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An Improved Black–Scholes Model to Determine the Optimal Boundary of Asset–Liability

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Abstract

The study addresses limitations of the Black–Scholes framework, specifically its reliance on a risk-neutral market and a self-financing hedging portfolio by proposing a generalized derivative pricing approach grounded in the efficient markets hypothesis. The research objective is to establish a valuation model in which a derivative's fair value equals a conditional expectation discounted at the underlying asset's drift, thereby explicitly retaining the asset's drift rather than abstracting it away under risk neutrality. Methodologically, the paper develops a partial differential equation (PDE) that replaces the risk-free rate with an efficiency-consistent discount rate, derives a pricing formula for European call options that incorporates the underlying's drift, and analyzes the optimal exercise boundary for American call options under varying parameters. Key findings show that the optimal exercise price increases with higher volatility and risk-free interest rates and decreases with higher dividend yield; moreover, it is never optimal to exercise an American call option early when the underlying pays no dividends. The study concludes that an efficiency-based discounting scheme offers a coherent alternative to risk-neutral valuation while preserving internal consistency with observed market dynamics. The contribution and implication are a drift-inclusive theoretical framework that refines PDE-based pricing, clarifies comparative statics for exercise policy, and provides practitioners with guidance for pricing and exercise decisions in settings where asset drift is informationally relevant.

Keywords: Option Pricing; Derivative Valuation; Efficient Markets Hypothesis; Asset Drift; European Call Options; American Call Options; Optimal Exercise Boundary; Black–Scholes Model

Introduction

The pricing of financial derivatives underwent a significant transformation in 1973 with the introduction of the groundbreaking Black-Scholes framework (Black & Scholes, 1973). This seminal model provides a linear parabolic partial differential equation (PDE) that governs the value of financial instruments, such as European call options. Notably, an explicit formula is available to calculate the option's value, contingent upon key parameters. To derive these formulas, the Black-Scholes PDE is typically transformed into a constant coefficient PDE, leveraging techniques like Fourier analysis. The essential parameters required for this calculation include: Risk-free interest rate, Volatility and Exercise price (strike price). The Black-Scholes model's performance has been scrutinized in empirical terms, as it often fails to accurately predict true market values of options. Despite its limitations, the model remains a widely used benchmark for traders, financial markets professionals, and risk managers. However, several studies, including those by Bergman (1982) and Musiela and Rutkowski (2005), have challenged the model's underlying assumptions. Specifically, they argue that the hedging portfolio in the original model is not self-financing. A more fundamental critique is that the model's premise is overly narrow, rendering the delta-hedging argument largely irrelevant to practitioners. One of the most counterintuitive aspects of the Black-Scholes model is the independence of the option's fair price from the underlying asset's drift. This result stems from the model's framework, which assumes a geometric Brownian motion with non-zero drift for the underlying asset. Paradoxically, the model suggests that the value of a call option is unaffected by the underlying asset's drift, contrary to the intuitive expectation that a higher positive drift would increase the option's value. The Black-Scholes model operates under the assumption of a risk-neutral market, where the asset's drift is effectively equivalent to the risk-free rate. The mathematical justification for the irrelevance of the true drift lies in the hedging portfolio approach. Specifically, the model involves shorting the underlying asset by an amount equal to the option's delta. As a result, any increase in the asset's value due to a higher positive drift is offset by a corresponding decrease in the short position's value, rendering the hedging portfolio instantaneously risk-free and yielding the risk-free rate. The option's value is then determined as a residual to maintain the portfolio's local risk-free status. However, this approach has two significant limitations. Firstly, the Black-Scholes model's fair price for a call option is only relevant to the hedging portfolio holder, not to other market participants. For instance, a speculator holding a long call option on a blue-chip company's stock is unlikely to hold a short position in the underlying asset that exactly offsets the option's delta. Consequently, the model's assumptions do not reflect the reality of most market participants' portfolios, limiting its applicability. Beyond the conceptual issues, the Black-Scholes model faces a well-known technical challenge related to self-financing portfolios. The model assumes that the hedging portfolio is self-financing, meaning that its value changes solely due to fluctuations in the option and underlying asset prices, without any external funding flows. However, this assumption has been disputed by researchers such as Bergman (1982) and Bartels (1995), who argue that it may not hold in practice. If the hedging portfolio is not truly self-financing, the rate of return may not be riskless, which would undermine the model's fundamental assumptions. While this flaw does not necessarily invalidate the Black-Scholes model entirely, as noted by Bana (2007), it does highlight significant concerns. Additionally, the model's assumption of geometric Brownian motion for the underlying asset may not perfectly capture real-world dynamics, although this issue is secondary to the self-financing problem. Research on imperfect hedging portfolios due to market incompleteness has shown that contingent claim values lie within certain hedging bounds (Hao, 2008). Other studies have explored the impact of costly short-selling on bid-ask spreads (Atmaz & Basak, 2019) and developed equal risk pricing rules for incomplete markets (Guo & Zhu, 2017; Marzban et al., 2022). However, these approaches do not address the fundamental issue of delta-hedging, where holding a long call option is unlikely to be perfectly offset by shorting the underlying asset. The authors propose a generalized framework for derivative pricing that builds on the efficient markets hypothesis and argues that the traditional Black-Scholes framework, which relies on hedging portfolios, is too narrow. The proposed model suggests that the fair value of a financial derivative is determined by a conditional expectation discounted at the asset's drift, which reflects market efficiency. The asset price (A) is modeled as a geometric Brownian motion:

$$\frac{dA}{dt} = (r+\delta+\pi)A - \sigma A\left(\frac{dW}{dt}\right) \qquad 0 \leq t \leq T \tag{1}$$

$r > 0$ is the risk-free rate , $\delta > 0$ is the dividend yield, $\pi > 0$ is the inflation rate $\sigma > 0$ here, σ represents the volatility, and W denotes a standard Brownian motion.

The fair value of a financial derivative with a terminal payoff function $g(A_T)$ is given by the conditional expectation discounted by the market discount function:

$$P(A(t),t) = E_t e^{-(r+\delta+\pi)(T-t)} , \tag{2}$$

where P(x(t),t) is the fair value of the derivative at time t when the asset has price x(t). The partial differential equation (PDE) for the evolution of the financial derivative's value is:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 P}{\partial A^2} + (r+\delta+\pi)A\frac{\partial P}{\partial s} - (r+\delta+\pi)P = A \tag{3}$$

The key difference from the canonical Black-Scholes PDE is that the risk-free rate is replaced with a discount rate that reflects the efficiency of the financial market. For a European call option with payoff $g(A_T) = \max(A_T - K, 0)$, the pricing formula is:

$$P(Xt,t) = P(A,t) = Ke^{-(r+\delta+\pi)(T-t)\emptyset(-d_2)} - A\emptyset(-d_1) \tag{4}$$

where $\emptyset$ is the cumulative distribution function of the standard normal variable, and:

$$d_1 = \frac{\ln(\frac{A}{K}) + (r+\delta+\pi) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

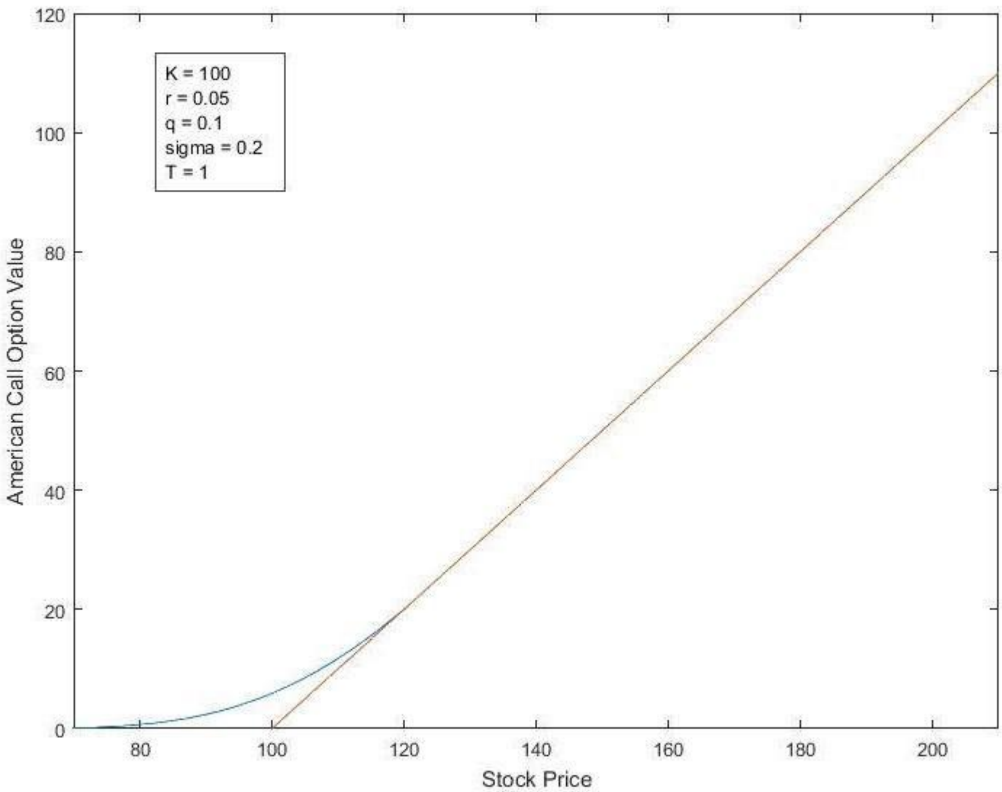
$$d_2 = \frac{\ln(\frac{A}{K}) + (r+\delta+\pi) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$= d_1 - A\sqrt{T-t}$$

This formula is similar to the one presented by James Boness in 1964. This new formula explicitly shows that the European call option's price depends on the asset's drift, which aligns with intuitive expectations. The main difference from the original Black-

Scholes model is the replacement of the risk-free rate with the underlying asset's drift. This adjustment, which affects the strike price, may help explain the volatility smile phenomenon.

Option	Value	Average Computational Time
90	0.6956	0.2134 seconds
95	1.3534	0.2143 seconds
100	2.3890	0.2165 seconds
105	3.8909	0.2180 seconds
110	5.9283	0.2210 seconds
115	8.5468	0.2374 seconds
120	11.7702	0.2646 seconds
125	15.6058	0.2871 seconds
130	20.0519	0.3079 seconds



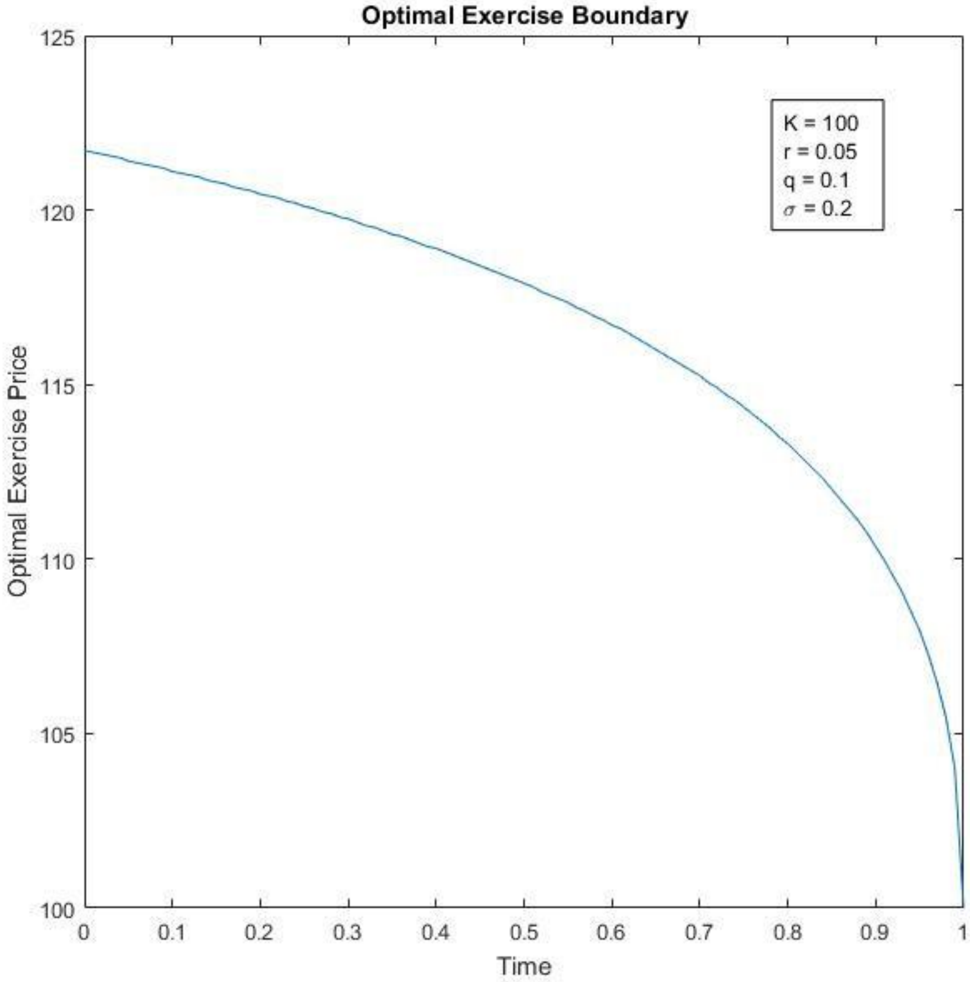
An illustration of the value of an American call option for varying asset prices at $t = 0$. The blue line is the value of the American option and the red line is the pay-off function.

The parameters of the option are the following: $K = 100, r = 0.05$, $q = 0.1$, $\delta = 0.05$, $\pi = 0.02$ $\sigma = 0.2$, $T = 1$, $\delta t = 0.0001$, $\delta S = 0.5$.The blue line in figure 2 resembles the value of an American call option and the red line is the pay-off function $max(S - K, 0)$. In the figure, it can be found that the value of an American call option converges with the pay-off function for increasing asset prices. It can also be found that the American call option is always worth more or equal to the pay-off function. This can be explained from the boundary condition for an American option:

The gain from an early exercise differs over time. The optimal prices for an early exercise are listed in table 2 for different times:

Optimal Exercise Prices	
t	Optimal Exercise Price
0	121.70
0.25	120.10
0.50	117.90
0.750	114.35
1.00	100.00

Table 3: Optimal exercise prices for an American call option with the parameters: $K = 100$, $r = 0.05$, $q = 0.1$, $\delta = 0.05$, $\pi = 0.02$ $\sigma = 0.2$, $T = 2$, $\delta t = 0.01$, $\delta S = 0.1$. As it can be observed in table 3, the optimal exercise price converges with the strike price. This can be explained by the loss of the option's time value as the maturity is approached. At expiry, the time value will be equal to zero, and hence the intrinsic value will determine the value of the option. Consequently, at expiration date, the optimal exercise price is equal to the strike price. The loss of time value as the option approaches the expiration date is illustrated in the plot of the optimal exercise boundary in figure 3.



Optimal Exercise Boundary - Varying Volatility

From figure 4, it can be found that the optimal exercise price increases with an increased volatility. The exercise price is equal to the strike price at maturity.

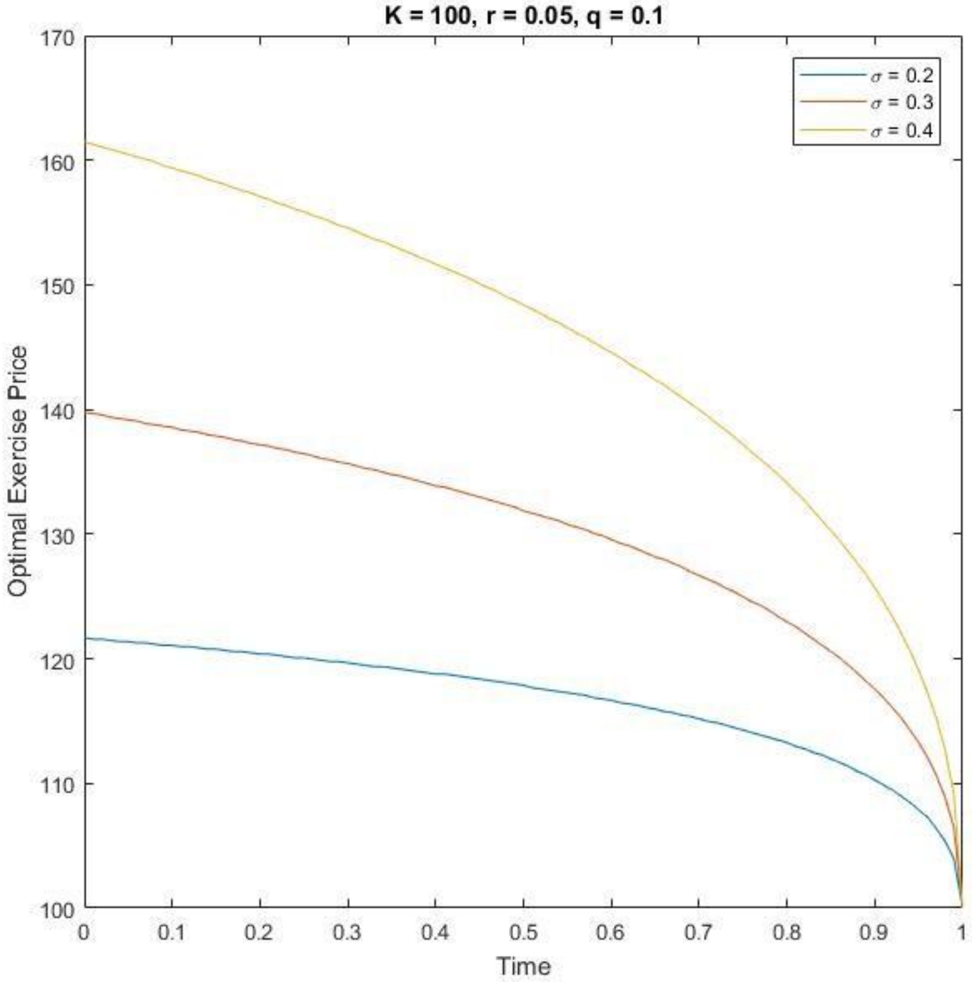


Figure 4: An illustration of the optimal exercise boundary of an American call option for varying volatilities. The following parameters are used: K = 100, r = 0.05, q = 0.1, $\delta = 0.05$, $\pi = 0.02$ T = 1, δt = 0.001, δA = 0.1. The blue line have the volatility σ = 0.2, the red line σ = 0.3 and the yellow line σ = 0.4. The explanation to why the optimal exercise price increase when the VOLATILITY increases is because the time value of the option will grow because of an increased probability of a profitable asset movements. This is true since the loss is limited to the premium. As the option approaches maturity, the time value will approach zero and hence the optimal exercise price at expiration date will be equal to the strike price.

Optimal Exercise Boundary - Varying Risk Free Interest Rate

From figure 5, it can be found that the optimal exercise price increases with an increased risk free interest rate. The exercise price is equal to the strike price at maturity.

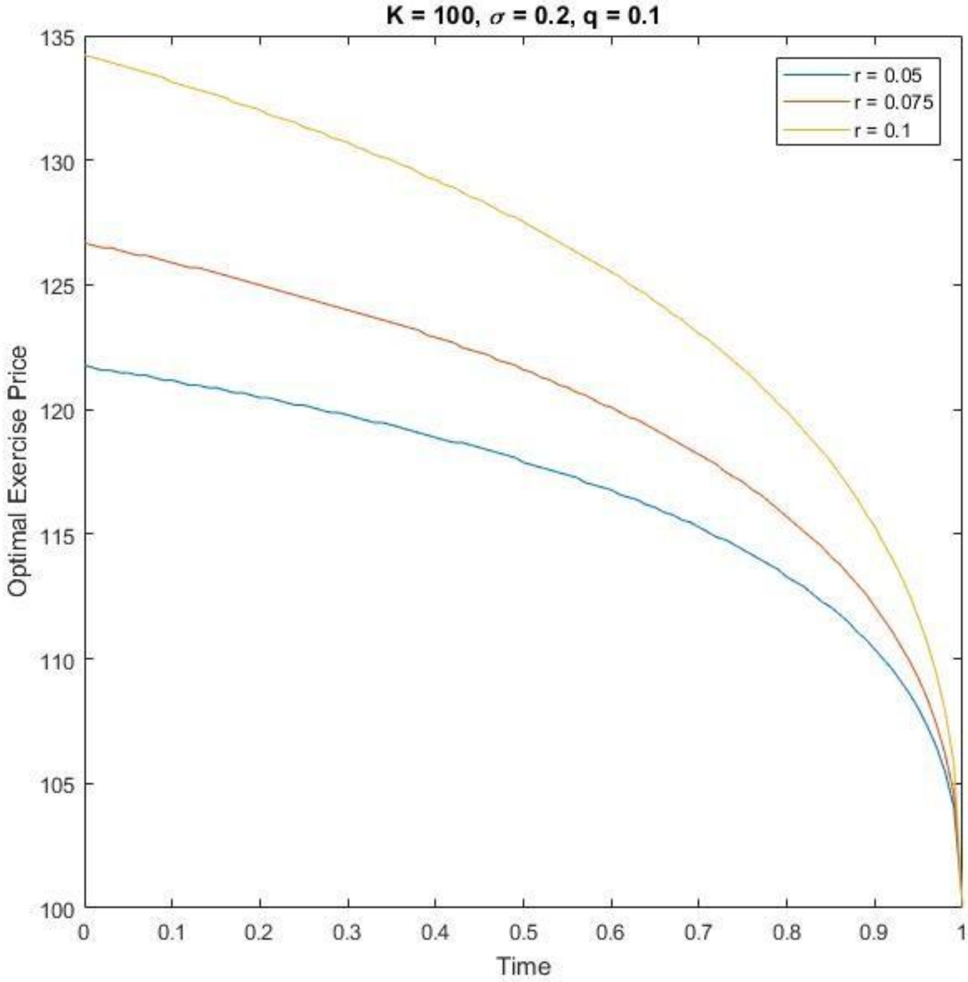


Figure 5: An illustration of the optimal exercise boundary of an American call option for varying risk free interest rates. The following parameters are used: K = 100, $\delta = 0.05$, $\pi = 0.02$ σ = 0.2, q = 0.1, T = 1, δt = 0.001, δA = 0.1. The blue line have the risk free interest rate r = 0.05, the red line r = 0.075 and the yellow line r = 0.1.The explanation to why the optimal exercise prices increase when the risk free interest rate increase is because the value of today's money increase. This means that the time value have increased. The logic behind this is that for a higher risk-free interest rate, if the option is exercised early, the cash from selling the option's underlying asset after an exercise can be invested in risk-free assets with a return equal to the risk-free interest rate. A higher risk-free interest rate will result in an increased return. As the option approaches maturity, the time value will approach zero and hence the optimal exercise price at expiration date will be equal to the strike price.

Conclusion

The conclusions that can be drawn from the analyses in this thesis are that it is never optimal to exercise an American call option if the underlying asset does not pay dividends. For American options paying dividends, it is always optimal to exercise an American call option at the first contact between the option value and its intrinsic value, i.e. $C(S,t) = S_f - K$, where S_f is the optimal exercise price. The optimal exercise price will change over time, since the time value will decrease as the time of expiration is approached. The optimal exercise boundary shows the different optimal exercise prices during a specific options' lifetime. Furthermore, it was found that the optimal exercise price increases when the following parameters increases: Volatility, risk-free interest rate, an increasing dividend yield rate did on the other hand decrease the optimal exercise price. A higher risk-free interest rate will decrease time value faster over time. The different optimal prices for a stock loan can be shown in a plot of the optimal exit boundary. The optimal exit boundary show the different optimal exit prices during the lifetime of the loan for a specific portfolio of stocks used as collateral. Furthermore, the optimal exit price will increase when the volatility increases and it will decrease when the dividend yield rate increases. When time $t = 0$, i.e. today, the optimal exit price will increase when the risk-free interest rate is increased and it will decrease when the stock loan value is increased. With time, the risk-free interest rate will decrease the time value and the loan interest rate will increase the time value. Consequently, at time $t = T$, i.e. at expiration date, the optimal exit price will decrease when the risk-free interest rate is increased and it will increase when the loan interest rate is increased.

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