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Alpha Power Transformed Ishita Distribution: Properties and Applications to Medical and Engineering Data

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Abstract

Modeling lifetime and reliability data in medicine and engineering often requires highly flexible statistical distributions capable of capturing skewed, kurtotic, and non-monotonic hazard behaviors, for which classical models such as the exponential, gamma, and Weibull distributions are often inadequate. To address this limitation, numerous generalized families of distributions have been developed, including the Alpha Power Transformed (APT) family, which has gained attention due to its simplicity and capacity to enhance the flexibility and tail behavior of some classical distributions, and the Ishita distribution, which has proven useful for modeling lifetime data with increasing or decreasing hazard rates in medical and reliability contexts. Building on these developments, this study proposes a new extension of the Ishita model known as the Alpha Power Transformed Ishita Distribution (APTID). The study derives and investigates important properties of this distribution, including its moments, moment-generating and characteristic functions, reliability measures, and order statistics, and estimates its parameters using the maximum likelihood method. The performance of the proposed APTID is evaluated using three real-life datasets, namely the remission times of bladder cancer patients, failure times of turbocharger, and body fat percentages of Australian athletes. Model selection criteria such as AIC, BIC, CAIC, and Kolmogorov–Smirnov tests indicate that the APTID consistently outperforms the transmuted Ishita, sine-Ishita, Ishita, Akash, and Lindley distributions. These results confirm that the proposed APTID will be a robust and versatile method for modeling diverse medical and engineering lifetime data.

Keywords: Alpha Power Transformation; Ishita Distribution; Lifetime Data; Moments and Generating Functions; Reliability Analysis; Survival Models; Maximum Likelihood Estimation

Introduction

Probability distributions play a crucial role in modeling real-world phenomena across diverse scientific and engineering disciplines. Traditional models such as the exponential, Weibull, and gamma distributions often fail to capture the complex shapes of hazard functions observed in real-world data (Reis et al., 2022; Xu et al., 2003).

In recent years, continuous efforts to enhance model flexibility and improve data fitting has led to the introduction of several generalized families of distributions. Among these families are; the quadratic rank transmutation map by Shaw and Buckley (2007), Exponentiated T-X by Alzaghal *et al.* (2013), Weibull-X by Alzaatreh *et al.* (2013), Weibull-G by Bourguignon *et al.* (2014), a Lomax-G family by Cordeiro *et al.* (2014), a new Weibull-G family by Tahir *et al.* (2016), a Lindley-G family by Cakmakyapan and Ozel (2016), a Gompertz-G family by Alizadeh *et al.* (2017), the Alpha Power Transformation (APT) method by Mahdavi and Kundu (2017), Odd Lindley-G family by Gomes-Silva *et al.* (2017), odd Lomax-G family by Cordeiro *et al.* (2019), an odd Chen-G family of distributions by Anzagra *et al.* (2022), a new sine family of generalized distributions by Benchiha *et al.* (2023), the Topp-Leone type II exponentiated half logistic-G family of distributions by Gabanakgosi and Oluyede (2023), a novel bivariate Lomax-G family of distributions Fayomi *et al.* (2023), X-exponential-G Family of Distributions by Mohammad (2024) and the new Frechet-G family of continuous probability distributions by Ieren *et al.* (2024). These families introduce additional shape parameters that significantly improve the ability to model skewness and kurtosis in empirical data, and have been used to study many compound distributions including the exponential-Lindley distribution by Ieren and Balogun (2021), transmuted Kumaraswamy distribution by Khan *et al.* (2016), transmuted normal distribution by Ieren and Abdullahi (2020), Lomax-Frechet distribution by Gupta *et al.* (2015), Weibull-exponential distribution by Oguntunde et al. (2015), a transmuted odd Lindley-Rayleigh distribution by Umar *et al.* (2021) and Weibull-Frechet by Afify *et al.* (2016), etc.

The Alpha Power Transformed (APT) family proposed by Mahdavi and Kundu (2017) has gained significant attention due to its simplicity, interpretability, and ability to generate a wide range of distributional shapes. The Alpha Power Transformed (APT) technique provides a simple but effective method of generating new distributions from existing ones by introducing an alpha parameter to enhance flexibility. Since its introduction, the APT family has been used to extended various families and distributions with applications to lifetime and reliability datasets (Ahmad et al., 2020; Elbatal et al., 2022; Ihtisham et al., 2019; Ali et al., 2021; Ahmad et al., 2021; Ceren et al., 2018; Eghwerido, 2021; Mohiuddin and Kannan, 2021).

The Ishita distribution introduced by Shanker and Shukla (2017) has shown potential for modeling lifetime data in medical and engineering systems, particularly due to its capability to describe data with increasing or decreasing hazard rates. The probability density function (pdf) of the Ishita distribution (ID) according to Shanker and Shukla (2017) is defined by

$$g(x)=\frac{\theta^3}{\theta^3+2}\left(\theta+x^2\right)\mathrm{e}^{-\theta x} \tag{1}$$

The corresponding cumulative distribution function (CDF) of ID is given by

$$G(x)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]\mathrm{e}^{-\theta x} \tag{2}$$

where, $x>0$ and $\theta>0$ is a scale parameter of the ID. According to Shanker and Shukla (2017), the ID performed better than the Akash distribution (AKD), Lindley distribution (LnD) and exponential distribution considering two applications to real life data. The model's flexibility has encouraged some of its extensions including the transmuted Ishita distribution (Gharaibeh & Al-Omari, 2019) and sine-Ishita distribution (Yakubu et al., 2025), which demonstrate improved goodness-of-fit to biomedical and reliability datasets.

Considering the capability of the APT family and the robust nature of the Ishita model, this study aims to introduce and explore the statistical properties of the Alpha Power Transformed Ishita distribution (APTID) and demonstrate its applicability to medical and engineering datasets. The model's flexibility and performance will be assessed using three real-life datasets, including bladder cancer remission times and reliability lifetime datasets, to illustrate its superiority over existing models.

The remaining sections of this paper are as follows: definition of the new distribution with its plots is provided in section 2. Section 3 presents the simplification of the pdf of the APTID. Section 4 derived some properties of the proposed distribution. The estimation of parameters using maximum likelihood estimation (MLE) is presented in section 5. An application of the new model with other existing distributions to real life data is done in section 6 and a conclusion is given in section 7.

Alpha Power Transformed Ishita distribution (APTID)

According to Mahdavi and Kundu (2017), the cumulative distribution function (cdf) and the probability density function (pdf) of the Alpha Power transformed family of distributions are defined as:

$$F(x) = \frac{\alpha^{G(x)} - 1}{\alpha - 1} \tag{3}$$

and

$$f(x) = \frac{\log(\alpha)}{(\alpha - 1)} g(x) \alpha^{G(x)} \tag{4}$$

"respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\alpha > 0$ and $\alpha \neq 1$ is the power or shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Putting equations (1) and (2) into equations (3) and (4) and simplifying, we obtain the cdf and pdf of the APTID given in equations (5) and (6) respectively as follows:

$$F(x) = (\alpha - 1)^{-1} \left(\alpha^{\left(1 - \left[1 + \frac{\theta x(\theta x + 2)}{\theta^3 + 2}\right] e^{-\theta x}\right)} - 1 \right) \tag{5}$$

and

$$f(x)=\frac{\theta^3\log(\alpha)}{(\alpha-1)(\theta^3+2)}(\theta+x^2)e^{-\theta x}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)} \tag{6}$$

where $\theta>0$ and $\alpha>0$ are the scale and shape parameters of the APTID respectively, and $x>0$.

Plots of the pdf and cdf of the APTID using some parameter values are presented in figure 1 as follows.

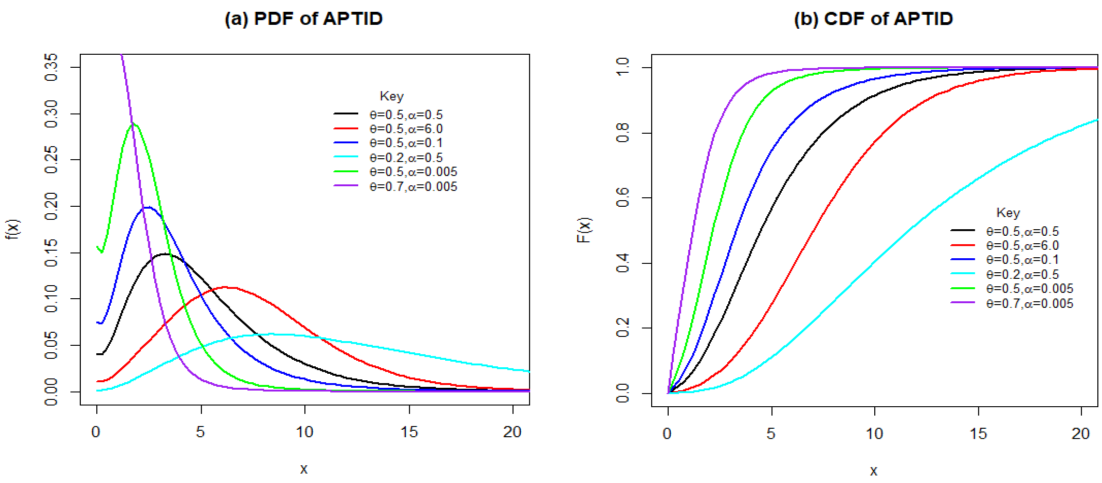

Figure 1: PDF and CDF of the APTID for different values of the parameters.

In figure 1(a) above, it can be seen that the pdf APTID is positively skewed and takes various shapes depending on the parameter values. Also, in figure 1(b) above, the plot of the cdf, it is clear that the value of the cdf equals one when X approaches infinity and equals zero when X tends to zero as normally expected.

Simplification of PDF of Alpha Power Transformed Ishita Distribution (APTID).

This section presents a simplification and linear representation of the PDF of the APTID to enable the derivation of some important properties of the distribution. Recall that the PDF of the APTID is given as:

$$f(x)=\frac{\theta^3\log(\alpha)}{(\alpha-1)(\theta^3+2)}(\theta+x^2)e^{-\theta x}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)} \tag{7}$$

For $\alpha > 0$ and $\alpha \neq 1$, equation (7) can be expanded using power series as:

$$\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)} = \sum_{k=0}^{\infty} \frac{(\log \alpha)^k}{k!} \left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)^k \quad (8)$$

Substituting the result in equation (8) into equation (7) and simplifying gives:

$$f(x) = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{k+1}\theta^3}{(\alpha-1)(\theta^3+2)k!}(\theta+x^2)e^{-\theta x}\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)^k \quad (9)$$

Using the generalized binomial expansion on the last term in equation (9) yields the following:

$$\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)^k = \sum_{l=0}^{\infty}(-1)^l \binom{k}{l}\left(\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)^l \quad (10)$$

Making use of the result in (10) above in equation (9) and simplifying, we obtain:

$$f(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l(\log \alpha)^{k+1}\theta^3}{(\alpha-1)(\theta^3+2)k!}\binom{k}{l}(\theta+x^2)e^{-\theta(l+1)x}\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]^l \quad (11)$$

Also, using binomial expansion on the last term in (11) gives:

$$\left(1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right)^l = \sum_{m=0}^{\infty}\binom{l}{m}\left(\frac{\theta x(\theta x+2)}{\theta^3+2}\right)^m \quad (12)$$

Making use of the result (12) in equation (11) and simplifying, we obtain:

$$f(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^l(\log \alpha)^{k+1}\theta^{m+3}}{(\alpha-1)(\theta^3+2)^{m+1}k!}\binom{k}{l}\binom{l}{m}x^m(\theta+x^2)e^{-\theta(l+1)x}(\theta x+2)^m \quad (13)$$

Also, using binomial expansion on the last term in (13) gives:

$$(\theta x+2)^m = \sum_{r=0}^{\infty}\binom{m}{r}2^{m-r}(\theta x)^r \quad (14)$$

Making use of the result in equation (14) in equation (13) and simplifying, we obtain:

$$f(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^l(\log \alpha)^{k+1}\theta^{m+r+3}2^{m-r}}{(\alpha-1)(\theta^3+2)^{m+1}k!}\binom{k}{l}\binom{l}{m}\binom{m}{r}x^{m+r}(\theta+x^2)e^{-\theta(l+1)x} \quad (15)$$

Now, let $\psi_{klmr} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{l}(\log\alpha)^{k+1}\theta^{m+r+3}2^{m-r}}{(\alpha-1)(\theta^{3}+2)^{m+1}k!}\binom{k}{l}\binom{l}{m}\binom{m}{r}$ be a constant,

which implies that the pdf in equation (15) can also be written in its simple and linear form as:

$$f(x)=\psi_{klmr}x^{m+r}\left(\theta+x^{2}\right)e^{-\theta(l+1)x} \tag{16}$$

Hence, equation (16) is the simplified pdf of the alpha power transformed Ishita distribution (APTID) which will be used subsequently to derive some properties of the APTID.

Derivation of Properties of Alpha Power Transformed Ishita Distribution (APTID)

In this section, some properties of the APTID are derived and discussed as follows:

Moments

Let X denote a continuous random variable the n^{th} moment of X is given by;

$$\mu_{n}^{'}=E\left(X^{n}\right)=\int_{0}^{\infty}x^{n}f(x)dx \tag{17}$$

Substituting the simplified pdf of the APTID from equation (16) into equation (17), the n^{th} ordinary moment of the APTID can be derived as follows:

$$\mu_{n}^{'}=E\left(X^{n}\right)=\int_{0}^{\infty}x^{n}f(x)dx=\psi_{klmr}\int_{0}^{\infty}x^{m+n+r}\left(\theta+x^{2}\right)e^{-\theta(l+1)x}dx \tag{18}$$

Making use of integration by substitution method in equation (18), we perform the following operations:

Let $u=\theta(l+1)x \Rightarrow x=u\left(\theta(l+1)\right)^{-1}$ such that $\frac{du}{dx}=\theta(l+1)\Rightarrow dx=\frac{du}{\theta(l+1)}$

Substituting for x, u and dx in equation (18) and simplifying; we have:

$$\mu_{n}^{'}=E\left(X^{n}\right)=\psi_{klmr}\int_{0}^{\infty}\left(\frac{u}{\theta(l+1)}\right)^{m+n+r}\left(\theta+\left(\frac{u}{\theta(l+1)}\right)^{2}\right)e^{-u}\frac{du}{\theta(l+1)}$$

$$\mu_n' = E\left(X^n\right) = \psi_{klmr}\left[\eta\int_0^\infty \frac{u^{m+n+r}}{\left[\theta(l+1)\right]^{m+n+r+1}} \mathrm{e}^{-u}du + \int_0^\infty \frac{u^{m+n+r+2}}{\left[\theta(l+1)\right]^{m+n+r+3}} \mathrm{e}^{-u}du\right] \tag{19}$$

Recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma\left(k\right)$ and that $\int_0^\infty t^k e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma\left(k+1\right)$

Considering the statement above, the n^{th} ordinary moment of X for the APTID is obtained as:

$$\mu_n' = E\left(X^n\right) = \psi_{klmr}\left[\frac{\theta\Gamma(m+n+r+1)}{\left[\theta(l+1)\right]^{m+n+r+1}} + \frac{\Gamma(m+n+r+3)}{\left[\theta(l+1)\right]^{m+n+r+3}}\right] \tag{20}$$

where $\psi_{klmr} = \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty\sum_{r=0}^\infty \frac{(-1)^l\left(\log\alpha\right)^{k+1}\theta^{m+r+3}2^{m-r}}{(\alpha-1)\left(\theta^3+2\right)^{m+1}k!}\binom{k}{l}\binom{l}{m}\binom{m}{r}$ is a constant.

Moment Generating Function

The moment generating function of a continuous random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^\infty e^{tx}f(x)dx \tag{21}$$

Using the nth ordinary moment of X in equation (20), the moment generating function of a random variable X for the APTID can be derived based on power series expansion as follows:

$$M_X\left(t\right) = E\left[\mathrm{e}^{tx}\right] = E\left[\sum_{n=0}^\infty \frac{(tx)^n}{n!}\right] = \sum_{n=0}^\infty \frac{t^n}{n!}\int_0^\infty x^n f\left(x\right)dx = \sum_{n=0}^\infty \frac{t^n}{n!}E\left(X^n\right) \tag{22}$$

Substituting the result from equation (20) into equation (22) and simplifying, the moment generating function of the APTID is obtained as:

$$M_X\left(t\right) = E\left[\mathrm{e}^{tx}\right] = \sum_{n=0}^\infty \frac{t^n\psi_{klmr}}{n!}\left[\frac{\theta\Gamma(m+n+r+1)}{\left[\theta(l+1)\right]^{m+n+r+1}} + \frac{\Gamma(m+n+r+3)}{\left[\theta(l+1)\right]^{m+n+r+3}}\right] \tag{23}$$

where $\psi_{klmr} = \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty\sum_{r=0}^\infty \frac{(-1)^l\left(\log\alpha\right)^{k+1}\theta^{m+r+3}2^{m-r}}{(\alpha-1)\left(\theta^3+2\right)^{m+1}k!}\binom{k}{l}\binom{l}{m}\binom{m}{r}$ is a constant.

Characteristics Function

The characteristics function for a continuous random variable X is defined as:

$$\phi_x(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} f(x)\,dx \tag{24}$$

The characteristics function for a continuous random variable X can be obtained based on the n[th] ordinary moment using power series expansion as follows:

$$\phi_X(t) = E\left[\mathrm{e}^{itx}\right] = E\left[\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\int_0^\infty x^n f(x)\,dx = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left(X^n\right) \tag{25}$$

Again, substituting for $E\left(X^n\right)$ in equation (25) and simplifying gives the characteristic function of the APTID as:

$$\phi_X(t) = E\left[\mathrm{e}^{itx}\right] = \sum_{n=0}^{\infty}\frac{(it)^n \psi_{klmr}}{n!}\left[\frac{\theta\Gamma(m+n+r+1)}{\left[\theta(l+1)\right]^{m+n+r+1}} + \frac{\Gamma(m+n+r+3)}{\left[\theta(l+1)\right]^{m+n+r+3}}\right] \tag{26}$$

where $\psi_{klmr} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^l(\log\alpha)^{k+1}\theta^{m+r+3}2^{m-r}}{(\alpha-1)\left(\theta^3+2\right)^{m+1}k!}\binom{k}{l}\binom{l}{m}\binom{m}{r}$ is a constant.

Reliability analysis of the APTID.

A derivation and study of the survival function and the hazard rate function is presented in this section. The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{27}$$

Applying the cdf of the APTID in (27), the survival function for the APTID is obtained as:

$$S(x) = 1 - \frac{\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]\mathrm{e}^{-\theta x}\right)} - 1}{(\alpha-1)} \tag{28}$$

A hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{29}$$

Meanwhile, the expression for the hazard rate of the APTID is given by:

$$h(x)=\frac{\theta^{3}\log(\alpha)\left(\theta+x^{2}\right)\mathrm{e}^{-\theta x}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^{3}+2}\right]\mathrm{e}^{-\theta x}\right)}}{\left(\theta^{3}+2\right)\left[(\alpha-1)-\left(\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^{3}+2}\right]\mathrm{e}^{-\theta x}\right)}-1\right)\right]} \tag{30}$$

The plot of the survival and hazard functions for some chosen parameter values are presented in Figure **2** as shown below:

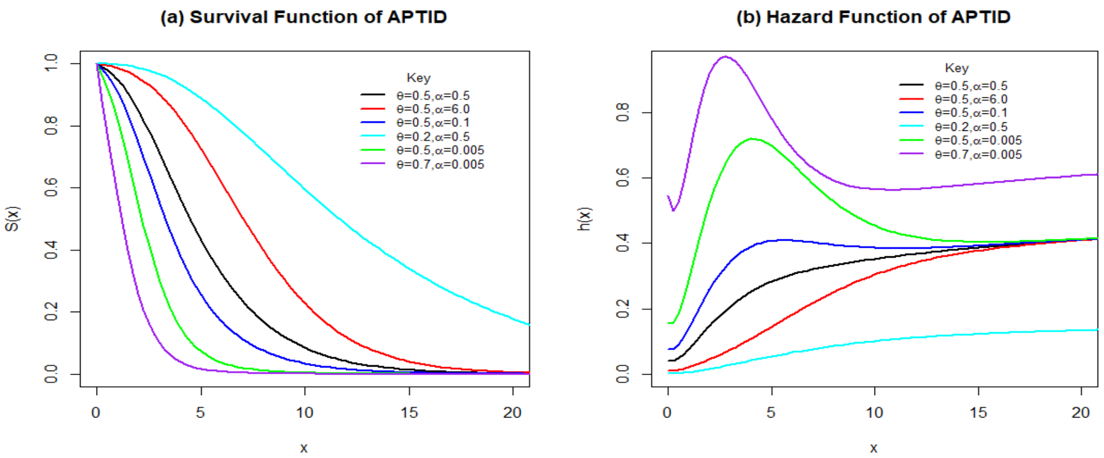

Figure 2: The Survival Function and Hazard Function of APTID.

The plot in Figure **2(a)** shows that the probability of survival is always sure at an initial time or early age and it decreases as time increases up to zero (0) at infinity.

Also, figure **2(b)** revealed that the APTID has an increasing and decreasing failure rate which implies that the probability of failure for any random variable following APTID could be increasing or decreasing with time.

Order Statistics

Suppose $X_1, X_2, \ldots.., X_n$ is a random sample from the APTID and let $X_{1:n}, X_{2:n}, \ldots.., X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}f(x)\left[F(x)\right]^{k+i-1} \tag{31}$$

Using (5) and (6), the pdf of the i^{th} order statistics $X_{i:n}$, can be expressed from (31) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}\frac{\theta^3\log(\alpha)(\theta+x^2)}{(\alpha-1)(\theta^3+2)e^{\theta x}}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}\left[\frac{\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}-1}{(\alpha-1)}\right]^{k+i-1} \tag{32}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the APTID are respectively given by;

$$f_{1:n}(x)=\frac{n!}{(n-1)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-1}{k}\frac{\theta^3\log(\alpha)(\theta+x^2)}{(\alpha-1)(\theta^3+2)e^{\theta x}}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}\left[\frac{\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}-1}{(\alpha-1)}\right]^{k} \tag{33}$$

And

$$f_{n:n}(x)=n\frac{\theta^3\log(\alpha)(\theta+x^2)}{(\alpha-1)(\theta^3+2)e^{\theta x}}\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}\left[\frac{\alpha^{\left(1-\left[1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right]e^{-\theta x}\right)}-1}{(\alpha-1)}\right]^{k+n-1} \tag{34}$$

Maximum Likelihood Estimation of the Parameters of the APTID

Let $X_1, X_2, \ldots\ldots\ldots., X_n$ be a sample of size 'n' independently and identically distributed random variables from the APTID with unknown parameter, θ defined previously.

The likelihood function of the APTID using the pdf in equation (6) is given by:

$$L(\underline{X}\mid\alpha,\theta)=\left(\frac{\theta^3\log(\alpha)}{(\alpha-1)(\theta^3+2)}\right)^n e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left((\theta+x_i^2)\alpha^{\left(1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^3+2}\right]e^{-\theta x_i}\right)}\right) \tag{35}$$

Let the natural logarithm of the likelihood function be, $l=\log L(\underline{X}\mid\eta)$, therefore, taking the natural logarithm of the function equation (35) above gives:

$$l=n\log\left(\frac{\log\alpha}{\alpha-1}\right)+3n\log\theta-n\log(\theta^3+2)+\sum_{i=1}^{n}\log\left[\theta+x_i^2\right]-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left(1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^3+2}\right]e^{-\theta x_i}\right)\log\alpha \tag{36}$$

Differentiating l partially with respect to α and θ gives the following result:

$$\frac{\partial l}{\partial \alpha}=\frac{n(\alpha-1)}{\ln \alpha}\left(\frac{\alpha^{-1}(\alpha-1)-\ln \alpha}{(\alpha-1)^{2}}\right)+\alpha^{-1}\sum_{i=1}^{n}\left(1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^3+2}\right]e^{-\theta x_i}\right) \quad (37)$$

$$\frac{\partial l}{\partial \theta}=\frac{3n}{\theta}-\frac{3n\theta^2}{(\theta^3+2)}+\sum_{i=1}^{n}\left(\theta+x_i^2\right)^{-1}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left\{\frac{\theta^2 x_i e^{-\theta x_i}\left(\theta^3+2\right)^{-2}\log(\alpha)}{\left(\theta^4+\theta^3x_i^2+3\theta x_i+8\theta+2x_i^2\right)^{-1}}\right\} \quad (38)$$

Equating (37) and (38) to zero and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α and θ. It is however difficult to solve these equations analytically and hence more appropriate to employ Newton-Raphson's iteration methods using computing software such as R in this case.

Applications to Medical and Engineering Datasets

This section presents applications of the proposed distribution, Alpha power transformed Ishita distribution (APTID), with other related distributions to three real-life datasets. The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution, APTID, are compared with other competitive models. For all the three datasets, the fits of the Alpha power transformed Ishita distribution (APTID), are compared to the sine-Ishita distribution (SID) by Yakubu et al. (2025), transmuted Ishita distribution (TrID) by Gharaibeh and Al-Omari (2019), Ishita distribution (ISD) by Shanker and Shukla (2017), Akash distribution (AKD) by Shanker (2015) and the Lindley distribution (LnD) by Lindley (1958).

The process of selecting the most fitted distribution was done based on the value of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Kolmogorov-smirnov (K-S) statistics. More information about these measures has also been discussed in Chen and Balakrishnan (1995). Also, it is important to note that the smaller the values of these measures for a particular distribution, the better the fit of that distribution.

Dataset I: This data represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by Lee and Wang

(2003), Rady *et al.* (2016), Ieren and Chukwu (2018) and Abdullahi *et al.* (2018). It is summarized as follows:

Table 1: Summary Statistics for data set I

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

Based on the descriptive statistics in table 1, it can be seen that the dataset on the remission times of a random sample of 128 bladder cancer patients is heavily skewed to the right which could be good with skewed models.

Table 2: Maximum Likelihood Parameter Estimates for dataset I

Distribution	Parameter Estimates	
APTID	$\hat{\theta} = 0.2949883$	$\hat{\alpha} = 0.4027298$
TrID	$\hat{\theta} = 0.2797652$	$\hat{\lambda} = 0.5266383$
SID	$\hat{\theta} = 0.2214239$	-
ID	$\hat{\theta} = 0.3205050$	-
AKD	$\hat{\theta} = 0.3104915$	-
LnD	$\hat{\theta} = 8.0324273$	-

Table 3: The statistics ℓ, AIC, CAIC, BIC K-S statistic and P-values for dataset I

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	K-S	P-Value (K-S)
APTID	442.8728	889.7457	889.8417	895.4497	0.1595432	0.002958474
TrID	444.7638	893.5276	893.6236	899.2316	0.1787437	0.0005609527
SID	479.4714	960.9428	960.9745	963.7948	0.2705085	1.463843e-08
ID	450.6839	903.3679	903.3996	906.2199	0.200951	6.478308e-05
AKD	443.9460	889.8920	889.9237	892.7440	0.1983555	8.446714e-05
LnD	9122.0736	18246.147	18246.179	18248.999	0.958367	1.53583e-102

The following figure presents a plot of estimated PDFs (densities) and CDFs of the fitted models to dataset I.

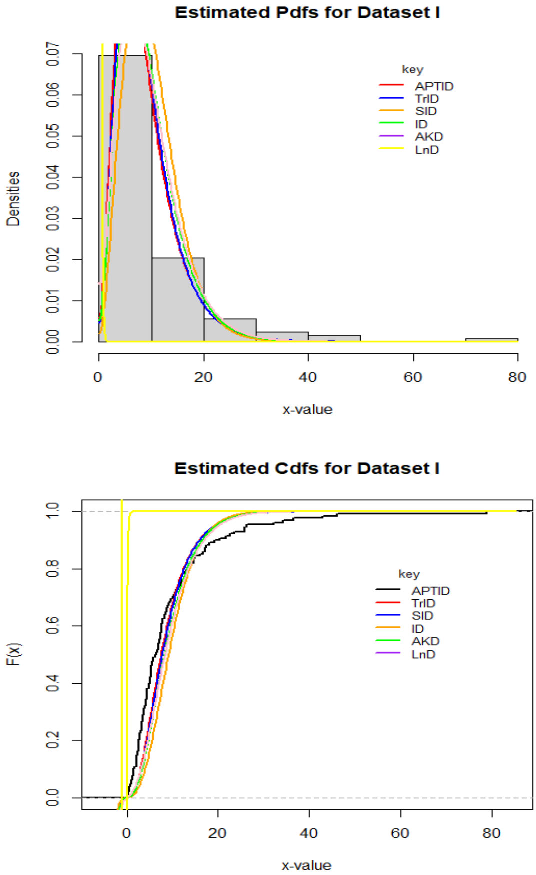

Figure 3: Plots of the estimated densities and CDFs of the fitted distributions to dataset I.

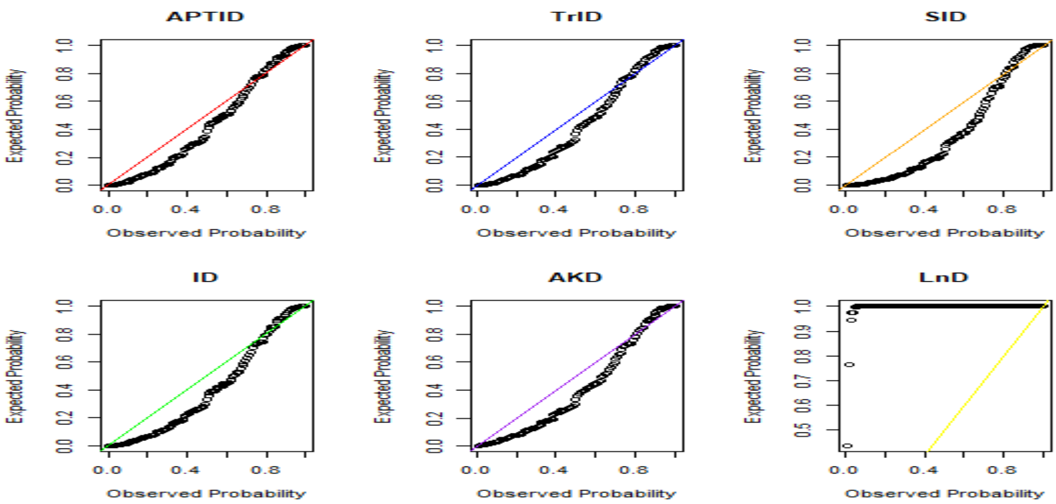

Figure 4: Probability plots for the fitted distributions to dataset I.

Data set II: This second data set has been used by Xu *et al.* (2003), Cordeiro *et al.* (2019), Reis *et al.* (2022) and Ieren et al. (2024). It consists of the time-to-failure of a particular type of engine (turbocharger data). The summary statistics is given as follows:

Table 4: Summary Statistics for data set II

Parameters	n	Mini	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	40	1.60	5.075	6.5	7.825	6.255	9.00	3.83587	-0.65791	-0.36311

Table 5: Maximum Likelihood Parameter Estimates for dataset II

Distribution	Parameter Estimates	
APTID	$\hat{\theta} = 0.6460854$	$\hat{\alpha} = 9.7452583$
TrID	$\hat{\theta} = 0.5833978$	$\hat{\lambda} = -0.7108202$
SID	$\hat{\theta} = 0.3502438$	-
ID	$\hat{\theta} = 0.4605034$	-
AKD	$\hat{\theta} = 0.4503106$	-
LnD	$\hat{\theta} = 5.8367095$	-

Table 6: The statistics ℓ, AIC, CAIC, BIC, K-S statistic and P-values for dataset II

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	K-S	P-Value (K-S)
APTID	90.25469	184.5094	184.8337	187.8871	0.1859731	0.1256844
TrID	91.64165	187.2833	187.6076	190.6611	0.2004468	0.08036051
SID	91.86311	185.7262	185.8315	187.4151	0.1927821	0.1022657
ID	95.70210	193.4042	193.5095	195.0931	0.2176064	0.04527195
AKD	96.88118	195.7624	195.8676	197.4512	0.2299689	0.02907977
LnD	1318.64854	2639.2971	2639.4023	2640.9860	0.9997919	3.731895e-35

The following figure presents a plot of estimated densities and CDFs of the fitted models to dataset II.

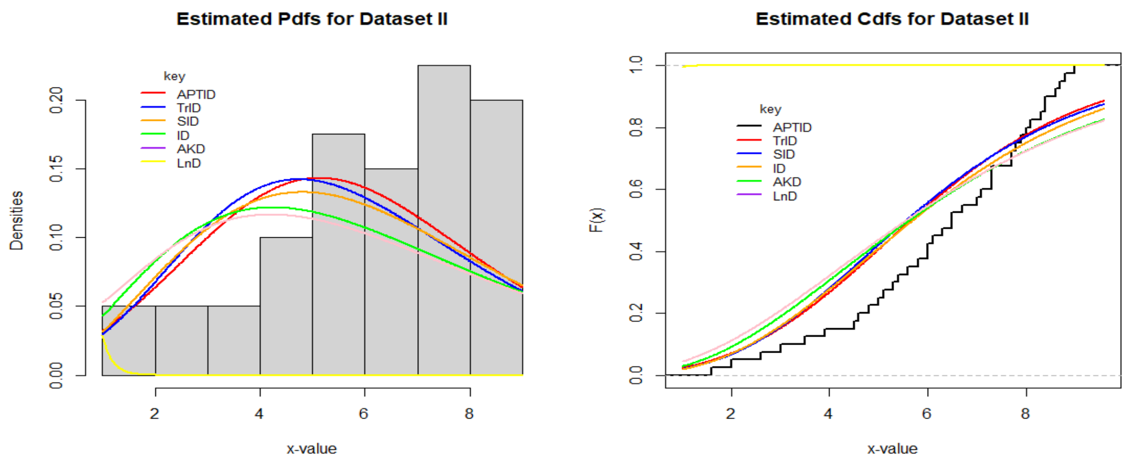

Figure 5: Plots of the estimated densities and CDFs of the fitted distributions to dataset II.

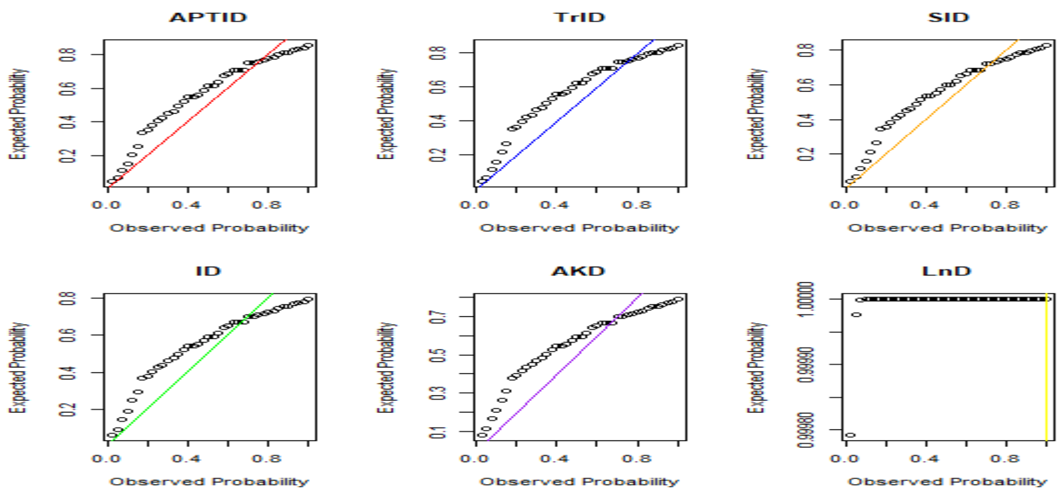

Figure 6: Probability plots for the fitted distributions to dataset II.

Data set III: This dataset reflects the body fat percentage of 202 Australian athletes, it was extracted from Oguntunde *et al.,* (2018) and has also been used by Al-Noor and Hadi (2021), Yakubu et al. (2025) and Ieren *et al.* (2024).

Table 7: Summary Statistics for data set III

Parameters	**n**	**Mini**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Values	202	5.630	8.545	11.650	18.080	13.507	35.520	38.31395	0.75955	-0.1733

Table 8: Maximum Likelihood Parameter Estimates for dataset III

Distribution	Parameter Estimates	
APTID	$\hat{\theta} = 0.2886459$	$\hat{\alpha} = 7.2547917$
TrID	$\hat{\theta} = 0.1729210$	$\hat{\lambda} = 0.6534666$
SID	$\hat{\theta} = 0.1636839$	-
ID	$\hat{\theta} = 0.2185217$	-
AKD	$\hat{\theta} = 0.2186436$	-
LnD	$\hat{\theta} = 0.1390133$	-

Table 9: The statistics ℓ, AIC, CAIC, BIC, K-S statistic and P-values for dataset III

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	K-S	P-Value (K-S)
APTID	642.1867	1288.373	1288.434	1294.990	0.09470003	0.05339799
TrID	647.4434	1298.887	1298.947	1305.503	0.1266825	0.003056686
SID	641.4227	1284.845	1284.865	1288.154	0.1085504	0.01712419
ID	647.9308	1297.862	1297.882	1301.170	0.1335562	0.001483863
AKD	650.1304	1302.261	1302.281	1305.569	0.1443257	0.0004429129
LnD	680.1069	1362.214	1362.234	1365.522	0.2326574	6.364183e-10

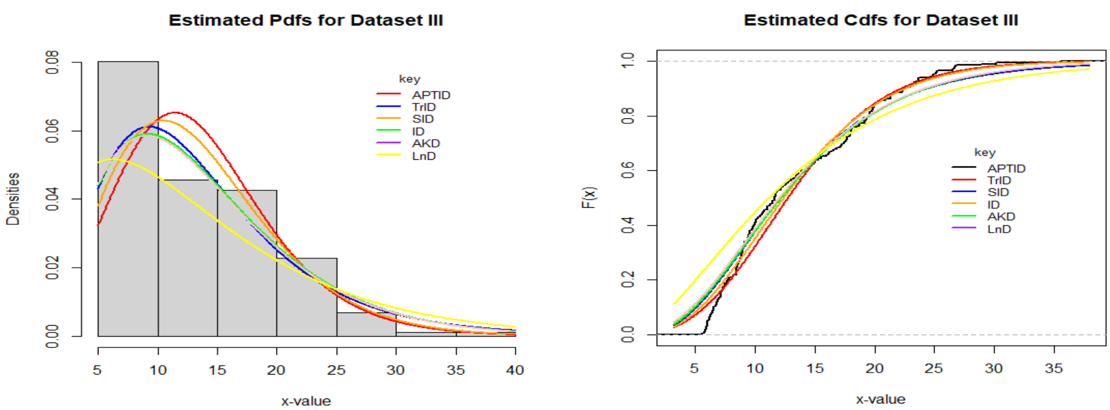

Figure 7: Plots of the estimated densities and CDFs of the fitted distributions to dataset III.

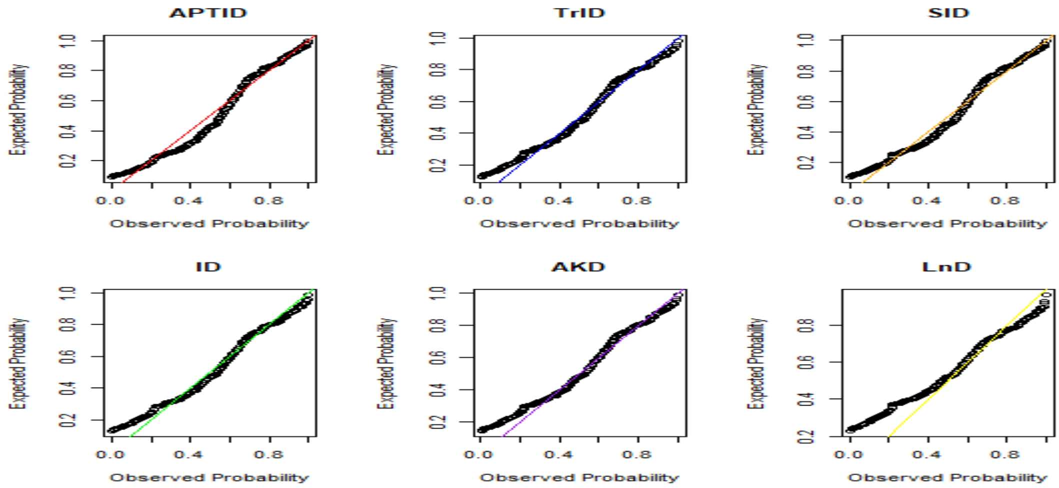

Figure 8: Probability plots for the fitted distributions based on dataset III.

From the results above, the maximum likelihood parameter estimates of the fitted APTID and the other five distributions (TrID, SID, ID, AKD and LnD) based on dataset I, dataset II, and dataset III are presented in Tables 2, 5 and 8 respectively and Tables 3, 6 and 9 list the values of $\hat{\ell}$, AIC, CAIC, BIC and Kolmogorov-Smirnov (K-S) statistics with P-value for the fitted six distributions based on dataset I, dataset II and dataset III respectively. The plots of the fitted densities and CDFs of the fitted six distributions for dataset I, dataset II and dataset III are also shown in Figures 3, 5 and 7 respectively and the probability plots of the fitted six distributions for dataset I, dataset II and dataset III are shown in figures 4, 6 and 8 respectively.

Using the previously listed model selection criteria, it clearly shows that the APTID performs better than the other five related distributions. This is due to the lowest values of the AIC, BIC, CAIC, K-S and the highest p-value of the KS test given in Tables 3, 6 and 9 for dataset I, dataset II and dataset III respectively. These results which are based on the values of AIC, CAIC, BIC and K-S is also in line with the plots of the estimated densities and CDFs of the fitted distributions shown in Figures 3, 5 and 7 as well as the probability plots in Figures 4, 6 and 8 for dataset I, dataset II and dataset III respectively.

In conclusion, this overall performance of the APTID is a proof that the alpha power transformed family of distributions is effective for generating continuous distributions as previously reported by other authors (Ceren et al., 2018; Ihtisham et al., 2019; Ahmad et al., 2021; Ahmad et al., 2020; Eghwerido, 2021; Mohiuddin and Kannan, 2021; Ali et al., 2021; Elbatal et al., 2022).

Conclusion

In this work, the Alpha Power Transformed family of distributions was used to study Alpha Power Transformed Ishita distribution (APTID). The study discussed some properties of the APTID such as the moments, generating functions, reliability analysis, and order statistics. The maximum likelihood method was used to estimate the parameter of the APTID. The proposed model was fitted to three real life datasets to demonstrate its performance and applicability over other existing distributions. The results of the applications show that the APTID has a better performance compared to the other five fitted distributions based on the datasets used. This shows that the APTID and its properties will attract applications in several areas of statistical theory and related fields.

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