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A Study of Some Geometric Structures on Inner Metric Spaces

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Abstract

Inner metric spaces, characterized by the approximate midpoint property, play an important role in metric geometry and in the analysis of length structures on metric spaces. This paper investigates structural properties of inner metric spaces, focusing on the relationship between the inner metric condition, approximate midpoints, and geodesicity. We first revisit the definition of inner metric spaces and establish that every inner metric space admits an approximate midpoint. We then show that, when an inner metric space is proper, it is geodesic. The arguments rely on the notions of length of curves and rectifiable curves to relate distance and curve length within this class of spaces. These results clarify how the inner metric property is linked to geodesic behavior and contribute to a deeper understanding of metric spaces that can be treated as length spaces.

Keywords: Inner Metric Spaces; Approximate Midpoint; Geodesic; Length of Curves; Rectifiable Curves.

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INTRODUCTION

A metric space (X, d) is known as inner metric space, if the distance between any two points of a given metric space is equal to the infimum of the length of the curves joining the points together. As we can see in (i), thereafter. This shows that inner metric spaces are intrinsic. Inner metric space is an important notion in mathematics, particularly in metric geometry and analysis. Inner metric space is a metric space (X, d), where the metric d (distance function) satisfies the following properties, see (Monsuru, A.M, *et al*, 2023).

(i) Intrinsic metric property: Here, the distance between two points is defined as the infimum of length of the curve joining them t, that is, $d(x_1, x_2) = \inf(l(\gamma))$, (Anderson, 2021). The metric space has no any path of finite length between two points x_1 and x_2, therefore, $d(x_1, x_2) = \inf(l(\gamma)) = \infty \Longrightarrow l(\gamma) = \infty$. For example, given two points $x_1 = [0,0]$ and $x_2 = [5,5]$, the by the intrinsic metric between two points x_1 and x_2 in a metric or topological space $\mathbb{R}^n$, $d(x_1, x_2) = \sqrt{(5-0)^2 + (5-0)^2} = \sqrt{25+25} = \sqrt{50}$.

(ii) Length structure property: An Inner metric space can be equipped with a length structure and allowing the measurement of curve lengths, (Fernandez- Unzueta, 2018). Note that if the intrinsic metric is known, the concepts of length structure is straight forward. The Euclidean space $\mathbb{R}^n$ is a length space, since the shortest path in the intrinsic metric is a straight line between two points x_1 and x_2, hence, for this the normal Euclidean metric $d(x_1, x_2)$ equals Intrinsic metric $d_{intr}(x_1, x_2)$.

Inner metric spaces provide an outline for studying geometric and analytic properties of spaces with applications in different areas of Mathematics and Physics.

Examples of Inner metric spaces are: Euclidean spaces, Riemannian manifolds (compact connected Riemannia Manifolds), (Knox ,2013), convex domain in Euclidean spaces and metric graph (Graphs with edge lengths) (Oleksiy and Dovgoshey, 2025). While the set $\{\mathbb{R}^2 - (0,0)\}$ equiped with Euclidean metric is not an inner metric spaces, i.e here path have to go around the missing element. (Jiwon, 2019).

The sphere $\mathbb{S}^1$ in $\mathbb{R}^2$ endowed with the Euclidean metric is not an Inner metric spaces. This can be observed by showing that paths in a sphere $\mathbb{S}^1$ in $\mathbb{R}^2$ are parts of the sphere, whereas the metric $d(x_1, x_2)$ called the Euclidean metric has what we call Chordals

between points outside the sphere and metric $d(x_1, x_2)$ equals Intrinsic metric $d_{intr}(x_1, x_2)$, provided that $x_1 \neq x_2$.

Also, if A is connected subset of the space $\mathbb{R}^2$ and if A is convex, then it is an Inner metric spaces but if it is not convex, is not an Inner metric spaces- there exists two points x_1 and x_2 where there is no a straight line between then, and is in A. Therefore, Euclidean metric $d(x_1, x_2)$ does not equal Intrinsic metric $d_{intr}(x_1, x_2)$, (Nicklas, P. 2011).

A metric space is said to be an inner metric if it has an approximate midpoint, that is for a metric space (X, d), and for every $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $x_3 \in X$ such that

$max(d(x_1, x_3),\ d(x_3, x_2)) \leq \frac{d(x_1,x_2)}{2} + \epsilon$. (Gromov,1980)

In this paper, we have the following geometric concepts to rely on. They are; length of a curve, Rectifiable curve and Geodesic, which their definitions are included in the following definitions below.

Definition 1 (Burago and Ivanov, 2001)**:** Suppose (X_1, d_1) and (X_2, d_2) are metric spaces, then the function $f: X_1 \to X_2$ is uniformly continuous of for every $\epsilon > 0$, if $\exists\ \delta > 0$ such that $d_1(x_1, x_2) < \delta \Rightarrow d_2(f(x_1),\ f(x_2)) < \epsilon$.

Definition 2 (Burago and Ivanov, 2001)**:** Suppose (X, d) is a metric space. If for $x, y \in X$, and there exists a map $f: [0, d(x, y)] \to X$ such that $f(0) = x$, $f(d(x, y)) = y$. then, its isometry $\big(d\big(f(s)\big), f(t)) = |s - t|\big)$ is called a geodesic from x to y.

Definition 3 (Micheal, 2016)**:** The metric space (X, d) is geodesic if there is a geodesic from x to y, for all x,$y \in X$.

Definition 4 (Ahmadu and Monsuru, 2021)**:** A metric space (X, d) is complete, if there is a sequence x_n in (X, d) which converges to a limit point L, say, in (X, d).

Definition 5 (Burago and Ivanov, 2001)**:** Given that (X, d) is a metric space, it is said to have approximate midpoints, if for every $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $x_3 \in X$ such that

$max(d(x_1, x_3),\ d(x_3, x_2)) \leq \frac{d(x_1,x_2)}{2} + \epsilon$.

Definition 6 (Burago and Ivanov, 2001)**:** A metric space (X, d) is an inner metric space if for any $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $t \in X$ such that $max\big(d(x_1, t), d(x_2, t)\big) \leq \frac{d(x_1,x_2)}{2} + \epsilon$.

Definition 7 (Micheal, 2016)**:** Let (X, d) be an inner metric space. A minimizing geodesic γ is continuous curve $\gamma: I \to X$ such that $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$, for any $t_1, t_2 \in I$, where I is a closed and bounded interval $[a, b]$.

Definitions 8 (Gromov,1980)**:** Given that γ is continuous curve $\gamma: I \to X$.Then, the length of this curve can be define as $l(\gamma) = sup \sum_{k=1}^{n} d(\gamma(t_{k-1}), \gamma(t_k))$.

For example; consider a topological space $\mathbb{R}^2$. The path or curve $\gamma: I \to \mathbb{R}^2$ is represented by

$$\gamma(x) = \begin{cases} a = x^2, x \in I \\ b = x^3, x \in I \end{cases}$$

is a continuous map between the closed and bounded intervals [*0,0*] and [*1,1*]. Therefore, $\gamma(x)$ is a path or curve between the closed and bounded intervals [*0,0*] and [*1,1*].

Definition 9 (Facundo, M, *et al*, 2023): A curve $\gamma: I \to X$ is said to be rectifiable, if it has a finite length, that is if $l(\gamma) < \infty$.

Definition 10 (Monsuru, A. M, *et al*, 2020): A topological space X is said to be locally compact if for every point $x \in X$, there is a compact neighborhood of x in X.

METHODOLOGY

Here, we are going to present some methods used in getting our results.

Note that if we are given a length space, our duty is to define a metric associated with the structure of the length space given to us by assuming that the topological space (X, d) bearing the length structure is a Hausdorff space. Now, for the two points $x_1, x_2 \in X$, we established the associated distance $d(x_1, x_2)$ between them to be the infimum of the lengths of paths joining the points. As

$$d_l(x_1, x_2) = \inf\{l(\gamma); \gamma: I \to X, I = [x, y], \gamma(x) = x_1, \gamma(y) = x_2\}.$$

Lemma 1 (Burago and Ivanov, 2001): Suppose X is an inner metric space. Then, for any $x \in X, r > 0$, $Cl(x,r) = Cl(B)(x,r)$.

Lemma 2 (Anderson, 2021): A path $p: x \to y$ in a metric space (X,d) is a geodesic if and only if its arc – length parametrized and hence, $l(f) = d(x_1, x_2)$, for $x_1, x_2 \in X$. This means that if p is a path from x to y such that $l(f) = d(x_1, x_2)$, for $x_1, x_2 \in X$, its arc- length reparametrization is geodesic.

Lemma 3 (Anderson, 2021): An Inner metric space (X,d) is proper if and only if there is a Cauchy sequence which converges in it and locally compact.

Theorem 1 (Micheal, 2016): Every proper inner metric space is geodesic.

Theorem 2 (Micheal, 2016): Every inner metric space has approximate midpoint.

From the fact that a metric space (X,d) has approximate midpoints, if for every $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $x_3 \in X$ such that

$max(d(x_1, x_3),\ d(x_3, x_2)) \leq \frac{d(x_1,x_2)}{2} + \epsilon$.

We can see that if a metric space has an approximate midpoint, is an inner metric space. By comparing two definitions below named as Definition 5 and Definition 6 above respectively, that is, if we replace t in definition 5 by x_3, Definition 5 and 6 are the same, which shows that, a metric space with approximate midpoint is an inner metric space.

Definition (Micheal, 2016)**:** Given that (X,d) is a metric space. It is said to have approximate midpoints, if for every $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $x_3 \in X$ such that

$max(d(x_1, x_3),\ d(x_3, x_2)) \leq \frac{d(x_1,x_2)}{2} + \epsilon$.

Definition (Burago and Ivanov, 2001)**:** A metric space (X,d) is an inner metric space if for any $x_1, x_2 \in X$ and $\epsilon > 0$, there exists $t \in X$ such that $max\big(d(x_1, t),\ d(x_2, t)\big) \leq \frac{d(x_1,x_2)}{2} + \epsilon$.

So, if we put $t = x_3$, we have from definition 6 that $max\big(d(x_1, x_3),\ d(x_2, x_3)\big) \leq \frac{d(x_1,x_2)}{2} + \epsilon$, which is expression for approximate midpoint. Hence, a metric with approximate midpoint is an inner metric space.

By the definition of inner metric space, we see that (1) every inner metric space is path connected, (ii) every geodesic space is an inner metric space, and (iii) the Riemannian manifold $\mathcal{R}^n$ with a point removed is not geodesic but an inner metric space. (Burago and Ivanov, 2001)

Theorem 3 (Burago and Ivanov, 2001): Given that (X, d) is a locally compact length space. Then, the following statements are equivalent:

(i) (X, d) is proper

(ii) (X, d) is complete

(iii) (X, d) is geodesically complete, that is every distance preserving map $g: [0, M) \longrightarrow X$ can be extended to $[0, M]$.

(iv) There is a point $x \in X$ such that every distance preserving map $g: [0, M) \longrightarrow X$ with $g(0) = x$ can be extended to the closed and bounded interval $[0, M]$.

RESULTS AND DISCUSSION

Here, we are going to make used of a Lemma 1 and Theorem 3 stated in the Methodology above to provide results to the Theorem 1 and Theorem 2 below;

Theorem 1: Let $cl(X)$ be the closed subset of compact metric space X. Then, for any $x_1, x_2 \in cl(X)$ and any $\epsilon > 0$, $\exists\, x_3 \in X$ such that $max\big(d(x_1, x_3),\ d(x_2, x_3)\big) \leq \frac{d(x_1, x_2)}{2} + \epsilon$, the space $cl(X)$ is an inner metric space.

Proof: Suppose $x_1, x_2 \in cl(X)$ and any $\epsilon > 0$, we can choose $\epsilon_k > 0$ such that $\sum_k \epsilon_k = c < \infty$, so that $\prod_{k=1}^{\infty}(1 + \epsilon_k) < \infty$.

Let $\delta = d(x_1, x_2)$, then by the statement of the Theorem, there is $\frac{x_3}{2} \in cl(X)$ such that

$$max\left(d\left(x_1, \frac{x_3}{2}\right),\ d\left(x_2, \frac{x_3}{2}\right)\right) \leq \frac{1}{2}\delta + \frac{1}{2}\delta\epsilon_1 = \frac{1}{2}\delta(1 + \epsilon_1) \quad (1)$$

By following equation (1), we can still choose $\frac{x_3}{4}, \frac{x_3'}{4} \in cl(X)$ such that

$$max\left(d\left(x_1, \frac{x_3}{4}\right),\ d\left(x_2, \frac{x_3'}{4}\right)\right) \leq \frac{1}{2}\left(\frac{1}{2}\delta(1 + \epsilon_1)\right) + \frac{\epsilon_2}{2}\left(\frac{1}{2}\delta(1 + \epsilon_2)\right) \quad (2)$$

and

$$max\left(d\left(\frac{x_3}{4},\frac{x_3'}{4}\right),\ d(\frac{x_3'}{4},x_2\ )\right) \leq \frac{1}{2}\left(\frac{1}{2}\delta(1+\epsilon_1)\right)+\frac{\epsilon_2}{2}(\frac{1}{2}\delta(1+\epsilon_2)). \quad (3)$$

Then, we have, $max\left(d\left(x_1,\frac{x_3}{4}\right),\ d\left(\frac{x_3}{4},\frac{x_3}{2}\right),d\left(\frac{x_3}{2},\frac{x_3'}{4}\right),d(x_2,\frac{x_3'}{4})\right) \leq \frac{1}{2}\left(\frac{1}{2}\delta(1+\epsilon_1)\right)+\frac{\epsilon_2}{2}\left(\frac{1}{2}\delta(1+\epsilon_1)\right) \leq \frac{1}{2}\left(\frac{1}{2}\delta(1+\epsilon_1)\right)+\frac{\epsilon_2}{2}\left(\frac{1}{2}\delta(1+\epsilon_1)\right)=\frac{1}{4}\delta(1+\epsilon_1)(1+\epsilon_2)=\frac{\delta}{2^2}(1+\epsilon_1)(1+\epsilon_2)$.

By continuing the process this way, we get the map $\varphi\colon \mathbb{Q}[2]\subset[0,1]\longrightarrow cl(X)$ such that

$$d\left(f\left(\frac{r}{2^n}\right),f(\frac{1+r}{2^n})\right) \leq \frac{\delta}{2^n}\prod_{k=1}^{\infty}(1+\epsilon_k) < \infty \quad (4)$$

And $\mathbb{Q}[2]$ is a set of rational number modulo 2. By completeness of the hyperspace $cl(X)$, we have the extension of the function f as $f^*\colon[0,1]\longrightarrow cl(X)$. Then, we can choose $c=\sum_k c_k$ to be arbitrary close to 1. Therefore, the space $cl(X)$ is an inner metric space.

Remarks on Theorem 1: (1) The space $cl(X)$ is also known as hyperspace of metric space X

(2) The metric space X can also be consider as Riemannian manifold.

(3) Riemannian manifold is a manifold which Riemannian metric is defined on it.

(4) Due to the fact that closed subset of Metric space is compact, this is showing that the space $cl(X)$ is compact.

Theorem 2: Let (X,d) be a complete inner metric space and let $x_1\in X$ and $0<\tau$. If $cl(x_1,r)$ is compact for every r satisfing $0\leq r<\tau$, then the closed ball $cl(B,\tau)$ is compact.

Proof: Suppose $\{x_n\}$ is a sequence in a closed ball $cl(B,\tau)$. Then, we shall proof that the sequence $\{x_n\}$ has a convergent subsequence in the closed ball $cl(B(x_1,\tau))$. If we assume that $\lim_{n\to\infty} d(x_1,x_n)=\tau$, there is a r_1 satisfies $0<r_1<\tau$ such that the closed ball $cl(B(x_1,r_1))$ contains infinitely many points of the sequence x_n.Then, provided the closed ball $cl(B(x_1,\tau))$ is bounded, then it is compact, which shows that the sequence $\{x_n\}$ has a convergent subsequence.

Next, choose a sequence $\{\epsilon_m\}$, where $\epsilon_m > 0$, $\lim_{m\to\infty} \epsilon_m = 0$, and assume that $0 < \epsilon_m < \tau$. Now, for each m, there is $M = M_m \in \mathbb{N}$ such that $|d(x_1, x_n) - \tau| < \epsilon_m, \forall n \geq \mathbb{N}$, since $\lim_{n\to\infty} d(x_1, x_n) = \tau$, then $\tau\backslash\epsilon_m < d(x_1, x_n) < \tau$. Thus, there is a point $y_n^m \in (B, \tau\backslash\epsilon_m)$ such that $d(x_n, y_n^m) < \epsilon_m$. Consequently, for fixed point m, the sequence $\{y_n^m\} \subset cl(B(x_1, \tau\backslash\epsilon_m))$, for $n \geq M_k$. Provided that the closed ball $cl(B(x_1, \tau\backslash\epsilon_m))$ is compact, certainly, the sequence $\{y_n^m\}$ must have a convergent subsequence $\{y_{n_i}^m\}$.

Thus, $d\left(x_{n_i}, x_{n_j}\right) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}^m, y_{n_i}^m) + d\left(y_{n_j}^m, x_{n_j}\right) < \epsilon_{n_i} + \epsilon_{n_j} + d(y_{n_i}^m, y_{n_j}^k)$, which shows that the sequence $\{x_{n_i}\}$ is a Cauchy sequence. Since the inner metric space (X, d) is complete, then, the sequence $\{x_{n_i}\}$ converges to a point $x \in cl(B(x_1, \tau))$, which shows that the closed ball $cl(B(x_1, \tau))$ is sequentially compact.

Remark on Theorem 2: Since a metric space is compact if it has a convergence subsequence, by the sequentially compactness property, this convergence subsequence is Cauchy sequence as mentioned in the Theorem 2 above.

In the next result, we are going to consider an inner metric space that is both complete and locally compact. For this, we are going to strongly make use of Theorem 3 in the methodology above to provide a proof to this result.

Theorem 3: Given that (X, d) is an inner metric space that is complete and locally compact. Then (i) Any closed set $A \subset X$ is compact (ii) for any $x_1, x_2 \in X$, there is a minimal geodesic curve $\gamma: I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$, where $I = [0,1]$.

Proof: (i) Since X is locally compact, for $k \in X$, the closed ball $cl(B(k, r))$ is compact. Then, we claim that $\sup\{r: cl(B(k, r)) \text{ is compact}\} = \tau < \infty$. The, by Theorem 2 above, we see that the closed ball $cl(B(k, \tau))$ is compact and is $S(k, \tau) = \{x \in X: d(x, k) = \tau\}$. For this, we see that the remainder is standard point – set topological space argument. From the choice of the compactness, if we consider the finite cover of the set $S(k, \tau)$, we can also have $\tau' > \tau$ such that the closed ball $cl(B(k, \tau'))$ is also compact, which proves the first part of the Theorem.

(ii) If X is compact, provided that X is an inner metric space, then for $x_1, x_2 \in X$, and there are geodesic curves $\gamma_n: I \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ such that the length of the curves γ_n implies that $l(\gamma_n) \leq d(x_1, x_2) + \frac{1}{n}$.

Next, let us assume that the curves γ_n is parametrized which is proportional to the arc – length. The set of the curves γ_n, $\{\gamma_n\}$ are equicontinuous. Provided that the metric space X is compact, then, by Arzel$\grave{a}$ - Ascoli Theorem, we have a subsequence of the sequence of the curves $\{\gamma_n\}$ that converges to a continuous map $\gamma: I \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Provided that $d(x_1, x_2) \leq l(\gamma) \leq \lim_{n\to\infty} \inf l(\gamma_n) = d(x_1, x_2)$, we have $d(x_1, x_2) = l(\gamma)$. If we consider the inner metric space X to be locally compact only, then the curves $\gamma_n \subset cl(B(k, 2d(x_1, x_2)))$, by Theorem 2 above, since the closed ball $cl(B(k, 2d(x_1, x_2)))$ is compact, then, $l(\gamma) = d(x_1, x_2)$ and $\gamma_n \subset cl(B(k, 2d(x_1, x_2)))$. This ends the proof.

Remarks on Theorem 3: (1) Since the closed subset of compact metric space is compact and compact subset of metric space is closed, this is showing that the closed set $A \subset X$ will be compact.

(2) Since the finite set is compact, this shows that finite cover of inner metric or topological space X is compact. S finite cover of the set $S(k, \tau)$ is compact.

From the fact that, a path or curve in a metric space is rectifiable if it has a finite length, that means if there is a path in metric space and we can find that its length is finite, such path is rectifiable. From this fact, we have the Theorem 4 below;

Theorem 4: Suppose (X, d) is an inner metric space with rectifiable path. Then it has approximate midpoint.

Proof: Let $x_1, x_2 \in X$ and $\epsilon > 0$. We assume that there is a path $f: I \longrightarrow X$ from x_1 to x_2 with $L(f) < d(x_1, x_2) + 2\epsilon$, if we replace I by $L(f)$ where $I = [0, k]$. We can now define a map $v: I \longrightarrow [0, l(f)]$, $v \to L(f \restriction I)$, then the map is a constant map on $I_1 \subseteq I$ if and only if the map f is constant on the closed and bounded interval I_1. Then the map $v: n \longrightarrow L(f \restriction I_1)$ is a surjective map to the interval $[0, l(f)]$, where $I_1 = [a, b]$. Thus, there exist a point $n \in I$ such that $v(n) = L(f \restriction I_2) = \frac{1}{2}L(f)$, where $I_2 = [0, n]$. By additive property of the length, $L(f \restriction I) = \frac{1}{2}L(f)$. Let $m = f(n)$, so that $d(x_1, m) \leq L(f \restriction I_2) = \frac{1}{2}L(f) < \frac{1}{2}d(x_1, x_2) + \epsilon$, hence, $d(m, x_2) < \frac{1}{2}d(x_1, x_2) + \epsilon$. Therefore, every inner metric space has an approximate midpoint.

Now, we are going to make use the statement of Lemma 2 and Theorem 3 in the methodology to provide solution to Theorem 5 and Theorem 6 below;

Theorem 5: Suppose (X, d) is a complete metric space with approximate midpoint, then is an inner metric space.

Proof: Let $x_1, x_2 \in X$ and $\epsilon_n > 0$, for all $n \in \mathbb{N}$ such that $c = \prod_{k=1}^{\infty}(1 + \epsilon_k) < \infty$. By approximate midpoints, let us pick $x_{\frac{1}{2}} \in X$ such that max $\left\{d\left(x_1, x_{\frac{1}{2}}\right), d(x_{\frac{1}{2}}, x_2)\right\} \leq (1 + \epsilon_1)\frac{d(x_1,x_2)}{2}$(1)

Next, we can also choose points $x_{\frac{1}{2}}, x_{\frac{3}{4}}$ such that

max $\left\{d\left(x_1, x_{\frac{1}{4}}\right), d\left(x_{\frac{1}{4}}, x_{\frac{1}{2}}\right), d\left(x_{\frac{1}{2}}, x_{\frac{3}{4}}\right), d(x_{\frac{3}{4}}, x_2)\right\} \leq (1 + \epsilon_1)(1 + \epsilon_2)$(2)

If we iterate constructions 1 and 2 above, we get $x_{n_1} \in X, \forall n \in D \cap I$ such that $d(x_n, x_{n_1}) \leq cd(x_1, x_2)|n_1 - n_2|, \forall n_1, n_2 \in \mathbb{D} \cap I$, where $I = [0,1]$. Then, the map $g: \mathbb{D} \cap I \to X, n_1 \to x_{n_1}$ is uniformly continuous, since $cl\{\mathbb{D} \cap I\} = I \; and \; by$ the fact that for metric spaces X and Y with metrics d_1 and d_2 respectively with Y complete. If $A \subseteq X$ and $g: (A, d_1) \longrightarrow (Y, d_2)$ uniformly continuous, then, the function g has a uniformly continuous extension $\hat{g}: cl(A) \longrightarrow Y$, and then we have a unique extension $\hat{g}: I \longrightarrow X$ such that $d(\hat{g}(n_1), \hat{g}(n_2)) \leq cd(x_1, x_2)|n_1 - n_2|, \forall n_1, n_2 \in I$. By the fact that if X is a metric space with metric d and $g: [0, k] \longrightarrow X$ is a path from x_1 to x_2, then if $d(g(n_1), g(n_2)) \leq c|n_1 - n_2|$, $\forall n_1, n_2 \in [0, k]$, then, the function g is rectifiable and $L(g) \leq cK$, showing that the function g is a rectifiable path of the length $L(g) \leq cd(x_1, x_2)$. Then, by appropriate choice of ϵ_k, we can bring $c > 1$ arbitrarily close to 1. Therefore, the metric space (X, d) is an inner metric space.

Let again use Lemma 3 and Theorem 3 in the methodology to provide a proof for the Theorem 6:

Theorem 6: If X is a proper inner metric space, then it is geodesic.

Proof: Let $x_1, x_2 \in X$, where $x_1 \neq x_2$. From the definition of inner metric space, for every $n \in \mathbb{N}$, we can pick a path p_n from x_1 to x_2 such that $L(p_n) \leq d(x_1, x_2) + \frac{1}{x}$. By the fact that the path $p: [0, k] \longrightarrow X$ be a rectifiable and $l(\tau) = L(p \restriction [0, \tau])$, there is a

rectifiable path $p: [0, L(p)] \longrightarrow X$ that is parametrized by arc – length and satisfies $p = kol$. Now, we can make the path parametrized by the arc – length and make it linear in such a way that it can be defined on the closed and bounded interval $[0,1]$. Then,

$$\frac{d(p_n(n_1), p_n(n_2))}{d(x_1,x_2)+1} \leq \frac{L(p_n \restriction [n_1,n_2])}{L(p_n)} = |n_1 - n_2|,$$

this implies that the family $\{p_n\}$ is uniformly – equicontinuous. So, every path p_n begins at x_1 have the length L such that $L(p_n) \leq d(x_1, x_2) + 1$, thus, they all lie in the closed ball $cl(B(x_1, d(x_1, x_2) + 1)) \subseteq Y$, this is compact due to properness of X and also satisfied the concept of Arzelà – Ascoli, then we get a subsequence $\{p_{n_k}\}$ of the sequence $\{p_n\}$ which converges to some $c \in ([0,1], X)$.

Due to the fact that if X is a metric space and the path $p: [0, k] \longrightarrow X$ if from x_1 to x_2, $\forall x_1, x_2 \in X$, then the map $c([0, k], X) \longrightarrow [0, \infty]$, $f \longrightarrow L(p)$ is a lower semicontinuous function with respect to pointwise or uniformly convergence of the path p.

So, the map $p \longrightarrow L(p)$ is a lower semicontinuous function, and we have $d(x_1, x_2) = L(c) \leq \lim\ inf L(p_{n_k})$. Since the opposite assertion of $d(x_1, x_2) \leq L(c)$ holds for every path p from x_1 to x_2, then, $d(x_1, x_2) \leq L(c)$. So, it is clear that c is parametrized by the arc – length that reformed to the closed and bounded interval $[0,1]$.

Let us assume that $c'(n) = c\left(\frac{n}{d(x_1,x_2)}\right)$. This shows that $c'(n)$ is also parametrized by arc – length, which left the length unchanged, so

$$d(x_1, x_2) = L(c) = L(c'(n)) \quad (1)$$

by the fact that any path p from x_1 to x_2 is geodesic if and only if it is parametrized by arc – length and

$$L(p) = d(x_1, x_2) \quad (2)$$

Comparing equations (1) and (2), we see that $c'(n)$ is a geodesic. This ends the proof.

It is clear that being geodesic of a metric space (X, d) implies that the metric space (X, d) is metrically convex. Even if the metric space (X, d) is complete, the reverse of this implication is also true. We can now use this notion to formulate the Theorem 7 below. Thus, by Theorem 2 and Theorem 3 in the methodology above, we have:

Theorem 7: Every complete inner metric space with approximate midpoints is geodesic.

Proof: Suppose a metric $d(x_1, x_2) = A$, for $x_1 \neq x_2$. Since the inner metric space X with metric d contains strong midpoints, then there is a point $x_{1\frac{A}{2}} \in A$ such that $d\left(x_1, x_{1\frac{A}{2}}\right) = d\left(x_{1\frac{A}{2}}, x_2\right) = \frac{A}{2}$, this implies that a point $x_{1\frac{A}{2}}$ satisfies the interval $x_1 < x_{1\frac{A}{2}} < x_2$. We use the same process to find the points $x_{1\frac{A}{4}}$, $x_{1\frac{3A}{4}}$, such that the metric $d\left(x_1, x_{1\frac{A}{4}}\right) = d\left(x_{1\frac{A}{4}}, x_{1\frac{A}{2}}\right) = d\left(x_{1\frac{A}{4}}, x_2\right) = \frac{A}{4}$. By the iteration, we get x_n for every $t \in (\mathbb{D} \cap I)A$ such that $d(n, m) = |n - m|, \forall n, m, t \in (\mathbb{D} \cap I)A$. Therefore, there is a distance preserving map such that $g: (\mathbb{D} \cap I)A \longrightarrow X$. Since, the map g is an isometry which is uniformly continuous and the set $Cl((\mathbb{D} \cap I)A) = [0, A]$. Then, there is isometry extension $\hat{g}$ of the function g, which is written as $\hat{g}(0) = x_1, \hat{g}(A) = x_2$. Therefore, the function $\hat{g}$ is a geodesic from x_1 to x_2.

Remark on Theorem 7: In the Theorem 7, (1) the distance preserving map is also known as isometry, and the extension of isometry is also an isometry, which is a geodesic from any two point in an inner metric space X, (2) Set $\mathbb{D}$ is dyadic rational, which is a rational number that can be express as $\frac{x}{2^y}$, where x can be any integer and y is non – negative integer. Also, dyadic rational are dense in the real numbers $\mathbb{R}$.

CONCLUSION

In this paper, we concluded that the Riemannian manifold $\mathcal{R}^n \backslash \{x\}$ and closed subset of Riemannian manifold is inner metric space where Riemannian manifold $\mathcal{R}^n \backslash \{x\}$ is not geodesic. Also, we proved that every inner metric space is path connected, hence, every geodesic space is an inner metric space.

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