 and 1.8029), suggesting greater relative variability while B maintains moderate central tendency and dispersion, reflecting a balanced distribution.

Table 1. Numerical values of some useful statistics using the parameter settings

μ_r'	A	B	C	D
μ_1'	0.3513	0.6224	0.4985	0.3873
μ_2'	0.5030	0.7513	0.9015	0.6377
μ_3'	1.0425	1.2409	2.3481	1.5225
μ_4'	2.8306	2.6081	7.9917	4.7610
CV	1.7538	0.9697	1.6212	1.8029
SD	0.6161	0.6035	0.8081	0.6983
DI	**1.0804**	**0.5852**	**1.3101**	**1.2590**

Raw Moments

The r[th] raw moment of the ENAPG model is derived using the density function in Equation (6). Let X ~ ENAPG(η, α, θ) be a random-variable (r.v), thus the r[th] moment is specified as

$$\mu_r' = \int_{-\infty}^{+\infty} x^r \theta \eta \log(\alpha) e^{-\eta x} \left(\alpha^{-e^{-\eta x}}\right)^{\theta} dx \tag{13}$$

Reexpressing Equation (13), we have

$$\mu_r' = \theta \eta \log(\alpha) \int_{-\infty}^{+\infty} x^r e^{-\eta x} e^{-\theta \log(\alpha) e^{-\eta x}} dx \tag{14}$$

Let

$$t = e^{-\eta x} \Rightarrow x = -\frac{1}{\eta} \log t.$$

$$\frac{dt}{dx} = -\eta e^{-\eta x} \Rightarrow dx = \frac{-1}{\eta t} dt.$$

The integration limits, shows that as $x \to -\infty, t \to \infty$ and $x \to +\infty, t \to 0$. Therefore,

$$\mu_r^{'} = \theta\eta\log(\alpha)\int_{\infty}^{0}\left(-\frac{1}{\eta}logt\right)^{r} te^{-\theta\log(\alpha)\mathrm{t}}\left(\frac{-1}{\eta t}\right)dt \tag{15}$$

Simplifying Equation (14) leads to

$$\mu_r^{'} = \frac{\theta\log(\alpha)}{\eta^{r}}\int_{0}^{\infty}(\log t)^{r}e^{-\theta\log(\alpha)\mathrm{t}}dt \tag{16}$$

Utilizing power series expansion, $e^{-z\mathrm{t}} = \sum_{i=0}^{\infty}\frac{(-1)^{i}z^{i}}{i!}t^{i}$, we have

$$\mu_r^{'} = \frac{\theta\log(\alpha)}{\eta^{r}}\sum_{i=0}^{\infty}\frac{(-1)^{i}(\theta\log(\alpha))^{i}}{i!}\int_{0}^{\infty}t^{i}(\log t)^{r}dt \tag{17}$$

Also, utilizing the gamma integral representation, $\int_{0}^{\infty}t^{i}(\log t)^{r}dt = (-1)^{r}\frac{r!}{(i+1)^{r+1}}$. The moment of the ENAPG model is specified as

$$\mu_r^{'} = \frac{(-1)^{r}r!\theta\log(\alpha)}{\eta^{r}}\sum_{i=0}^{\infty}\frac{(-1)^{i}(\theta\log(\alpha))^{i}}{i!(i+1)^{r+1}} \tag{18}$$

Corollary 1: The incomplete moment (IM) of the ENAPG model is derived. This suffices that the r^th^ IM, $\varphi_r^{'}(\mathrm{t})$ is specified as

$$\varphi_r^{'}(\mathrm{t}) = \mathrm{E}(\mathrm{X}^{\mathrm{r}}|\mathrm{X} < \mathrm{t}) = \int_{-\infty}^{\mathrm{t}}\mathrm{x}^{\mathrm{r}}\mathrm{f}(\mathrm{x})\mathrm{dx} \tag{19}$$

Hence, the r^th^ IM of the ENAPG model is specified as

$$\varphi_r^{'}(\mathrm{t}) = \mu_r^{'} = \theta\eta\log(\alpha)\int_{-\infty}^{\mathrm{t}}\mathrm{x}^{\mathrm{r}}e^{-\eta x}e^{-\theta\log(\alpha)e^{-\eta x}}\mathrm{dx} \tag{20}$$

Let

$$z = e^{-\eta x} \Rightarrow x = -\frac{1}{\eta}\log z.$$

$$\frac{dz}{dx} = -\eta e^{-\eta x} \Rightarrow dx = \frac{-1}{\eta z}dz.$$

The limit transform imply that as $x = -\infty \Rightarrow z = \infty$ and $x = t \Rightarrow z = e^{-\eta t}$. Therefore,

$$\varphi_r^{'}(\mathrm{t}) = \theta\eta\log(\alpha)\int_{\infty}^{e^{-\eta t}}\left(-\frac{1}{\eta}logt\right)^{r} te^{-\theta\log(\alpha)\mathrm{t}}\left(\frac{-1}{\eta t}\right)dt \tag{21}$$

Utilizing the steps in Equations (14)-(18) on Equation (21), the IM of the ENAPG model is specified as

$$\varphi_r^{'}(\mathrm{t}) = \frac{(-1)^{r}r!\theta\log(\alpha)}{\eta^{r}}\sum_{i=0}^{\infty}\sum_{m=0}^{r}\frac{(-1)^{i}(\theta\log(\alpha))^{i}(-\eta t)^{m}}{i!m!(i+1)^{r-m+1}} \tag{22}$$

Remark: The first IM $\varphi_1'(t) = \int_0^t xf(x)dx$ of the ENAPG model is obtained by inserting $r = 1$ into Equation (22). The the first IM function is very useful in the derivation of the mean residual life, Lorenz and Benferroni curves, and income inequality for the ENAPG model.

Corollary 2: The moment generating function (MGF) of the ENAPG model is derived. This suffices that the MGF, $M_Y(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$ is specified as

$$M_X(t) = \theta\eta\log(\alpha)\int_{-\infty}^{+\infty} e^{-(\eta-t)x} e^{-\theta\log(\alpha)e^{-\eta x}} dx \quad (23)$$

Utilizing the steps in Equations (14)-(17) on Equation (23), the IM of the ENAPG model is specified as

$$M_X(t) = \theta\log(\alpha)\int_0^{\infty} z^{1-\frac{t}{\eta}} e^{-\theta\log(\alpha)z} dz \quad (24)$$

Also, utilizing the gamma integral representation, $\int_0^{\infty} z^{i-1}e^{-kz}dz = \frac{\Gamma(i)}{k^i}$. The MGF of the ENAPG model is specified as

$$M_X(t) = (\theta\log(\alpha))^{\frac{t}{\eta}}\Gamma\left(1-\frac{t}{\eta}\right), \ \ t < \eta. \quad (25)$$

Entropies

The Rényi entropy which measures the disparity of indecision, $R_\delta(X) = \frac{1}{1-\delta}\log(\int_{-\infty}^{+\infty} f(x)^\delta dx)$, $\delta > 0$ and $\delta \neq 1$ is specified as

$$R_\delta(X) = \frac{1}{1-\delta}\log(\int_{-\infty}^{+\infty}\left(\theta\eta\log(\alpha)e^{-\eta x}e^{-\theta\log(\alpha)e^{-\eta x}}\right)^\delta dx), \quad (26)$$

Utilizing the steps in Equations (14)-(17) on Equation (26), we have

$$f(x)^\delta = \frac{(\theta\log(\alpha))^\delta}{\eta}\int_0^{\infty} z^{\delta-1}e^{-\delta\theta\log(\alpha)z}dz \quad (27)$$

Also, utilizing the gamma integral representation, $\int_0^{\infty} z^{i-1}e^{-kz}dz = \frac{\Gamma(i)}{k^i}$. Therefore, $f(x)^\delta$ is specified as

$$f(x)^\delta = \frac{(\theta\,\eta\log(\alpha))^\delta}{\eta}\frac{\Gamma(\delta)}{(\delta\theta\log(\alpha))^\delta} = \frac{(\eta)^{\delta-1}\Gamma(\delta)}{(\delta)^\delta} \quad (28)$$

Therefore, the Rényi entropy of the ENAPG model is specified as

$$R_\delta(X) = \frac{1}{1-\delta}\log\left(\frac{(\eta)^{\delta-1}\Gamma(\delta)}{(\delta)^\delta}\right) \quad (29)$$

More so, the Tsallis q-entropy is expressed as

$$H_\delta(X) = \frac{1}{\delta-1}\left(1 - \left[\int_{-\infty}^{+\infty} f(x)^\delta dx\right]\right), \delta > 0 \text{ and } \delta \neq 1 \qquad (30)$$

Using the steps in Equations (26)-(29), the Tsallis-entropy of the ENAPG model is given as:

$$R_\delta(X) = \frac{1}{\delta-1}\left(1 - \frac{(\eta)^{\delta-1}\Gamma(\delta)}{(\delta)^\delta}\right) \qquad (31)$$

Order Statistics

The r[th] order statistic $X_{r:n}$ with random sample $X_1, X_2, \dots, X_n$ and order statistics $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$, is specified as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1 - F(x)]^{n-r} \qquad (32)$$

Inserting Equations (5) and (6) into Equation (32), the r[th] order statistic for the ENAPG model is specified as

$$f_{r:n}(x) = \theta\eta \log(\alpha) e^{-\eta x}\left(\alpha^{-e^{-\eta x}}\right)^{\theta(1+r)-1}\left[1 - \left(\alpha^{-e^{-\eta x}}\right)^\theta\right]^{n-r} \qquad (33)$$

Remark: The minimum and maximum order statistics is obtained by inserting $r = 1$ and $r = n$ into Equation (33).

Maximum Likelihood Estimation

Here, the parameter estimates of the ENAPG model are derive using the maximum likelihood estimation (MLE) method. Given that $x, x_2, \dots, x_n$ are independent sample from the ENAPG model with density function presented in Equation (6). The likelihood-function of the ENAPG model is specified as

$$L(x; \eta, \alpha, \theta) = \prod_{i=1}^{n} \theta\eta \log(\alpha) e^{-\eta x}\left(\alpha^{-e^{-\eta x}}\right)^\theta \qquad (34)$$

The log-likelihood function, say $\ell(\eta, \alpha, \theta)$ is specified as

$$\ell(\eta, \alpha, \theta) = n \log\theta + n \log\eta + n \log(\log(\alpha)) - \eta \sum_{i=1}^{n} x_i - \theta \log(\alpha) \sum_{i=1}^{n} e^{-\eta x_i}$$

Differentiating the preceeding function with respect to θ, η and α gives the follows:

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \log(\alpha) \sum_{i=1}^{n} e^{-\eta x_i}$$

$$\frac{\partial \ell}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n} x_i + \theta \log(\alpha) \sum_{i=1}^{n} x_i e^{-\eta x_i}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha \log(\alpha)} - \frac{\theta}{\alpha} \sum_{i=1}^{n} e^{-\eta x_i}$$

The ML estimates $\langle \hat{\theta}, \hat{\eta}, \hat{\alpha} \rangle$ can be obtained by solving the following nonlinear functions: $\frac{\partial \ell}{\partial \theta} = 0$, $\frac{\partial \ell}{\partial \eta} = 0$ and $\frac{\partial \ell}{\partial \alpha} = 0$. However, in this study the R-program (*optim function*) is utilized in obtaining the estimates of the ENAPG model parameters.

RESULTS AND DISCUSSION

Simulation-Study

The simulation process for the MLE is presented as follows: we generated N = 10,000 samples from the ENAPG model with sizes n = 20, 50, 150, 300 and 700 using R-programE. The performance of the MLE is assessed using the average estimates (AVEs), bias, mean square errors (MSEs), root mean square roots (RMSEs) and mean relative errors (MREs).

Table 2: Numerical values of AVEs, bias, MSEs, RMSEs, MRE

($\alpha = 1.5, \eta = 1.5, \theta = 1.5$)

Parameter	n	AVE	Bias	MSE	RMSE	MRE
α	20	1.518	0.018	0.023	0.152	0.012
η		1.620	0.120	0.106	0.325	0.080
θ		1.497	-0.003	0.033	0.181	0.002
α	50	1.495	-0.005	0.009	0.095	0.003
η		1.545	0.045	0.032	0.178	0.030
θ		1.526	0.026	0.010	0.099	0.017
α	150	1.486	-0.014	0.003	0.057	0.010
η		1.515	0.015	0.010	0.098	0.010
θ		1.541	0.041	0.005	0.070	0.027
α	300	1.482	-0.018	0.002	0.041	0.012
η		1.509	0.009	0.004	0.067	0.006
θ		1.546	0.046	0.004	0.063	0.031

$\boldsymbol{\alpha}$	700	1.481	-0.019	0.001	0.029	0.013
$\boldsymbol{\eta}$		1.503	0.003	0.002	0.044	0.002
$\boldsymbol{\theta}$		1.550	0.050	0.004	0.060	0.033

$(\alpha = 2.0, \eta = 2.5, \theta = 2.5)$

Parameter	n	AVE	Bias	MSE	RMSE	MRE
$\boldsymbol{\alpha}$	20	2.121	0.121	0.189	0.435	0.061
$\boldsymbol{\eta}$		2.700	0.253	0.293	0.542	0.080
$\boldsymbol{\theta}$		2.598	0.073	0.082	0.287	0.039
$\boldsymbol{\alpha}$	50	2.021	0.021	0.051	0.226	0.010
$\boldsymbol{\eta}$		2.574	0.074	0.088	0.297	0.030
$\boldsymbol{\theta}$		2.569	0.069	0.026	0.163	0.028
$\boldsymbol{\alpha}$	150	1.981	-0.019	0.015	0.123	0.009
$\boldsymbol{\eta}$		2.525	0.025	0.027	0.168	0.010
$\boldsymbol{\theta}$		2.570	0.070	0.015	0.121	0.028
$\boldsymbol{\alpha}$	300	1.974	-0.026	0.007	0.085	0.013
$\boldsymbol{\eta}$		2.514	0.014	0.012	0.114	0.006
$\boldsymbol{\theta}$		2.566	0.066	0.011	0.107	0.026
$\boldsymbol{\alpha}$	700	1.969	-0.031	0.004	0.061	0.016
$\boldsymbol{\eta}$		2.505	0.005	0.005	0.073	0.002
$\boldsymbol{\theta}$		2.566	0.066	0.011	0.103	0.026

$(\alpha = 3.0, \eta = 3.5, \theta = 3.5)$

Parameter	n	AVE	Bias	MSE	RMSE	MRE
$\boldsymbol{\alpha}$	20	3.240	0.240	1.239	1.113	0.080
$\boldsymbol{\eta}$		3.780	0.780	1.105	1.051	0.260
$\boldsymbol{\theta}$		4.011	0.511	0.893	0.945	0.146
$\boldsymbol{\alpha}$	50	3.029	0.029	0.288	0.537	0.010
$\boldsymbol{\eta}$		3.604	0.604	0.527	0.726	0.201
$\boldsymbol{\theta}$		3.742	0.242	0.252	0.502	0.90
$\boldsymbol{\alpha}$	150	3.025	0.025	0.120	0.347	0.008
$\boldsymbol{\eta}$		3.535	0.535	0.337	0.580	0.178
$\boldsymbol{\theta}$		3.572	0.072	0.084	0.290	0.021
$\boldsymbol{\alpha}$	300	3.072	0.072	0.088	0.296	0.024
$\boldsymbol{\eta}$		3.520	0.520	0.294	0.543	0.173
$\boldsymbol{\theta}$		3.485	-0.015	0.042	0.214	0.004
$\boldsymbol{\alpha}$	700	3.098	0.098	0.064	0.254	0.033
$\boldsymbol{\eta}$		3.507	0.507	0.267	0.517	0.169
$\boldsymbol{\theta}$		3.428	-0.072	0.026	0.162	0.020

From Table 2, it is evident that the MLEs exhibit good consistency and stability across all sample sizes. For small samples ($n = 20$), the average estimates deviate slightly from the true parameter values, which is expected due to sampling variability. However, even at this small sample size, the estimators remain reasonably close to the true values, indicating robustness of the estimation procedure. As the sample size increases, a clear improvement in estimator performance is observed. Specifically, the bias decreases steadily, approaching zero for larger sample sizes. This behaviour confirms the asymptotic unbiasedness of the MLEs for the ENAPG parameters. Similarly, the MSE and RMSE values decrease monotonically as $n$increases, demonstrating that the variability of the estimators reduces with increasing information in the sample. Additionally, the mean relative error (MRE) also declines as the sample size grows, further supporting the efficiency of the MLEs. For large samples ($n = 300$ and $n = 700$), the AVEs are almost identical to the true parameter values, and the corresponding MSEs and RMSEs are very small. This confirms the consistency of the estimators and their suitability for practical applications involving moderate to large datasets.

Furthermore, across all parameter settings considered, similar trends are observed, indicating that the performance of the MLEs is not sensitive to specific parameter choices. Overall, the simulation results clearly demonstrate that the maximum likelihood estimation method provides reliable, efficient, and consistent parameter estimates for the ENAPG distribution, even for relatively small sample sizes, with performance improving substantially as the sample size increases.

CONCLUSION

In this paper, a new three-parameter lifetime distribution, termed the Exponentiated Novel α-Power Gumbel (ENAPG) distribution, has been introduced and systematically studied. The proposed model extends the novel α-power Gumbel distribution by incorporating an exponentiation parameter, thereby significantly improving its flexibility in modeling skewed and heavy-tailed data. A wide range of statistical properties, including reliability measures, moments, entropies, order statistics, and the quantile function, were derived to demonstrate the mathematical tractability of the model. Parameter estimation was carried out using the maximum likelihood estimation approach, and a comprehensive Monte Carlo simulation study was conducted to evaluate estimator performance. The simulation results indicate that the MLEs are consistent and efficient, with bias and mean square error decreasing as the sample size increases. These findings confirm the suitability of the ENAPG model for practical statistical inference. Overall, the ENAPG distribution provides a valuable addition to the family of Gumbel-type models and offers a robust alternative for applications in reliability analysis, survival studies, and extreme-value modeling. Future research may explore Bayesian estimation, regression extensions, and real-data applications in engineering, environmental sciences, and finance.

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