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Properties and Application of Alpha Power Transformed Perks Distribution to Engineering Data

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Abstract

This study introduces the Alpha Power Transformed Perks Distribution (APTPD), obtained by applying the alpha power transformation method to the classical Perks distribution in order to enhance its flexibility and enable it to accommodate different data structures. Several analytical properties of the proposed distribution are established, including a simplified expression for its probability density function, ordinary moments, moment-generating and characteristic functions, reliability measures, and order statistics. The parameter estimation for the APTPD is conducted using the method of maximum likelihood. To demonstrate its practical relevance, the APTPD is fitted to an engineering dataset on aircraft windshield service times and compared with several existing generalizations of the Perks distribution. Based on multiple goodness-of-fit statistics and information criteria, the APTPD provides the best fit among the considered competing models. Overall, the results indicate that the proposed distribution is a useful and flexible model for analyzing positively skewed lifetime data in reliability and survival studies.

Keywords: Alpha Power Transformation; Perks Distribution; Lifetime Modeling; Reliability Analysis; Survival Function; Maximum Likelihood Estimation

Introduction

Power transformations are a cornerstone of modern statistical modeling because they can reduce skewness, stabilize variance, and increase model flexibility when fitting real-life data (Box & Cox, 1964; Yeo & Johnson, 2000). The alpha power transformation method introduced by Mahdavi and Kundu (2017) as a general method for generating new univariate distributions has recently become a widely used formula for deriving highly flexible lifetime distributions, producing models with richer shapes for densities and hazard rates than many classical alternatives (Mahdavi & Kundu, 2017; Nassar et al., 2017).

Researchers have applied alpha power transformed family across many base distributions and it has been showing systematic improvements in fit and interpretability for applied problems in reliability, survival analysis, and other empirical domains (Ahmad et al., 2020; Elbatal et al., 2022; Ihtisham et al., 2019; Ali et al., 2021; Ahmad et al., 2021; Ceren et al., 2018; Eghwerido, 2021; Mohiuddin and Kannan, 2021; Dey et al., 2019; Agegnehu et al., 2024; Dugasa, 2024).

The Perks distribution originally proposed in actuarial/mortality work provides a logistic-type alternative to Gompertz-Makeham mortality laws and has been adapted into multiple generalized forms for actuarial and reliability applications (Perks, 1932).

The cumulative distribution function (cdf) and probability density function (pdf) of the Perks distribution are respectively defined as:

$$G(x) = 1 - \frac{1+a}{1+ae^{bx}} \tag{1}$$

and

$$g(x) = abe^{bx}\frac{1+a}{\left(1+ae^{bx}\right)^{2}} \tag{2}$$

where $x, a, b > 0$, and a and b are the scale and shape parameters of the Perks distribution respectively while X is the random variable.

More recent methodological work has produced weighted and T–X generalized Perks variants that increase flexibility without excessive parameter proliferation, demonstrating advantages on real datasets (e.g., pensioner mortality and failure-time data) and motivating further generalizations for applied modeling (Ugwu et al., 2024). Some researchers have used other families to developed extensions of the Perks distribution such as the Burr X-Perks distribution by Ieren and Umar (2025), the exponentiated Perks distribution by Singh and Choudhary (2017), the Kumaraswamy-Perks distribution by Oguntunde *et al.* (2018) and the Chen-Perks distribution by Mendez-Gonzalez *et al.* (2023). Combining Perks-type baselines with modern generators or transformation techniques is therefore a natural step for practitioners who need models that are able to capture complex hazard shapes (nonmonotone, bathtub, plateauing) found in many empirical datasets.

This paper develops and studies the Alpha Power Transformed Perks distribution (APTPD). The remaining sections of this paper are as follows: definition of the new distribution with its plots is provided in section 2. Section 3 presents the simplification of the pdf of the APTPD. Section 4 derived some properties of the proposed distribution. The estimation of parameters using maximum likelihood estimation (MLE) is presented in section 5. An application of the new model with other existing distributions to a real-life engineering data is captured in section 6 and the conclusion is given in section 7.

Alpha Power Transformed Ishita distribution (APTID)

The cumulative distribution function (cdf) and the probability density function (pdf) of the Alpha Power transformed family of distributions by Mahdavi and Kundu (2017), are defined as:

$$F(x) = \frac{\alpha^{G(x)} - 1}{\alpha - 1} \tag{3}$$

and

$$f(x) = \frac{\log(\alpha)}{(\alpha - 1)} g(x) \alpha^{G(x)} \tag{4}$$

"respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\alpha > 0$ and $\alpha \neq 1$ is the power or shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Substituting equations (1) and (2) into equations (3) and (4) and simplifying, we obtain the cdf and pdf of the APTPD (for $x > 0$) are given in equations (5) and (6) respectively as follows:

$$F(x) = (\alpha - 1)^{-1} \left(\alpha^{\left(1 - \frac{1+a}{1+ae^{bx}}\right)} - 1 \right) \tag{5}$$

and

$$f(x) = \frac{(1+a)\log(\alpha)}{(\alpha - 1)\left(1 + ae^{bx}\right)^2} abe^{bx} \alpha^{\left(1 - \frac{1+a}{1+ae^{bx}}\right)} \tag{6}$$

where $a > 0$ and $b > 0$ are the scale and shape parameters of the APTPD respectively, and $\alpha > 0$ is the alpha power transformed parameter.

Figure 1 below presents the plots of the pdf and cdf of the APTPD using some parameter values as follows.

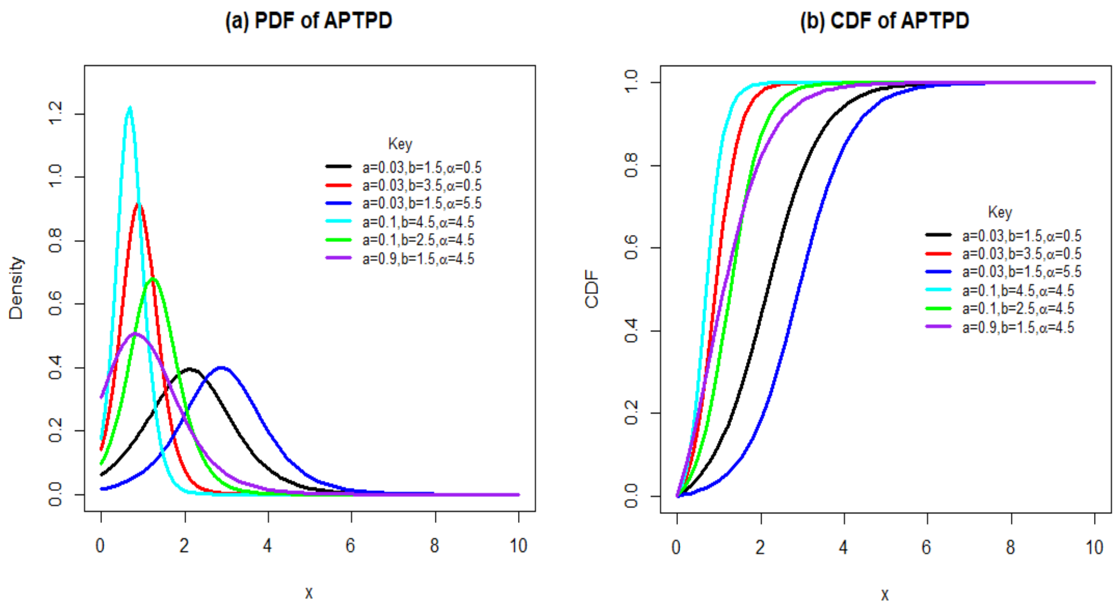

Figure 1: PDF and CDF of the APTPD.

The plot of the pdf in figure 1(a) above revealed that the pdf APTPD is positively skewed and flexible with different shapes depending on the chosen values of the parameters. Also, the plot of the cdf in figure 1(b) show that it is a valid distribution because the value of the cdf equals one when X approaches infinity and equals zero when X tends to zero as normally expected.

Simplification of PDF of APTPD.

This section presents a simplification of the pdf of APTPD to enable the derivation of some important properties of the distribution. Recall that the pdf of APTPD is given as:

$$f(x)=\frac{(1+a)\log(\alpha)}{(\alpha-1)\left(1+a\mathrm{e}^{bx}\right)^{2}}ab\mathrm{e}^{bx}\alpha^{\left(1-\frac{1+a}{1+a\mathrm{e}^{bx}}\right)} \quad (7)$$

For $\alpha>0$ and $\alpha\neq 1$, equation (7) can be expanded using power series as:

$$\alpha^{\left(1-\frac{1+a}{1+a\mathrm{e}^{bx}}\right)}=\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k}}{k!}\left(1-\frac{1+a}{1+a\mathrm{e}^{bx}}\right)^{k} \quad (8)$$

Substituting the result in equation (8) into equation (7) and simplifying gives:

$$f(x)=\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}(1+a)}{(\alpha-1)k!\left(1+a\mathrm{e}^{bx}\right)^{2}}ab\mathrm{e}^{bx}\left(1-\frac{1+a}{1+a\mathrm{e}^{bx}}\right)^{k} \quad (9)$$

Using the generalized binomial expansion on the last term in equation (9) yields the following:

$$\left(1-\frac{1+a}{1+a\mathrm{e}^{bx}}\right)^{k}=\sum_{l=0}^{\infty}(-1)^{l}\binom{k}{l}\left(\frac{1+a}{1+a\mathrm{e}^{bx}}\right)^{l} \quad (10)$$

Making use of the result in (10) above in equation (9) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l}(\log\alpha)^{k+1}ab}{(1+a)^{-(l+1)}(\alpha-1)k!}\mathrm{e}^{bx}\left(1+a\mathrm{e}^{bx}\right)^{-(l+2)} \quad (11)$$

Also, using binomial expansion on the last term in (11) gives:

$$\left(1+a\mathrm{e}^{bx}\right)^{-(l+2)}=\sum_{m=0}^{\infty}\binom{-l-2}{m}a^{m}\mathrm{e}^{mbx} \quad (12)$$

Making use of the result in equation (12) and simplifying, equation (11) becomes:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{l}\binom{-l-2}{m}\frac{(-1)^{l}(\log\alpha)^{k+1}a^{m+1}b}{(1+a)^{-(l+1)}(\alpha-1)k!}\mathrm{e}^{b(m+1)x} \tag{13}$$

Now, let $\eta_{klm}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{l}\binom{-l-2}{m}\frac{(-1)^{l}(\log\alpha)^{k+1}a^{m+1}b}{(1+a)^{-(l+1)}(\alpha-1)k!}$ be a constant, which implies that the pdf in equation (13) can also be written in its simple and linear form as:

$$f(x)=\eta_{klm}\mathrm{e}^{b(m+1)x} \tag{14}$$

Hence, equation (14) is the simplified pdf of the alpha power transformed Perks distribution (APTPD) which will be used to derive some properties of the APTPD in the next section.

Derivation of Properties of Alpha Power Transformed Perks Distribution (APTPD)

In this section, some properties of the APTPD are derived and discussed as follows:

Moments

Let X denote a continuous random variable the n^{th} moment of X is given by;

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)\,dx \tag{15}$$

Substituting the simplified pdf of the APTPD from equation (14) into equation (15), the n[th] ordinary moment of the APTPD can be derived as follows:

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)\,dx=\eta_{klm}\int_0^{\infty}x^n\mathrm{e}^{b(m+1)x}dx \tag{16}$$

Making use of integration by substitution method in equation (16), we have:

Let $b(m+1)x=-u\Rightarrow x=-u\left(b(m+1)\right)^{-1}$ such that $\frac{du}{dx}=b(m+1)\Rightarrow dx=-\frac{du}{b(m+1)}$

Substituting for x, u and dx in equation (18) and simplifying; we have:

$$\mu_n^{'} = E\left(X^n\right) = \left(-\tfrac{1}{b(m+1)}\right)^{n+1} \eta_{klm} \int_0^\infty u^{n+1-1} \mathrm{e}^{-u} du \tag{17}$$

Recall that $\int_0^\infty t^{k-1} e^{-t} dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t} dt = \int_0^\infty t^{k+1-1} e^{-t} dt = \Gamma(k+1)$

Considering the statement above, the n^{th} ordinary moment of X for the APTPD is given as:

$$\mu_n^{'} = E\left(X^n\right) = \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \binom{k}{l} \binom{-l-2}{m} \frac{(-1)^{l+n+1} (\log \alpha)^{k+1} a^{m+1} b \Gamma(n+1)}{(1+a)^{-(l+1)} (\alpha-1) k! \left[b(m+1)\right]^{n+1}} \tag{18}$$

Moment Generating Function

The moment generating function of a continuous random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^\infty e^{tx} f(x) dx \tag{19}$$

Using the nth ordinary moment of X in equation (18), the moment generating function of a random variable X for the APTPD can be derived based on power series expansion as follows:

$$M_X(t) = E\left[\mathrm{e}^{tx}\right] = E\left[\sum_{n=0}^\infty \frac{(tx)^n}{n!}\right] = \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^\infty x^n f(x) dx = \sum_{n=0}^\infty \frac{t^n}{n!} E\left(X^n\right) \tag{20}$$

Substituting the result from equation (18) into equation (20) and simplifying, the moment generating function of the APTPD is obtained as:

$$M_X(t) = E\left[\mathrm{e}^{tx}\right] = \sum_{n=0}^\infty \frac{t^n}{n!} \left[\sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \binom{k}{l} \binom{-l-2}{m} \frac{(-1)^{l+n+1} (\log \alpha)^{k+1} a^{m+1} b \Gamma(n+1)}{(1+a)^{-(l+1)} (\alpha-1) k! \left[b(m+1)\right]^{n+1}}\right] \tag{21}$$

Characteristics Function

The characteristics function for a continuous random variable X is defined as:

$$\phi_x(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} f(x) dx \tag{22}$$

The characteristics function for a continuous random variable X can be obtained based on the n^{th} ordinary moment using power series expansion as follows:

$$\phi_X(t) = E\left[e^{itx}\right] = E\left[\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\int_0^{\infty}x^n f(x)\,dx = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left(X^n\right) \quad (23)$$

Again, substituting for $E\left(X^n\right)$ in equation (23) and simplifying gives the characteristic function of the APTPD as:

$$\phi_X(t) = E\left[e^{itx}\right] = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{l}\binom{-l-2}{m}\frac{(-1)^{l+n+1}(\log\alpha)^{k+1}a^{m+1}b\Gamma(n+1)}{(1+a)^{-(l+1)}(\alpha-1)k!\left[b(m+1)\right]^{n+1}}\right] \quad (24)$$

Reliability analysis of the APTPD.

A derivation and study of the survival function and the hazard rate function is presented in this section. The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \quad (25)$$

Applying the cdf of the APTPD in (25), the survival function for the APTPD is obtained as:

$$S(x) = 1 - \frac{\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)} - 1}{(\alpha-1)} \quad (26)$$

A hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \quad (27)$$

Meanwhile, the expression for the hazard rate of the APTPD is given by:

$$h(x) = \frac{(\alpha-1)^{-1}(1+a)\log(\alpha)abe^{bx}\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}}{\left(1+ae^{bx}\right)^2\left[(\alpha-1)-\left(\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}-1\right)\right]} \quad (28)$$

The plot of the survival and hazard functions for some chosen parameter values are presented in Figure **2** as shown below:

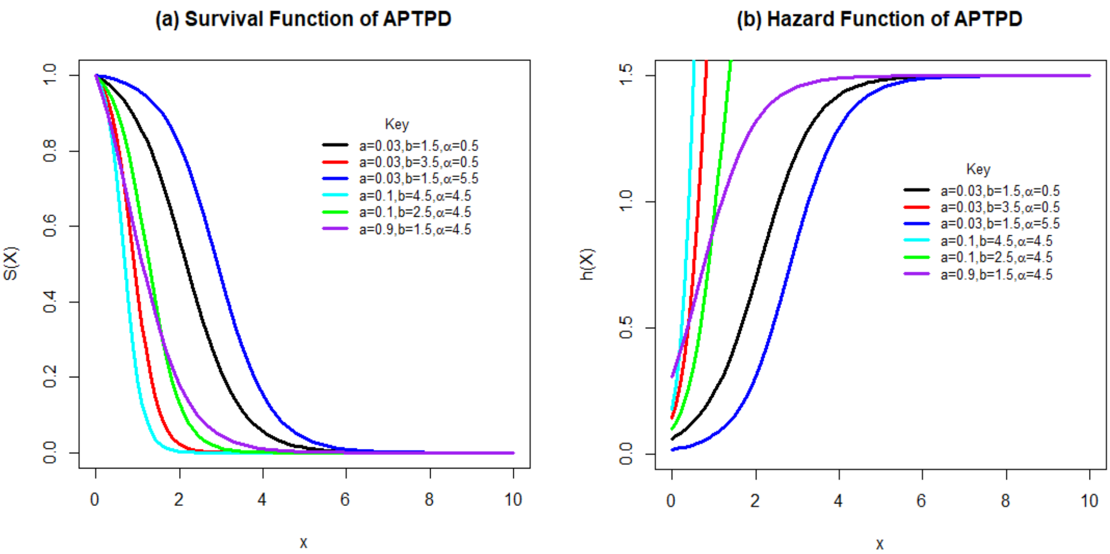

Figure 2: The Survival Function and Hazard Function of APTPD.

The plot in Figure **2(a)** shows that the probability of survival is always sure at an initial time or early age and it decreases as time increases up to zero (0) at infinity.

Also, figure **2(b)** revealed that the APTPD has an increasing failure rate which implies that the probability of failure for any random variable following APTPD could be increasing with time.

Order Statistics

Suppose $X_1, X_2, \ldots\ldots, X_n$ is a random sample from the APTPD and let $X_{1:n}, X_{2:n}, \ldots\ldots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k \binom{n-i}{k} f(x)\left[F(x)\right]^{k+i-1} \qquad (29)$$

Using (5) and (6), the pdf of the i^{th} order statistics $X_{i:n}$, can be expressed from (29) as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k \binom{n-i}{k} \frac{(1+a)\log(\alpha)abe^{bx}}{(\alpha-1)\left(1+ae^{bx}\right)^2}\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}\left[\frac{\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}-1}{(\alpha-1)}\right]^{k+i-1} \qquad (30)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the APTPD are respectively given by;

$$f_{1:n}(x)=\frac{n!}{(n-1)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-1}{k}\frac{(1+a)\log(\alpha)abe^{bx}}{(\alpha-1)\left(1+ae^{bx}\right)^2}\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}\left[\frac{\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}-1}{(\alpha-1)}\right]^k \quad (31)$$

And

$$f_{n:n}(x)=n\frac{(1+a)\log(\alpha)abe^{bx}}{(\alpha-1)\left(1+ae^{bx}\right)^2}\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}\left[\frac{\alpha^{\left(1-\frac{1+a}{1+ae^{bx}}\right)}-1}{(\alpha-1)}\right]^{k+n-1} \quad (32)$$

Maximum Likelihood Estimation of the Parameters of the APTPD

Let $X_1, X_2, ..., X_n$ be a sample of size 'n' independently and identically distributed random variables from the APTPD with unknown parameter, a, b and α defined previously.

The likelihood function of the APTPD using the pdf in equation (6) is given by:

$$L(\underline{X}\mid a,b,\alpha)=\left(\frac{(1+a)\log(\alpha)ab}{(\alpha-1)}\right)^n e^{b\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(\alpha^{\left(1-\frac{1+a}{1+ae^{bx_i}}\right)}\left(1+ae^{bx_i}\right)^{-2}\right) \quad (33)$$

Let the natural logarithm of the likelihood function be, $l=\log L(\underline{X}\mid a,b,\alpha)$, therefore, taking the natural logarithm of the function equation (35) above gives:

$$l=n\log\left(\frac{\log\alpha}{\alpha-1}\right)+n\log a+n\log b+n\log(1+a)-2\sum_{i=1}^{n}\log\left[1+ae^{bx_i}\right]+\sum_{i=1}^{n}\left(1-\frac{1+a}{1+ae^{bx_i}}\right)\log\alpha \quad (34)$$

Differentiating l partially with respect to parameters a, b and α gives the following result:

$$\frac{\partial l}{\partial\alpha}=\frac{n(\alpha-1)}{\ln\alpha}\left(\frac{\alpha^{-1}(\alpha-1)-\ln\alpha}{(\alpha-1)^2}\right)+\alpha^{-1}\sum_{i=1}^{n}\left(1-\frac{1+a}{1+ae^{bx_i}}\right) \quad (35)$$

$$\frac{\partial l}{\partial a}=\frac{n}{a}+\frac{n}{1+a}-2\sum_{i=1}^{n}\left[\frac{e^{bx_i}}{1+ae^{bx_i}}\right]+\sum_{i=1}^{n}\left[\frac{e^{bx_i}-1}{\left(1+ae^{bx_i}\right)^2}\right]\log\alpha \quad (36)$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\left[\frac{\mathrm{a}x_i\mathrm{e}^{bx_i}}{1+a\mathrm{e}^{bx_i}}\right]+\sum_{i=1}^{n}\left[\frac{\mathrm{a}\left(1+a\right)x_i\mathrm{e}^{bx_i}}{\left(1+a\mathrm{e}^{bx_i}\right)^2}\right]\log\left(\alpha\right) \quad (37)$$

By equating (35), (36) and (37) to zero and solving for the solution of the non-linear system of equations gives the maximum likelihood estimates of parameters a, b and α.

Application to Engineering dataset

This section evaluates the capability of the proposed APTPD compared to other extensions of the Perks distribution such as the exponentiated Perks distribution (ExpPD) by Singh and Choudhary (2017), the Kumaraswamy-Perks distribution (KumPD) by Oguntunde *et al.* (2018), Burr X distribution, exponential distribution and the Perks distribution (PD) by Perks (1932), using a real-life dataset on the failure and service times of aircraft windshield.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, AIC, Consistent Akaike Information Criterion, CAIC, Bayesian Information Criterion, BIC, Hannan Quin Information Criterion, HQIC, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in Chen and Balakrishnan (1995). Meanwhile, the smaller these statistics are, the better the fit of the distribution is.

Data: This dataset contains the failure and service times of an aircraft windshield (the unit for measurement is 1000 hours), previously utilized by Kundu and Raqab (2009). It has also been used by Tahir et al (2015) and Shanker et al. (2025). The values are:

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483,1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 1: Summary Statistics of the dataset

parameters	**n**	**Minimum**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Vvalues	63	0.046	1.122	2.065	2.820	2.08527	5.14	1.55059	0.43959	-0.26741

The descriptive statistics in table 1 above show that dataset on the service times of windshield of aircraft is unimodal and positively skewed.

Table 2: Maximum Likelihood Parameter Estimates for the service times data

Distribution	**Parameter estimates**			
APTPD	$\hat{a} = 0.2672845$	$\hat{b} = 1.0570255$	$\hat{\alpha} = 2.607397$	-
KumPD	$\hat{a} = 1.8987705$	$\hat{b} = 0.5631136$	$\hat{\alpha} = 1.920286$	$\hat{\beta} = 1.935879$
ExpPD	$\hat{a} = 0.4608782$	$\hat{b} = 0.9671413$	$\hat{\alpha} = 1.436995$	-
PD	$\hat{a} = 0.1476737$	$\hat{b} = 1.0971146$	-	-
BXD	$\hat{a} = 2.0020799$	-	-	-
ExD	$\hat{a} = 0.4794002$	-	-	-

Table 3: The statistics ℓ, AIC, CAIC, BIC and HQIC for the service times data

Distribution	$\hat{\ell}$	***AIC***	***CAIC***	***BIC***	***HQIC***
APTPD	98.94770	203.8954	204.3022	210.3248	206.4241
KumPD	101.96012	211.9202	212.6099	220.4928	215.2919
ExpPD	100.37467	206.7493	207.1561	213.1787	209.2780
PD	100.54199	205.084	205.284	209.3702	206.7698
BXD	285.81751	573.6350	573.7006	575.7782	574.4779
ExD	109.29859	220.5972	220.6628	222.7403	221.4401

Table 4: The A*, W*, K-S statistic and P-values for the service times data

Distribution	*A**	*W**	*K-S*	*P-Value (K-S)*
APTPD	0.04682294	0.2930816	0.06774376	0.9157175
KumPD	0.133804	0.8134708	0.1240379	0.2643994
ExpPD	0.09836014	0.6003667	0.1008918	0.5105678
PD	0.04625393	0.2900577	0.06731094	0.9193069
BXD	0.05381951	0.3330396	0.5368086	6.661338e-16
ExD	0.1861535	1.126509	0.2077025	0.007325983

The following figure presents a plot of estimated PDFs and CDFs of the fitted models to the service times dataset.

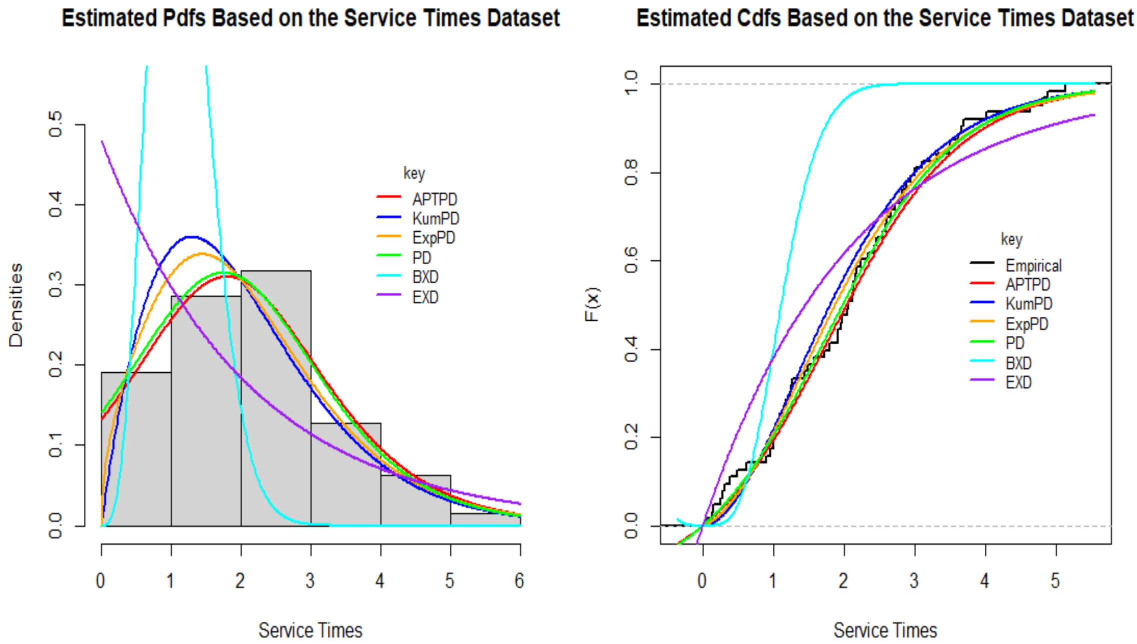

Figure 3: Plots of the estimated densities and CDFs of the fitted distributions to the Data.

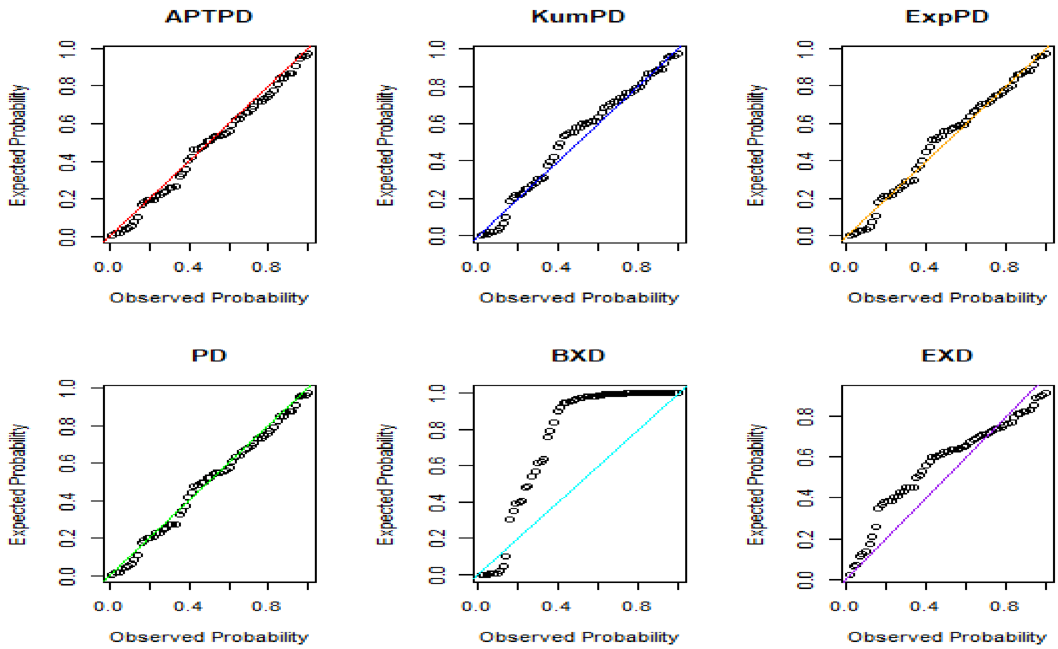

Figure 4: Probability plots for the fitted distributions to the Data.

Tables 2 presents the values of the Maximum Likelihood Estimates of the model parameters based on the service time data, while table 3 presents the values of AIC, CAIC, BIC and HQIC for all the distributions fitted to the data and the values of A*, W* and K-S for the fitted distributions based on the same dataset are provided in Tables 4.

From the results in tables 3 and 4 based on all the model selection measures, it has been found that APTPD has the lowest values of the measures compared to the other fitted distributions and therefore it is considered as the best model to fit the dataset used. The results show that the APTPD is better than the other fitted distributions (the exponentiated Perks distribution (ExpPD), the Kumaraswamy-Perks distribution (KumPD), Burr X distribution (BXD), exponential distribution (EXD) and the Perks distribution (PD)). These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in figures 3 and 4 respectively.

In conclusion, this overall performance of the APTPD is a proof that the alpha power transformed family of distributions is an efficient method for generating continuous distributions as found in Ceren et al. (2018), Ihtisham et al. (2019), Ahmad et al. (2021), Ahmad et al. (2020), Eghwerido (2021), Mohiuddin and Kannan (2021), Ali et al. (2021) as well as Elbatal et al. (2022).

Conclusion

This research employed the Alpha Power Transformed family of distributions to study Alpha Power Transformed Perks distribution (APTPD). The study discussed some properties of the Alpha Power Transformed Perks distribution such as the moments, moment generating function, characteristic function, survival and hazard functions and order statistics. The parameters of the APTPD were estimated using maximum likelihood method. The proposed model was fitted to a real-life dataset on failure and service times of aircraft windshield to demonstrate its usefulness over other existing distributions. The result of the application shows that the APTPD has a better performance compared to the other five fitted distributions based on the dataset used. This implies that the proposed model and its properties will be applied various areas of Science and Engineering.

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