Mikailalsys Journal of
Mathematics and Statistics

e-ISSN : 3030-816X
p-ISSN : 3030-8399
Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJMS.v3i3.7258

On the Generalized Ulam–Hyers Stability for Caputo Fractional Derivatives with Nonlocal Conditions

Jackson Efiong Ante[1], Runyi Emmanuel Francis[2], Samuel Okon Essang[3], Ede Moses Aigberemhon[4], Samuel Adamu[5], Michael Ogar-Abang[6], Stephen Ikenna Okeke[7]

[1]Topfaith University, Mkpatak, Nigeria; [2]Federal Polytechnic, Ugep, Nigeria; [3,6]Arthur Jarvis University, Akpabuyo, Nigeria; [4]University of Cross River State, Calabar, Nigeria; [5]Nigerian Army University, Biu, Nigeria; [7]David Umahi Federal University of Health Sciences, Uburu, Ebonyi State, Nigeria.

jackson.ante@topfaith.edu.ng; runyifrancis@fedpolyugep.edu.ng

Article Info:

Submitted:	Revised:	Accepted:	Published:
Aug 10, 2025	Sep 3, 2025	Sep 15, 2025	Sep 20, 2025

Abstract

This paper addresses the need for rigorous stability criteria in fractional integro-differential systems by investigating generalized Ulam–Hyers stability for equations involving a fractional-order derivative. The research objective is to establish sufficient conditions that guarantee generalized Ulam–Hyers stability for Caputo fractional differential equations. Methodologically, the study employs concise mathematical arguments together with an application of the Gronwall inequality to derive the required conditions. The key findings demonstrate that these conditions ensure stability in the generalized Ulam–Hyers sense for the considered class of equations. The paper concludes that the obtained results extend and improve upon existing findings in the literature. The contribution lies in refining the stability theory for fractional-order models by providing tractable criteria grounded in Gronwall-type estimates and Caputo derivatives.

Keywords: Ulam–Hyers Stability; Generalized Stability; Caputo Derivative; Fractional Integro-Differential Equations; Gronwall Inequality; Non-Local Conditions

Introduction

Ulam-Hyers stability is an aspect of stability theory that deals with the stability of functional equations. Unarguably, this aspect of stability theory of differential equations have attracted a lot of attention in recent times since the existence of analytical solutions of the given problems involving Ulam stabilities are guaranteed through the appropriate assumptions on the approximate solutions[35]. Ulam's concept of stability actually began with S. M. Ulam, whom in the year 1940 during an inaugural speech at the University of Wisconson, defined the stability of a functional equation. The idea was however extended by Hyers in [16] via the additive function specified on the Banach space. This discovery further prompted Rassias to generalize and broaden the stability notion, producing the Ulam-Hyers–Rassias stability (see [35]). Since then, research interest have sparked off in the area giving way to massive research inputs by mathematicians on the generalization of the Hyers' result in terms of control conditions used to defined the concept of an approximate solution (see [23] and the references therein). It was again observed that the Ulam stability theory plays an important role in realistic problems in Biology, economics, and especially numerical analysis where the explicit solution is difficult to find [34]. Qualitative results on the existence, uniqueness and Ulam stability have been investigated in [1,2,8,14,24]. Also, fundamental results on the Ulam stability, Ulam-Hyers stability as well as Ulam-Hyers-Rassias stability for fractional order derivatives have been examined in the literature (see [6,9,13,14,15,16, 29,30,31,32,33,34] and the references therein).

The theory of fractional calculus - which is a generalization of the classical calculus, is as old as the corresponding theory of the derivative of integer order and spans over 300 centuries [22]. As observed in [34], the theory of fractional calculus arise naturally in physics and engineering such as electrical circuits, chemical sciences, biology, control theory, fitting of experimental data, biophysics, etc. thereby providing viable instrument in the modeling of many physical phenomena [9].

Recently, its relevance have also cut across several other fields like the mathematical physics, engineering, control theory, etc. The systematic studies of the concept started about the 19th century by Liouville, Riemann, Leibnitz, Caputo, etc. [26].

However, in the qualitative theory of fractional differential equations (FrDEs), one of the main properties of interest is that of stability of solutions, although the study of the stability property of (FrDEs) is relatively new. Fundamental results on the existence and uniqueness of solutions of FrDE using scalar Lyapunov function was examined in [18], and a definition for the Caputo fractional Dini derivative of the Lyapunov function was proposed. Due to some difficulties encountered in the application of this definitions pointed out in [11], a new definition was proposed in [3] and some sufficient conditions for the stability properties of nonlinear system using a scalar Lyapunov-like functions were obtained. In [7], qualitative results for scalar FrDEs were obtained using the Lyapunov functional and the matrix inequality. Moreover, as observed in [10], other methods for examining stability of fractional order systems using Lyapunov-like functions abound, though marked by several difficulties and restrictions.

The novelty of this paper, as motivated by [23,33] is in the establishment of the sufficient conditions for the generalized Ulam-Hyers stability for a given Caputo FrDEs. By the mathematical application of the Gronwall inequality, the generalized stability results in the sense of Ulam-Hyers is obtained.

Basic Notations and Definitions

Let R^n be the n-dimensional Euclidean space with norm $\|.\|$. Let Ω be a domain in R^n containing the origin; $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, $t_0 \in R_+$, $t > 0$ and let $J \subset R_+$.

Now, fractional calculus is seen as a natural generalization of the classical calculus of integer order and thereby allows for the extension of the traditional concepts of derivative and integral to functions with fractional orders. By this extension, functions with noninteger orders are much more flexible in describing real world systems (see [14, 25, 26,34,37]).

There are several definitions of fractional derivatives and fractional integrals.

General case. Let the number $n-1<\beta<n,\beta>0$ be given, where n is a natural number, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.1.

The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}_{t_0}^{RL}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t\geq t_0 $$

Definition 2.2.

The Caputo fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}_{t_0}^{C}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t\geq t_0. $$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3.

The Grunwald-Letnikov fractional derivative of order β of $\gamma(t)$ is given by (See [1])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0. $$

and

Definition 2.4. The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (See [1])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0, $$

where ${}^{\beta}C_r$ are the binomial coefficients and $\left[\frac{t-t_0}{h}\right]$ denotes the integer part of $\frac{t-t_0}{h}$.

Particular case (when n=1). In most applications, the order of β is often less than *1*, so that $\beta\in(0,1)$. For simplicity of notation, we will use ${}^{C}D^{\beta}$ instead of ${}_{t_0}^{C}D^{\beta}$ and the Caputo fractional derivative of order β of the function $\gamma(t)$ is

$$ {}^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{-\beta}\gamma'\,ds,\ t \geq t_0. \tag{2.1} $$

Main Results

(Ulam-Hyers Stability Results)

Let ε be positive real number, $f:[a,b)\times E \to E$ be a continuous operator and $\sigma:[a,b)\times E \to R_+$ be a continuous function.

We consider the following differential equations

$$ {}^{C}D^{\beta}\gamma(t) = f(t,\gamma(t)),\quad \beta(0,1)\ or\ (1,2),\ t\in[a,b) \tag{3.1} $$

And the following inequalities

$$ \left|{}^{C}D^{\beta}\omega(t) = f(t,\omega(t))\right| \leq \varepsilon,\ t\in[a,b) \tag{3.2} $$

$$ \left|{}^{C}D^{\beta}\omega(t) = f(t,\omega(t))\right| \leq \alpha(t),\ t\in[a,b) \tag{3.3} $$

$$ \left|{}^{C}D^{\beta}\omega(t) = f(t,\omega(t))\right| \leq \varepsilon\alpha(t),\ t\in[a,b) \tag{3.4} $$

Definition 3.1.

The equation (3.1) is Ulam- Hyers stable if there exists a real number $\wp_0 > 0$ such that for each $\varepsilon > 0$ and for each solution $\omega \in C^1([a,b),E)$ of the inequality (3.2) there exists a solution $\gamma \in C^1([a,b),E)$ of the equation (3.1) with

$$ |\omega(t)-\gamma(t)| \leq \wp_0\varepsilon,\ t\in[a,b). $$

Definition 3.2.

The equation (3.1) is generalized Ulam- Hyers stable if there exists a real number $\lambda_0 \in C(R_+,R_+), \lambda_0(0)=0$ such that for each solution $\omega \in C^1([a,b),E)$ of the inequality (3.2) there exists a solution $\gamma \in C^1([a,b),E)$ of the equation (3.1) with

$$ |\omega(t)-\gamma(t)| \leq \lambda_0\varepsilon,\ t\in[a,b). $$

Definition 3.3.

The equation (3.1) is Ulam-Hyers-Rassias stable with respect to α if there exists $\wp_{0,\alpha} > 0$ such that for each $\varepsilon > 0$ and for each solution $\omega \in C^1([a,b),E)$ of the inequality (3.4) there exists a solution $\gamma \in C^1([a,b),E)$ of the equation (3.1) with

$$\left|\omega(t) - \gamma(t)\right| \leq \wp_{0,\alpha}\varepsilon\alpha(t),\ t \in [a,b).$$

Definition 3.4.

The equation (3.1) is generalized Ulam-Hyers-Rassias stable with respect to α if there exists a $\wp_{0,\alpha} > 0$ such that for each solution $\omega \in C^1([a,b),E)$ of the inequality (3.3) there exists a solution $\gamma \in C^1([a,b),E)$ of the equation (3.1) with

$$\left|\omega(t) - \gamma(t)\right| \leq \wp_{0,\alpha}\alpha(t),\ t \in [a,b).$$

Remark 3.1.

A function $\omega \in C^1([a,b),E)$ is a solution of the inequality (3.2) if and only if there exists a function $\kappa_\omega \in C^1([a,b),E)$ (which depends on ω) such that

(i) $\left|\kappa_\omega\right| \leq \varepsilon,\ t \in [a,b)$

(ii) ${}^C D^\beta \omega(t) = f(t,\omega(t)) + \kappa_\omega,\quad t \in [a,b)$

Remark 3.2.

A function $\omega \in C^1([a,b),E)$ is a solution of the inequality (3.3) if and only if for each $\alpha(t)$ there exists a function $\kappa_\omega \in C^1([a,b),E)$ (which depends on ω) such that

(i) $\left|\kappa_\omega\right| \leq \alpha(t),\ t \in [a,b)$

(ii) ${}^C D^\beta \omega(t) = f(t,\omega(t)) + \alpha(t),\quad t \in [a,b)$

Remark 3.3.

A function $\omega \in C^1([a,b),E)$ is a solution of the inequality (3.4) if and only if for each $\varepsilon > 0$ *and* $\alpha(t)$ there exists a function $\kappa_\omega \in C^1([a,b),E)$ (which depends on ω) such that

(i) $|\kappa_\omega| \le \varepsilon\alpha(t), \ \ t \in [a,b)$

(ii) ${}^{C}D^{\beta}\omega(t) = f(t,\omega(t)) + \varepsilon\alpha(t), \quad t \in [a,b)$

Lemma 3.2.

Let $0 < \beta < 0,$ if $\omega \in C^1([a,b),E)$ is a solution of the inequality (3.3), then ω is a solution of the following integral equation

$$\left|\omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds\right| \le \alpha(t)\frac{(t-t_0)^{\beta}}{\Gamma(\beta+1)}, \quad t \in [t_0,\infty)$$

Proof.

By the statement of Remark 3.2 we have that,

$${}^{C}D^{\beta}\omega(t) = f(t,\omega(t)) + \alpha(t), \quad t \in [a,b)$$

Integrating both sides in the Caputo sense,

$$I^{-\beta}({}^{C}D^{\beta}\omega(t)) = I^{-\beta}(f(t,\omega(t))) + I^{-\beta}\alpha(t), \quad t \in [a,b)$$

$$\omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\kappa_\omega(s)ds, \ t \in [t_0,\infty)$$

$$\omega(t) = \omega(t_0) + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\kappa_\omega(s)ds, \ t \in [t_0,\infty)$$

Taking the norm of both sides,

$$\left|\omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\omega(t))ds\right| = \left|\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\kappa_\omega(s)ds\right|, \ t \in [t_0,\infty)$$

$$\le \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}|\kappa_\omega(s)|ds$$

$$\le \frac{\alpha(t)}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}ds$$

$$\le \frac{\alpha(t)}{\Gamma(\beta)}\cdot\frac{(t-t_0)^{\beta}}{\beta}$$

$$\le \frac{\alpha(t)}{\beta\Gamma(\beta)}.(t-t_0)^{\beta} = \frac{\alpha(t)}{\Gamma(\beta+1)}.(t-t_0)^{\beta} \qquad \square$$

Theorem 3.1.

Let $0 < \beta < 0$ and consider (3.1) in relation to (3.3).

Assume that:

(i) $f \in C([t_0, \infty) \times E, E)$;

(ii) The function f is locally Lipschitz with respect to the second argument υ, that is, there exists $\ell > 0$ such that

$|f(t,\upsilon_1) - f(t,\upsilon_2)| \le \ell |\upsilon_1 - \upsilon_2|$, $t \in [t_0, \infty)$ and all $\upsilon_1, \upsilon_2 \in E$;

(iii) Let $\alpha \in C([t_0, \infty), R_+)$ be an increasing function. Suppose there exists a positive function $\hat{\lambda}_0 \in C(R_+, R_+), \hat{\lambda}_0(0) = 0$ for each $\varepsilon > 0$ such that

$$\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \alpha(s)) ds \le \varepsilon \hat{\lambda}_0, \;\; t \in [t_0, \infty).$$

Then (3.1) is generalized Ulam-Hyers stable.

Proof.

Let $\omega \in C^1([a,b), E)$ is a solution of the inequality (3.3), and let γ be the unique solution of the initial value problem (IVP),

$$\begin{aligned} {}^{C}D^{\beta}\gamma(t) &= f(t, \gamma(t)), \quad t \in [t_0, \infty) \\ \gamma(t_0) &= \omega(t_0) \end{aligned} \tag{3.5}$$

Integrating both sides in the Caputo sense,

$$I^{-\beta}({}^{C}D^{\beta}\gamma(t)) = I^{-\beta}(f(t, \gamma(t))) \quad t \in [a,b)$$

$$\gamma(t) = \gamma(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(t)) ds, \; t \in [t_0, \infty)$$

$$\gamma(t) = \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(t)) ds, \; t \in [t_0, \infty) \tag{3.6}$$

Using the inequality (3.3), we have

$$\omega(t) - \omega(t_0) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \le\le \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \alpha(s)) ds$$

$$\leq \varepsilon \qquad (3.7)$$

Combining equations (3.6) and (3.7) we have that

$$|\omega(t) - \gamma(t)| \leq | \omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(t)) ds \,|$$

$$\leq | \omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds$$
$$- \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \gamma(s)) ds \,|$$

$$\leq | \omega(t) - \omega(t_0) - \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, \omega(s)) ds \,| +$$
$$\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \,| f(s, \omega(s)) - f(s, \gamma(s)) |\, ds \qquad (3.8)$$

Substituting (3.7) into (3.8) gives,

$$\leq \varepsilon + \frac{\ell}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} \,| \omega(s) - \gamma(s) |\, ds$$

Then there exists a constant $\ell^* > 0$ independent of $\wp_{0,\alpha}\alpha(t)$ such that

$$| \omega(s) - \gamma(s) | \leq \ell^* \varepsilon := \lambda_0 \varepsilon \,, \text{ where } \wp_0 \; and \;\; t \in [t_0, \infty) \qquad \square$$

Hence, the equation (3.1) is generalized Ulam-Hyers stable.

Conclusion

Ulam-Hyers stability is a concept in mathematics that deals with the stability of functional equations. The uniqueness of this concept is premised on the idea as to whether a small perturbation in the equation can lead to a small change in the solution.

In this paper, an attempt is made at investigating the stability of nonlinear Caputo fractional order derivative in the sense of generalized Ulam-Hyers. By adopting the simple mathematical proofs, and using the Gronwall inequality, the generalized Ulam-Hyers stability result is established.

References

[1] Abbas, S., Benhohra, M. (2017). Existence and Ulam stability for Fractional Differential Equations of Hilfer-Hadamard type. *Adv. Differ. Equ.* 2017,180.

[2] Abbas, S., Benhohra, M. (2015). Existence and Ulam stability results for Quadratic Integral Equations. *Lib. Math. (N. S.) 35(2),* 83-93.

[3] Achuobi, J. O., Akpan, E. P., George, R., Ofem, A. E. (2024). Stability analysis of Caputo fractional time-dependent systems with delay using vector lyapunov functions, *AIMS Mathematics. 9,* 28079-28099 . doi: 10.3934/math.20241362

[4] Ante, J. E., Itam, O. O., Atsu, J. U., Essang, S. O., Abraham, E. E., Ineh, M. P. (2024). On the Novel Auxiliary Lyapunov Function and Uniform Asymptotic Practical Stability of Nonlinear Impulsive Caputo Fractional Differential Equations via New Modelled Generalized Dini Derivative. African *Journal of Mathematics and Statistics Studies 7(4),* 11-33. Doi: 10.52589/AJMSS-VUNAIOBC.

[5] Ante, J. E., Atsu, J.U., Abraham, E.E., Itam, O. O., Oduobuk, E. J., & Inyang, A. B. (2024). On a Class of Piecewise Continuous Lyapunov functions and asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative, *IEEE-SEM Journal Publication 12*, 2320-9151.

[6] Atsu, J. U., Ante, J. E., Inyang, A. B. & Akpan, U. D. (2024). A Survey on the Vector Lyapunov Functions and Practical Stability of Nonlinear Impulsive Caputo Fractional Differential Equations via new Modelled Generalized Dini Derivative. *IOSR Journal of Mathematics (IOSR-JM), 20(4),* 28-42. DOI: 10.9790/5728-2004012842.

[7] Agarwal, R., O'Regan, D. & Hristova, S. (2015). Stability of Caputo Fractional Differential Equations. *Applications of Mathematics. 60(6)*, 653-676.

[8] Akpan, E. P. & Akinyele, O. (1992). On the ϕ_0-Stability of Comparison Differential Systems. *Journal of Mathematical Analysis and Applications, 164(2),* 307-324.

[9] Baleanu, D. & Mustafa, O. G. (2010). On the Global Existence of Solutions to a Class of Fractional Differential Equations. *Comput. Math. Appl. 59*. 1835-1841.

[10] Basci, Y., Ogrekci, S. & Misir, A. (2019). On Hyers-Ulam stability for Fractional Differential Equations including the new Caputo-Fabrizio Fractional Derivative. *Mediterr. J. Math. 16:131*. https://doi.org/10.1007/s00009-019-1407-x1660-5446/19/050001-14

[11] Burton, T. A. (2011). Fractional Differential Equations and Lyapunov functionals. Nonlinear Analysis: Theory, Methods and Applications, *74(16)*, 5648-5662.

[12] Chalishajar, D., Kumar Existence, A. (2018). Uniqueness and Ulam's Stability of Solutions for a Coupled System of Fractional Differential Equations with Integral Boundary Conditions. *Electron. J. Qual. Theory differ. Equ. 52,* 1-16.

[13] Dai, Q., Gao, R., Li, Z. and Wang, C. (2020). Stability of Ulam-Hyers and Ulam-Hyers-Rassias for a Class of Fractional Differential Equations. *Advances in Difference Equations 2020:103*. https://doi.org/10.1186/s13662-020-02558-4.

[14] Diethelm, K. (2010). The Analysis of Fractional Differential Equations. An Application Oriented Exposition Using Differential Operators of Caputo type. Springer, Berlin.

[15] Devi, J. V., Mc Rae, F. A. & Drici, Z. (2012). Variational Lyapunov method for Fractional Differential Equations, *Comput. Math. Appl,* 64, 2982-2989.

[16] Hyers, D. H. (1941). On the Stability of the Linear Functional Equation. *Proc. Natl. Acad. Sci. USA 27,* 222-224.

[17] Ibrahim, R. W. (2013). Ulam Stability of Boundary Value Problem. *Kragujevac J. Math. 7,* 287-297.

[18] Igobi, D. K. & Ante, J. E. (2018). Results on Existence and Uniqueness of Solutions of Impulsive Neutral Integro-Differential System. *Journal of Mathematics Research, 10(4),* http://doi:10.5539/jmr.v10n4p165.

[19] Igobi, D. K. & Ineh, M. P. (2024). Results on Existence and Uniqueness of Solutions of Dynamic Equation on Time Scale via Generalized Ordinary Differential Equations. *International Journal of Applied Mathematics, 37(1),* 1-20.

[20] Ineh, M.P., Akpan, E.P., & Nabwey, H. (2024). On Lyapunov Stability of Caputo Fractional Dynamic Equations on Time Scale using a New Generalized Derivative, Preprints.

[21] Jung, S. M. (2004). Hyers-Ulam Stability of Linear Differential Equations of First Order. *Appl. Math. Lett. 17(10)*, 1135-1140.

[22] Jung, S. M. (2005). Hyers-Ulam Stability of Linear Differential Equations of First Order III. *J. Math. Anal. Appl. 311(1)*, 139-146.

[23] Jung, S. M. (2004). Hyers-Ulam Stability of Linear Differential Equations of First Order II. *Appl. Math. Lett. 19(9),* 854-858.

[24] Lakshmikantham, V. & Vatsala, A. S. (2008). Basic Theory of Impulsive Differential Equations, *Nonlinear Analysis: Theory, Methods & Applications, 69(8)*, 2677-2682.

[25] Lakshmikantham, V., Leela, S. & Sambandham, M. (2008). Lyapunov Theory for Fractional Differential Equations, *Communications in Applied Analysis, 12 (4)*, 365.

[26] Lakshmikantham, V., Leela, S. & Vasundara, D. J. (2009). Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers.

[27] Li, Y., Chen, Y., & Podlubny, I. (2010). Stability of fractional order nonlinear dynamic systems: Lyapunov Direct Method and Generalized Mittag-leffler Stability. *Comput. Math. Appl. 59,* 1810-1821.

[28] Li, C., Qian, D. & Chen,Y. (2011). On Riemann-Liouville and Caputo derivatives. *Discrete Dyn. Nat. Soc,* Article ID 562494, 15 pages, 2011.

[29] Li, C. P. & Zhang, F. R. (2011). A survey on the Stability of Fractional Differential Equations, *Eur. Phys. J. Special Topics 193,* 27-47, doi:10.1140/epjst/e2011-01379-1.

[30] Murali, R., Selvan, A. P., Baskaran, S., Park, C. & Lee, J. R. (2021). Hyers-Ulam Stability of First-Order Linear Differential Equations using Aboodh Transform. *Journal of Inequalities and Applications, 2012(133)*, https://doi.org/10.1186/s13660-021-02670-3

[31] Peng, S., Wang, J. (2015). Existence and Ulam-Hyers Stability of ODEs involving two Caputo Fractional Derivative. *Electron. J. Qual. Theory differ. Equ. 52,* 1-16.

[32] Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier.

[33] Kilbas, A. A., Marichev, O. I., & Samko, S. G. (1993). Fractional integrals and Derivatives (Theory and Applications).

[34] Rassias, Th. M. (1978). On the Stability of the Linear Mapping in Banach Spaces, *Proc. Amer. Math. Soc. 72,* 297-300.

[35] Ulam, S. M. (1960). A Collection of the Mathematical Problem. Interscience, New York.

[36] Vanterler da C. Sousa, J., Capelas de Oliveira, E. (2018). Ulam-Hyers Stability of a Nonlinear Fractional Volterra Integro-Differential Equation. *Appl. Math. Lett. 81,* 50-56. https://doi.org/10.1016/j.aml.2018.01.016.

[37] Khan, A., Shan, K.Li, Y., et al. (2017). Ulam type Stability for a Coupled System of Boundary Value problems of Nonlinear Fractional Differential Equation. *J. Funct. Spaces 2,* 1-8.

[38] Tunc, O., Tunc, C. (2024). On Ulam Stabilities of Iterative Fredholm and Volterra Integral Equations with Multiple Time-Varying delays. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* https://doi.org/10.1007/s13398-024-01579-y.

[39] Wang, J., Lv, L., Zhou, Y. (2011). Ulam Stability and Data Dependence for Fractional Differential Equations with Caputo Derivative. *Electronic J. Qual. Theory differ. Equ. 2011,* 63.

[40] Wang, J., Lv, L. and Zhou, Y. (2011). Ulam Stability and Data Dependence for Fractional Differential Equations with Caputo Derivative. *Electronic Journal of of Qualitative Theory Differential Equations. No. 63,* 1-10; http://www.math.u-szeged.hu/ejqtde/

[41] Vu, H., Hoa, V. N. (2022). Hyers-Ulam Stability of Fractional Integro-differentialEquation with a Positive Constant Coefficient involving the Generalized Caputo Fractional Derivative. *Filomat 36(18)*, 6299-6316, https://doi.org/10.2298/FIL2218299V.