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# Quantitative Assessment of Interest Rate Fluctuation Sensitivity in Nigerian Insurance Asset-Liability Management

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**Abstract**

This study investigates the sensitivity of insurance portfolios to interest rate fluctuations in Nigerian insurance companies, with particular focus on the implications for asset and liability valuation. The objective is to assess how interest rate variability affects the relative sensitivities of assets and liabilities, and the resulting solvency risks. A quantitative approach was adopted, using a sample of ten insurance companies selected based on asset base and data availability. Data covering a ten-year period (2013–2023) were obtained from published financial statements and Central Bank of Nigeria interest rate bulletins. Analytical techniques included stochastic simulations and regression modeling, applying the Vasicek and Heston frameworks, with visualization performed using Python 3.12.3. The results show that liabilities exhibit greater sensitivity to interest rate fluctuations than assets, with pronounced volatility under stress scenarios, thereby creating significant solvency challenges. These findings validate the importance of dynamic stochastic models in capturing the complexities of interest rate effects, as opposed to static mathematical assumptions. The study concludes that effective asset–liability management (ALM) requires robust dynamic interest rate modeling. Theoretical contributions include extending the application of stochastic differential equations to emerging market contexts, while practical recommendations urge insurance regulators and investment managers to adopt interest rate-sensitive frameworks for risk management and capital adequacy assessments. Future research is recommended on macroeconomic stress factors and stochastic volatility models tailored to African financial markets.

**Keywords**: Insurance Portfolios; Interest Rate Fluctuations; Asset–Liability Management; Vasicek Model; Heston Model; Stochastic Simulation; Solvency Risk; Nigeria

## INTRODUCTION

The mathematical evaluation of interest rate fluctuation sensitivity in insurance portfolios has emerged as a critical focus in the context of financial mathematics, especially in developing economies such as Nigeria. This focus arises from the importance of interest rate fluctuations in Asset and Liability Management (ALM), within which significant mismatches between the returns on Assets and the obligations on Liabilities can dangerously expose insurers to financial risk. A significant portion of the concern has to do with the impact of micro and macroeconomic fluctuations and regulatory changes on interest rate and insurance portfolio value solvency, and the valuation of insurance portfolios.

There has been considerable work in mathematical finance on stochastic control-based investment strategies to tackle asset and liability imbalance, with some of the most foundational works applying optimal control theory (Akpanibah, 2017) and martingale approaches (Zhang, 2023) to develop investment strategies maximizing an insurer's terminal return or utility in a stochastic environment. A considerable portion of this literature seems to work with a deterministic or GBM-type rate and a stochastic interest rate and asset modeling. However, Chiarella (2022) provides evidence that such assumptions tend to miss capturing real market volatilities, specifically during periods of severe economic disruption such as the COVID-19 pandemic.

To capture reality more closely, Edikan *et al.* (2021) research has employed stochastic volatility frameworks for risky assets, particularly the Constant Elasticity of Variance (CEV) and Heston models, as well as stochastic interest rate frameworks like Vasicek (1977), and the CIR (Cox-Ingersoll-Ross) processes. These frameworks yield more accurate descriptions of not only the behavior of asset prices but also the term structure of interest rates. For example, He (2020) and Akpanibah (2020) examined the optimal investment strategies with CEV driven risky assets and CIR modeled risk-free rates, underscoring the intricate relationship between asset prices and changing interest rates.

However, in the context of the Nigeria insurance sector, very few empirical and quantitative research works have focused on analyzing the impact of interest rate fluctuations on insurance portfolios, especially considering the local economic and regulatory framework. Most of the literature has been done under the assumption of constant or exogenously stable interest rates which ignores the prevailing interest rate volatility in the context of emerging economies and the impact of inflation, exchange rate fluctuations, and uncertainty in monetary policy.

The gap in literature has been identified in the current research and seeks to quantify the degree of interest rate fluctuation sensitivity in Nigeria insurance portfolios by applying a stochastic approach based on the Vasicek model. The primary purpose is to determine the reaction of insurance portfolios to external economic fluctuations in the form of changing interest rates, regulatory shifts, and to quantify the degree of portfolio sensitivity by applying stress-testing and stochastic simulation techniques. This contributes to the adequacy of risk, capital, and the ability to absorb macro-financial fluctuations.

This study improves existing literature by using an advanced stochastic framework with the combined use of Vasicek and Heston models with the Hamilton–Jacobi–Bellman Equation, which extensively analyzes the solvency dynamics and optimizes ALM with respect to Nigerian insurance portfolios in the face of volatile interest rates, thus extending the theoretical frontier and applied actuarial work in emerging markets.

## METHODS

This study adopted a quantitative research approach, which is necessary for precision, replicability, and generalizability when investigating financial phenomena such as interest rate sensitivity in insurance portfolios. Quantitative methods enable researchers to establish relationships among variables and apply mathematical models to real-world data (Creswell, 2014). This approach is particularly suitable for this research because it facilitates the application of stochastic financial models—namely, the Vasicek and Heston models—to empirical data, thus allowing for the measurement of sensitivity to interest rate fluctuations with a high degree of statistical reliability and interpretability.

The study employed an ex post facto survey design, which is appropriate for observing the effects of variables that cannot be manipulated, such as historical interest rate fluctuations and corresponding portfolio performances. This design aligns with those employed in similar financial and insurance risk studies (Ibe, 2017; Adegboye & Akinlabi, 2021), where secondary data were analysed to understand systemic financial behaviours over time. Unlike purely experimental designs, the ex post facto approach allows for real-world economic dynamics to be captured without imposing artificial controls, thereby preserving the ecological validity of the study.

This study adopts a quantitative and analytical research approach to develop and evaluate a modified pricing model for life insurance companies. The methodology involves mathematical modeling, numerical simulation, and risk analysis using financial stochastic models. The core models include the Diffusion Equation, Heston Model, and Vasicek Model, which are enhanced to incorporate additional financial risks such as market liquidity, interest rate sensitivity, and regulatory risk.

The target population consisted of all licensed insurance companies operating under the regulation of the National Insurance Commission (NAICOM) in Nigeria as of December 2023. Sampling technique was adopted to select a data sample of ten leading life insurance firms based on their portfolio size, data accessibility, and consistency of financial reporting over the last decade. According to Sugiyono (2019), purposive sampling is justified when the research aims to extract meaningful insights from a subgroup with the most relevant characteristics. The selected firms collectively represent over 65% of the Nigeria insurance market share, thus offering a reliable basis for generalising findings within the sector. The duration for this research process was within three months.

Data collection was based primarily on secondary data obtained from publicly available financial statements, NAICOM annual reports, and the Nigeria Exchange Group (NGX) interest rate datasets from 2013 to 2023. To ensure instrument validity and reliability, the study employed a triangulation process by cross-checking financial data with

Central Bank of Nigeria (CBN) reports and industry audits. Historical asset and liability entries were extracted for each insurance firm and classified into interest-sensitive categories based on NAICOM's solvency and capital adequacy guidelines. Previous studies (Okafor & Onuoha, 2020; Umeh & Obiora, 2023) have also adopted secondary financial data to model interest rate effects within regulated industries, validating the methodological soundness of this approach.

For data analysis, the study applied both descriptive statistics and inferential modeling techniques using Python 3.12.3 and SPSS v28. The descriptive statistics provided insights into portfolio structure trends, while the Vasicek and Heston stochastic models were used to quantify interest rate sensitivity over time. These models were calibrated using historical term structures and yield curve data, allowing for the simulation of mean-reversion effects and volatility clustering. The models' outputs were interpreted using standard performance indicators such as duration, convexity, and portfolio value-at-risk (VaR). Inferential analysis included regression diagnostics and Monte Carlo simulations to test the robustness of interest rate sensitivity estimates. The use of these advanced analytical tools aligns with recommendations by Miles, Huberman, and Saldaña (2014) for extracting complex patterns in numerical datasets.

**Model Formulation**

To formulate the mathematical model, existing pricing and interest rate models are extended by using Itô's Lemma to introduce dynamic risk factors:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \alpha \dfrac{\partial^2 A}{\partial t^2} + \sigma \cdot \psi - \beta \cdot e \\ dA_t = \mu A_t dt + \sqrt{V_t} A_t dW_t^A + \ell \cdot \kappa \\ dV_t = \theta_V (\sigma^2 - V_t) dt + \sigma_V \sqrt{V_t} dW_t^V + \delta \cdot \psi \\ dS_t = \mathcal{S}(L_t - S_t) dt + \sigma_R dW_t + \delta \cdot \xi \\ a_{K_x} = \displaystyle\sum_{K=1}^{\infty} \left( \sum_{j=0}^{k-1} v^{j+1} \right) {}_k q_x \\ W_t = \dfrac{1-\lambda}{1-\lambda^K} \lambda^{K-t} \end{cases} \tag{1}$$

$$\begin{cases} t = 0, A(0) = A_0, \left.\dfrac{\partial A}{\partial t}\right|_{t=0} = A_1, V(0) = V_0, V_t \geq 0 \\ S(0) = S_0, L(0) = L_0, \ 0 \leq t \leq K \end{cases} \tag{2}$$

where,

the annuity product of a life insurer's liability is modeled with discounting pricing

$$a_{K_x} = \sum_{K=1}^{\infty} \left( \sum_{j=0}^{k-1} v^{j+1} \right) k^{q_x} \tag{3}$$

The definitions of some variables and parameters are described as follows

Table 1: Parameters/Variables and Definitions

| **Parameters/variables** | **Description** |
|---|---|
| A(t) | Asset risk level at time, t |
| α | Rate of diffusion for the asset class |
| σ | Volatility of assets |
| ψ | Liquidity risk |
| β | Interest rate sensitivity |
| e | Inflation rate |
| $A_t$ | Asset price at time t |
| μ | Expected rate of return of the asset |
| $V_t$ | Variance of the asset price |
| $\theta_V$ | Long-run variance |
| $\sigma_V$ | Volatility of variance |
| $W_t^A$ , $W_t^V$ | Two correlated Wiener processes |
| ϰ | Correlation between asset returns and volatility |
| $\ell$ | Market liquidity |
| δ | Default risk premium |
| $S_t$ | Short-term interest rate at time t |
| § | Speed of reversion. |
| $L_t$ | Long-term mean interest rate |
| $\sigma_R$ | Volatility of interest rates |
| $W_t$ | Wiener process |
| ξ | Regulatory risk |
| $\lambda$ | Decay factor |
| $K$ | Risk evaluation period |
| $[A_{tg}]_{HE}$ | Diffusion Equation model |
| $[A_{tg}]_{HM}$ | Heston Model |
| $[R_t]_{VM}$ | Vasicek Model |

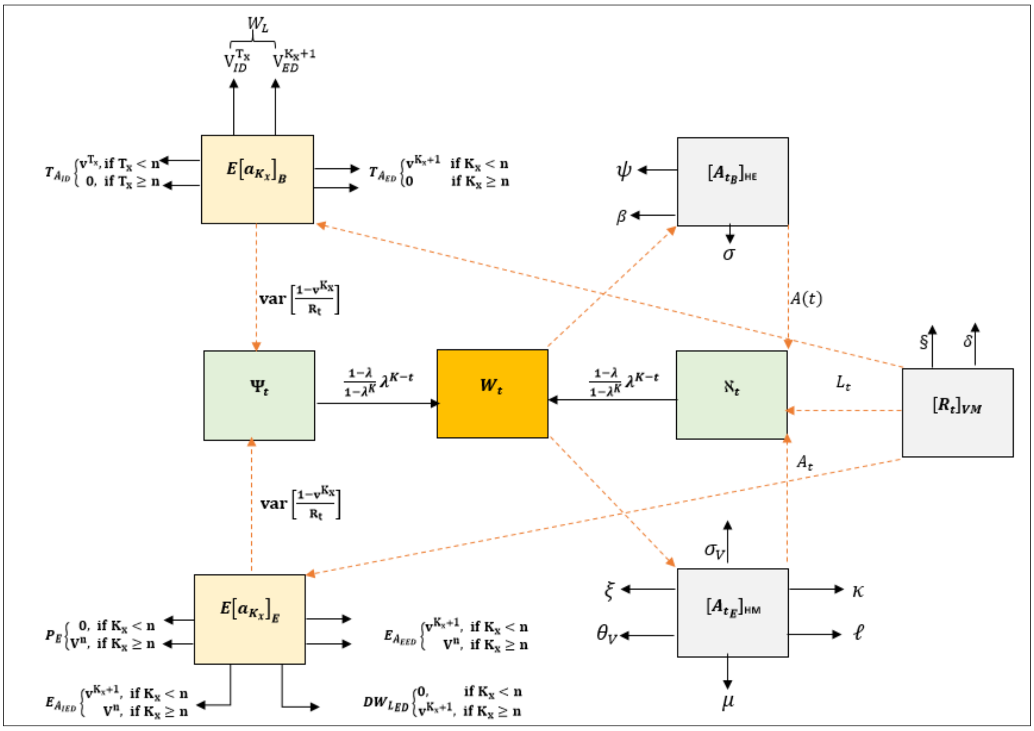



Figure 1: Schematic Diagram that shows total Asset ($\Psi_t$) and total Liability ($\aleph_t$)

## Mathematical Analysis of the Model

The financial market under study is assumed to operate continuously over the interval $t \in [0, \mathcal{T}]$, where $\mathcal{T}$ denotes the expiration horizon of the insurance investment portfolio (Edikan, 2021). Let $\{W_0(t), W_1(t), W_2(t): t \geq 0\}$ represent a system of standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, in which $\Omega$ is the sample space, $\mathcal{P}$ is a probability measure, and $\mathcal{F}$ is the filtration capturing all information available up to time $t$ (Karatzas & Shreve, 1991). The growth of the **risk-free asset**, denoted as $A(t)$, is governed by the differential Equation:

$$\frac{dA(t)}{A(t)} = \rho(t)dt, \qquad A(0) = \sigma_0 > 0 \tag{4}$$

where $\rho(t)$ is the instantaneous risk-free rate. This rate follows the **Vasicek** stochastic differential Equation (SDE), commonly used in interest rate modeling (Vasicek, 1977):

$$d\rho(t) = \S\big(\alpha - \rho(t)\big)dt + \varepsilon dW_0(t) + \delta \cdot \xi, \;\; \rho(0) = \rho 0 > 0 \tag{5}$$

Here, $\alpha$=long-term mean interest rate and $\varepsilon > 0$=interest rate volatility are positive constants, with Feller's condition $\varepsilon^2 < 2\alpha$ imposed to guarantee non-negativity of the interest rate process. $(\delta \cdot \xi) =$ exogenous structural fluctuations.

The market also includes two risky assets, $A_1(t)$ and $A_2(t)$, which represent components of the insurer's investment in financial instruments such as equity or bonds. The asset dynamics can be described using a stochastic differential Equation (SDE) process and the variance process $\nu(t)$ is assumed to follow a (Heston, 1993) model, commonly applied in stochastic volatility modeling, defined by:

$$\begin{cases} dA_1(t) = \mu_1 A_1(t)dt + \sqrt{V_1(t)}A_1(t)dW_t^{A_1} + \ell \cdot \varkappa_1 \\ dV_1(t) = \theta_{V_1}(\sigma_1^2 - V_1(t))dt + \sigma_{V_1}\sqrt{V_1(t)}dW_t^{V_1} + \delta \cdot \psi_1 \end{cases} \quad (6)$$

$$\begin{cases} dA_2(t) = \mu_2 A_2(t)dt + \sqrt{V_2(t)}A_2(t)dW_t^{A_2} + \ell \cdot \varkappa_2 \\ dV_2(t) = \theta_{V_2}(\sigma_2^2 - V_2(t))dt + \sigma_{V_2}\sqrt{V_2(t)}dW_t^{V_2} + \delta \cdot \psi_2 \end{cases} \quad (7)$$

where $\mu$ is the expected asset growth rate, $V(t)$ represents stochastic variance, $\kappa$ denotes the intensity of jumps, $\ell$ accounts for exposure to structural fluctuations, $\theta_V$ is the speed of mean reversion, $\sigma^2$ is the long-run variance, $\sigma_V$ is the volatility of variance, and $\delta \cdot \psi$ captures the effect of external fluctuations on the variance process.

The **total asset value** of the insurance firm, denoted $A(t)$, evolves according to a second-order partial differential Equation (PDE) that incorporates sensitivity to interest rate fluctuations, internal market volatility, and external macroeconomic disturbances:

$$\frac{\partial A}{\partial t} = \alpha \cdot \frac{\partial^2 A}{\partial t^2} + \sigma \cdot \psi(t) - \beta \cdot \epsilon(t) \quad (8)$$

Here, $\alpha$ represents the reversion or diffusion coefficient, $\sigma$ is asset volatility, $\psi(t)$ is a stochastic influence variable, $\beta$ is a sensitivity coefficient, and $\epsilon(t)$ captures policy or cost-driven fluctuations.

The surplus process, defined as $S(t) = A(t) - L(t)$, where $L(t)$ is the liability value (Zhang *et al.*, 2021), evolves as follows:

$$dS(t) = \big(L(t) - S(t)\big)dt + \eta dW_0(t) + \delta\xi(t) \quad (9)$$

where $\eta$ represents the standard deviation of claim payouts and $\xi(t)$ models idiosyncratic stress events in the liability stream.

To account for the decaying memory of past risks or reserves, the model incorporates an exponential weighting function:

$$\omega(t) = \frac{1-\lambda}{1-\lambda^K} \cdot \lambda^{K-t}, \quad \lambda \in (0,1), \quad K \in \mathbb{N} \qquad (10)$$

This function captures how insurers de-emphasize older liabilities over time, consistent with regulatory frameworks and dynamic capital adequacy practices.

**Optimisation Analysis**

According to Edikan *et al.*, (2021), let $\pi = (\pi_0, \pi_1, \pi_2)$ be the optimal investment strategy, where $\pi_0, \pi_1, and\ \pi_2$ denote the proportions of capital invested in the risk-free asset and two risky assets, respectively. We define the utility function $\mathcal{U}$ attained by the insurer from capital $h$ at time $t$ as:

$$\mathcal{V}^\pi(t,\rho,\sigma_1,\sigma_2,h) = \mathbb{E}^\pi\left[\mathcal{U}(H(T)) \middle| \begin{array}{c} \rho(t) = \rho, \\ A_1(t) = \sigma_1, \quad A_2(t) = \sigma_2, \quad H(t) = h \end{array}\right] \qquad (11)$$

To determine the optimal investment strategy $\pi^*$ and the corresponding value function $\mathcal{V}^\pi(t,\rho,\sigma_1,\sigma_2,h)$ such that:

$$\mathcal{V}(t,\rho,\sigma_1,\sigma_2,h) = {}^{sup}_{\pi}\mathcal{V}^\pi(t,\rho,\sigma_1,\sigma_2,h) \qquad (12)$$

$$\mathcal{V}^{\pi^*}(t,\rho,\sigma_1,\sigma_2,h) = \mathcal{V}(t,\rho,\sigma_1,\sigma_2,h) \qquad (13)$$

Let $H(t)$ denote the insurer's capital at time $t$, and the dynamics of capital follow:

$$dH(t) = H(t)\left[\pi_0 \frac{dA_0(t)}{A(t)} + \pi_1 \frac{dA_1(t)}{A_1(t)} + \pi_2 \frac{dA_2(t)}{A_2(t)}\right] \qquad (14)$$

Substituting the asset dynamics from the CIR and CEV models into (16), we obtain:

$$dH(t) = H(t)\begin{bmatrix} (\pi_1(\mu_1 - \rho) + \pi_2(\mu_2 - \rho) + \rho)dt \\ +(\pi_1 v_{11} A_1^\gamma + \pi_2 v_{21} A_2^\gamma)dW_1(t) \\ +(\pi_1 v_{12} A_1^\gamma + \pi_2 v_{22} A_2^\gamma)dW_2(t) \end{bmatrix} \qquad (15)$$

with initial capital $H(0) = h_0$, and $\pi_0 = 1 - \pi_1 - \pi_2$.

Applying Itô's lemma and the dynamic programming principle (Ihedioha *et al.*, 2020), we derive the associated Hamilton-Jacobi-Bellman (HJB) Equation for the value function $\mathcal{V}(t,\rho,\sigma_1,\sigma_2,h)$, which incorporates the partial derivatives $\mathcal{V}_t, \mathcal{V}_\rho, \mathcal{V}_{\sigma i}, \mathcal{V}_h$, and the second derivatives accordingly (Zhu, 2022). The HJB PDE maximizes over the control variables $\pi_1$ and $\pi_2$ to yield the optimal strategy as follows:

$$\begin{cases} V_t + \mu_1\sigma_1 V_{\sigma_1} + \mu_2\sigma_2 V_{\sigma_2} + \varrho h V_h + \frac{1}{2}P_1\sigma_1^{2\gamma+2}V_{\sigma_1\sigma_1} + \frac{1}{2}P_2\sigma_2^{2\gamma+2}V_{\sigma_2\sigma_2} \\ +P_3\sigma_1^{\gamma+1}\sigma_2^{\gamma+1}V_{\sigma_1\sigma_2} + (a-b\varrho)V_\varrho + \frac{1}{2}\varrho\xi^2 V_{\varrho\varrho} + P_4\xi\delta\sqrt{\varrho}\sigma_1^{\gamma+1}V_{\sigma_1\varrho} + P_5\xi\delta\sqrt{\varrho}\sigma_2^{\gamma+1}V_{\sigma_2\varrho} \\ +\sup_{\pi_1,\pi_2}\begin{Bmatrix} \left[\frac{1}{2}P_1\pi_1^2\sigma_1^{2\gamma} + P_3\pi_1\pi_2\sigma_1^{\gamma}\sigma_2^{\gamma} + \frac{1}{2}P_2\pi_2^2\sigma_2^{2\gamma}\right]h^2V_{hh} \\ +\left[(\mu_1-\varrho)\pi_1 + (\mu_2-\varrho)\pi_2\right]hV_h \\ +\left[P_4\xi\delta\sqrt{\varrho}\pi_1\sigma_1^{\gamma} + P_5\xi\delta\sqrt{\varrho}\pi_2\sigma_2^{\gamma}\right]hV_{h\varrho} \\ +\left[P_1\sigma_1^{2\gamma+1}\pi_1 + P_3\sigma_1^{\gamma+1}\sigma_2^{\gamma}\pi_2\right]hV_{h\sigma_1} \\ +\left[P_3\sigma_1^{\gamma}\sigma_2^{\gamma+1}\pi_1 + P_2\sigma_2^{2\gamma+1}\pi_2\right]hV_{h\sigma_2} \end{Bmatrix} = 0 \end{cases} \qquad (16)$$

where,

Differentiating (16) with respect to $\pi_1^*$ and $\pi_2^*$, we obtain the first-order maximizing condition for Equation (16) as

$$\pi_1^* = \frac{\left[P_3\sigma_1^{\gamma}(\mu_2-\varrho) - P_2\sigma_2^{\gamma}(\mu_1-\varrho)\right]hV_h - \sigma_1V_{h\sigma_1} - (P_2P_4 - P_3P_5)\sqrt{\varrho}\xi\delta hV_{h\varrho}}{(P_1P_2 - P_3^2)\sigma_1^{\gamma}\sigma_2^{2\gamma}V_{hh}} \qquad (17)$$

$$\pi_2^* = \frac{\left[P_3\sigma_2^{\gamma}(\mu_2-\varrho) - P_1\sigma_1^{\gamma}(\mu_2-\varrho)\right]hV_h - \sigma_2V_{h\sigma_2} - (P_1P_5 - P_3P_4)\sqrt{\varrho}\xi\delta hV_{h\varrho}}{(P_1P_2 - P_3^2)\sigma_1^{\gamma}\sigma_2^{2\gamma}V_{hh}} \qquad (18)$$

Substituting (17) and (18) into (16) and simplifying will give Malitsky & Tam (2022)'s Equation of the form:

$$V_t + \mu_1\sigma_1V_{\sigma_1} + \mu_2\sigma_2V_{\sigma_2} + (\varrho + \omega_1)hV_h + \frac{1}{2}P_1\sigma_1^{2\gamma+2}V_{\sigma_1\sigma_1} + \frac{1}{2}P_2\sigma_2^{2\gamma+2}V_{\sigma_2\sigma_2}$$

$$+P_3\sigma_1^{\gamma+1}\sigma_2^{\gamma+1}V_{\sigma_1\sigma_2} + (a - b\varrho)V_\varrho + \frac{1}{2}\varrho\xi^2 V_{\varrho\varrho} + P_4\xi\delta\sqrt{\varrho}\sigma_1^{\gamma+1}V_{\sigma_1\varrho} + P_5\xi\delta\sqrt{\varrho}\sigma_2^{\gamma+1}V_{\sigma_2\varrho}$$

$$+\frac{1}{2}\left(\omega_2\sigma_1^{-2\gamma} + \omega_3\sigma_2^{-2\gamma}\right)\frac{V_\varrho^2}{V_{hh}} + \left(\omega_4 - (\mu_1 - \varrho)\sigma_1\right)$$

$$+\left(\omega_5 - (\mu_2 - \varrho)\sigma_2\right)\frac{V_h V_{h\sigma_2}}{V_{hh}} + \left(\omega_6\sigma_1^{-\gamma} + \omega_7\sigma_2^{-\gamma}\right)\frac{V_h V_{h\varrho}}{V_{hh}}$$

$$-\frac{1}{2}P_1\sigma_1^{2\gamma+2}\frac{V_{h\sigma_1}^2}{V_{hh}} - \frac{1}{2}P_2\sigma_2^{2\gamma+2}\frac{V_{h\sigma_2}^2}{V_{hh}} - \frac{1}{2}P_{11}\delta^2\xi^2\varrho\frac{V_{h\varrho}^2}{V_{hh}}$$

$$-P_3\sigma_1^{\gamma+1}\sigma_2^{\gamma+1}\frac{V_{h\sigma_1}V_{h\sigma_2}}{V_{hh}} = 0 \qquad (19)$$

where:

$$P_1 = \nu_{11}^2 + \nu_{12}^2, \quad P_2 = \nu_{21}^2 + \nu_{22}^2,$$

$$P_3 = \nu_{11}\nu_{21} + \nu_{12}\nu_{22}, \quad P_4 = \nu_{11} + \nu_{12}, \quad P_5 = \nu_{21} + \nu_{22},$$

$$P_6 = \frac{2P_3(\mu_1 - \varrho)(\mu_2 - \varrho)}{(P_1P_2 - P_3^2)}, P_7 = \frac{P_2(\mu_1 - \varrho)^2}{(P_1P_2 - P_3^2)},$$

$$P_8 = \frac{P_1(\mu_2 - \varrho)^2}{(P_1P_2 - P_3^2)}, \quad P_9 = \frac{P_2P_4 - P_3P_5}{(P_1P_2 - P_3^2)}, P_{10} = \frac{P_1P_5 - P_3P_4}{(P_1P_2 - P_3^2)},$$

$$P_{11} = \frac{P_2P_4^2 + P_1P_5^2 - 2P_3P_4P_5}{(P_1P_2 - P_3^2)} \qquad (20)$$

From (Zhao, 2012), we assumed that the optimal investment plan for risky assets' prices is known based on the assumption that

$$\mu_1\pi_1^* + \mu_2\pi_2^* = \beta(\text{constant}) \qquad (21)$$

Substituting (17) and (18) into (21), we derive an expression for $\frac{1}{\sigma_1^\gamma\sigma_2^\gamma}$ as

$$\frac{1}{\sigma_1^{\gamma}\sigma_2^{\gamma}} = \left\{ \frac{P_1P_2 - P_3^2}{P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)} \left[ \begin{array}{c} \beta \cdot h \cdot \frac{V_{hh}}{V_h} + \frac{P_2\mu_1(\mu_1 - \varrho)}{(P_1P_2 - P_3^2)\sigma_1^{2\gamma}} + \mu_1\sigma_1 \frac{V_{h\sigma_1}}{V_h} \\ + \frac{(P_2P_4 - P_3P_5)\mu 1\sqrt{\varrho}\xi\delta}{(P_1P_2 - P_3^2)\sigma_1^{\gamma}} \frac{V_{h\varrho}}{V_h} + \frac{P_1\mu_2(\mu_2 - \varrho)}{(P_1P_2 - P_3^2)\sigma_2^{2\gamma}} \\ + \mu_2\sigma_2 \frac{V_{h\sigma_2}}{V_h} + \frac{(P_1P_5 - P_3P_4)\mu 2\sqrt{\varrho}\xi\delta}{(P_1P_2 - P_3^2)\sigma_2^{\gamma}} \frac{V_{h\varrho}}{V_h} \end{array} \right] \right. \tag{22}$$

Substituting Equation **(22)** into the non-linear HJB Equation **(19),** we obtain:

$$0 = \left\{ \begin{array}{c} V_t + \mu_1\sigma_1 V_{\sigma_1} + \mu_2\sigma_2 V_{\sigma_2} + (\varrho + \omega_1)hV_h + \frac{1}{2}P_1\sigma_1^{2\gamma+2}V_{\sigma_1\sigma_1} \\ + \frac{1}{2}P_2\sigma_2^{2\gamma+2}V_{\sigma_2\sigma_2} + P_3\sigma_1^{\gamma+1}\sigma_2^{\gamma+1}V_{\sigma_1\sigma_2} + (a - b\varrho)V_{\varrho} \\ + \frac{1}{2}\varrho\xi^2 V_{\varrho\varrho} + P_4\xi\delta\sqrt{\varrho}\sigma_1^{\gamma+1}V_{\sigma_1\varrho} + P_5\xi\delta\sqrt{\varrho}\sigma_2^{\gamma+1}V_{\sigma_2\varrho} \\ + \frac{1}{2}\left(\omega_2\sigma_1^{-2\gamma} + \omega_3\sigma_2^{-2\gamma}\right)\frac{V_{\varrho}^2}{V_{hh}} + \left(\omega_4 - (\mu_1 - \varrho)\sigma_1\right)\frac{V_hV_{h\sigma_1}}{V_{hh}} \\ + \left(\omega_5 - (\mu_2 - \varrho)\sigma_2\right)\frac{V_hV_{h\sigma_2}}{V_{hh}} + \left(\omega_6\sigma_1^{-\gamma} + \omega_7\sigma_2^{-\gamma}\right)\frac{V_hV_{h\varrho}}{V_{hh}} \\ -\frac{1}{2}P_1\sigma_1^{2\gamma+2}\frac{V_{h\sigma_1}^2}{V_{hh}} - \frac{1}{2}P_2\sigma_2^{2\gamma+2}\frac{V_{h\sigma_2}^2}{V_{hh}} \\ -\frac{1}{2}P_{11}\delta^2\xi^2\varrho\frac{V_{h\varrho}^2}{V_{hh}} - P_3\sigma_1^{\gamma+1}\sigma^{2\gamma+1}\frac{V_{h\sigma_1}V_{h\sigma_2}}{V_{hh}} = 0 \end{array} \right. \tag{23}$$

Where the parameters $\omega_i$ are defined as:

$$\omega_1 = \frac{P_6(P_1P_2 - P_3^2)\beta}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}$$

$$\omega_2 = \frac{P_2P_6\mu_1(\mu_1 - \varrho)(P_1P_2 - P_3^2)}{P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)} - P_7$$

$$\omega_3 = \frac{P_1P_6\mu_2(\mu_2 - \varrho)(P_1P_2 - P_3^2)}{P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)} - P_8$$

$$\omega_4 = \frac{P_6\mu_1\sigma_1(\mu_2 - \varrho)(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}$$

$$\omega_5 = \frac{P_6\mu_2\sigma_2(\mu_2 - \varrho)(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}$$

$$\omega_6 = \frac{(P_2P_4 - P_3P_5)P_6\mu_1\xi\delta\sqrt{\varrho}}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)} - P_9\xi\delta\sqrt{\varrho}(\mu_1 - \varrho)$$

$$\omega_7 = \frac{(P_1P_5 - P_3P_4)P_6\mu_2\xi\delta\sqrt{\varrho}}{2P_3\left(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho\right)} - P_{10}\xi\delta\sqrt{\varrho}(\mu_2 - \varrho)$$

## RESULTS

To proceed, we solve the non-linear partial differential equation in equation (23) under the assumption of exponential utility (Edikan, 2021). Let the **exponential utility function** be defined as:

$$U(h) = -\frac{1}{\lambda}e^{-\lambda h} \qquad ,\lambda > 0 \tag{24}$$

where,

$U(h) = V(t, \rho, \sigma_1, \sigma_2, h)$

$U(h) =$ denotes the marginal utility function of the insurer (Li, 2017) and (Zhao, 2012).

**$\lambda =$ coefficient of absolute risk aversion**

$$\begin{cases} V(t, \varrho, \sigma_1, \sigma_2, h) = -\frac{1}{\lambda}\exp(-\lambda[v(t, \varrho, \sigma_1) + w(t, \varrho, \sigma_2) + h \cdot g(t)]\,) \\ v(T, \varrho, \sigma 1) = 0, w(T, \varrho, \sigma 2) = 0, g(T) = 1 \end{cases} \tag{25}$$

$$\begin{cases} V_t = -\lambda V\left(v_t + \omega_t + hg_t\right), \\ V_h = -\lambda V g(t),\ \ V_{hh} = \lambda^2 V g(t)^2, \\ V_{\sigma_1} = -\lambda V v_{\sigma_1},\quad V_{\sigma_2} = -\lambda V \omega\sigma_2, \\ V_\varrho = -\lambda V(v\varrho + \omega\varrho),\ \ V\varrho\varrho = \lambda^2 V\left(v_\varrho + \omega_\varrho\right)^2 - \lambda V\left(v_{\varrho\varrho} + \omega_{\varrho\varrho}\right), \\ V_{\sigma_1\sigma_1} = \lambda^2 V v\sigma_1^2 - \lambda V v\sigma_1\sigma_1,\ \ V_{\sigma_2\sigma_2} = \lambda^2 V\omega_{\sigma_2}^2 - \lambda V\omega_{\sigma_2\sigma_2}, \\ V_{\sigma_1\sigma_2} = \lambda^2 V g(t) v_{\sigma_1}\omega_{\sigma_2}, \\ V_{h\sigma_1} = \lambda^2 V g(t) v_{\sigma_1}\quad V_{h\sigma_2} = \lambda^2 V g(t)\omega_{\sigma_2} \\ V_{h\varrho} = \lambda^2 V g(t)\left(v_\varrho + \omega_\varrho\right),\quad V_{\varrho\sigma_1} = \lambda^2 V\left(v_\varrho v_{\sigma_1} + \omega_\varrho v_{\sigma_1}\right) - \lambda V, \\ V_{\varrho\sigma_2} = \lambda^2 V\left(v_\varrho\omega_{\sigma_2} + \omega_\varrho\omega_{\sigma_2}\right) - \lambda V\left(\omega_{\varrho\sigma_2}\right). \end{cases} \tag{26}$$

Cited from Cisneros (2013)'s work

Substituting (26) into (23),

$$
\begin{aligned}
&h\left[g_t + (\varrho + \omega_1)g\right] \\
&+ \left\{ v_t + \omega_t + (\varrho + \omega_4)\sigma_1 v_{\sigma_1} + (\varrho + \omega_5)\sigma_2 \omega_{\sigma_2} - \frac{1}{2}\lambda\xi^2\varrho\left(1 - P_{11}\delta^2\right)\left(v_\varrho + \omega_\varrho\right)^2 \right. \\
&+ \frac{1}{2}\xi^2\varrho\left(v_{\varrho\varrho} + \omega_{\varrho\varrho}\right) + \frac{\omega^2}{2\lambda}\sigma_1^{-2\gamma} + \frac{\omega^3}{2\lambda}\sigma_2^{-2\gamma} + \left(v_\varrho + \omega_\varrho\right)\left(a - b_\varrho - \omega_6\sigma_1 - \gamma - \omega_7\sigma_2^{-\gamma}\right) \\
&+ \frac{1}{2}P_1\sigma_1^{2\gamma+2} v_{\sigma_1\sigma_1} + P_4\xi\delta\sqrt{\varrho}\sigma_1^{\gamma+1}\left(v_{\varrho\sigma_1} - \lambda\left(v_\varrho v_{\sigma_1} + \omega_\varrho v_{\sigma_1}\right)\right) + \frac{1}{2}P_2\sigma_2^{2\gamma+2}\omega_{\sigma_2\sigma_2} \\
&\left. + P_5\xi\delta\sqrt{\varrho}\sigma_2^{\gamma+1}\left(\omega_{\varrho\sigma_2} - \lambda\left(v_\varrho\omega_{\sigma_2} + \omega_\varrho\omega_{\sigma_2}\right)\right)\right\} \\
&= 0
\end{aligned}
\tag{27}
$$

Splitting (27) we have

$$
\begin{cases}
g_t + (\rho + \omega)g = 0, \\
g(T) = 1
\end{cases}
\tag{28}
$$

$$
\begin{cases}
\quad v_t + w_t + (\varrho + \omega_4)\,\sigma_1 v_{\sigma_1} + (\varrho + \omega_5)\,\sigma_2 w_{\sigma_2} \\
\quad -\frac{1}{2}\lambda^2\xi^2\varrho\left(1 - P_{11}\delta^2\right)\left(v_\varrho + w_\varrho\right)^2 + \frac{1}{2}\xi^2\varrho\left(v_{\varrho\varrho} + w_{\varrho\varrho}\right) + \frac{\omega_2}{2\lambda}\sigma_1^{-2\gamma} + \frac{\omega_3}{2\lambda}\sigma_2^{-2\gamma} \\
\quad +\left(v_\varrho + w_\varrho\right)\left(a - b\varrho - \omega_6\sigma_1^{-\gamma} - \omega_7\sigma_2^{-\gamma}\right) + \frac{1}{2}P_1\sigma_1^{2\gamma+2} v_{\sigma_1\sigma_1} \\
\quad +P_4\xi\delta\sqrt{\varrho}\sigma_1^{\gamma+1}\left(v_{\varrho\sigma_1} - \lambda\left(v_\varrho v_{\sigma_1} + w_\varrho v_{\sigma_1}\right)\right) + \frac{1}{2}P_2\sigma_2^{2\gamma+2} w_{\sigma_2\sigma_2} \\
\quad +P_5\xi\delta\sqrt{\varrho}\sigma_2^{\gamma+1}\left(w_{\varrho\sigma_2} - \lambda\left(v_\varrho w_{\sigma_2} + w_\varrho w_{\sigma_2}\right)\right) = 0 \\
v(T,\rho,\sigma_1) = 0, \quad \omega(T,\rho,\sigma_2 = 0
\end{cases}
\tag{29}
$$

Solving Equation (28) for g, we obtain

$$
g_t = e^{(\rho+\omega_1)(T-t)}
\tag{30}
$$

Recall that:

$$
\omega_1 = \frac{P_6\left(P_1P_2 - P_3^2\right)\beta}{2P_3\left(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho\right)}
$$

Substituting into Equation (30), we get:

$$
g_t = e^{\left(\rho + \frac{P_6\left(P_1P_2 - P_3^2\right)\beta}{2P_3\left(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho\right)}\right)(T-t)}
\tag{31}
$$

**Lemma 1**: The solution of Equation (30) is given as

$$
v(t,\varrho,\sigma_1) + \omega(t,\varrho,\sigma_2) = f(t,\varrho,y,z) = f_1(t,\varrho,y,z) + \sqrt{\varepsilon}f_2(t,\varrho,y,z) + \varepsilon f_3(t,\varrho,y,z)
$$

where,

$$y = \sigma_1^{-2\gamma}, \quad z = \sigma_2^{-2\gamma},$$

$\varepsilon =$ asymptotic expansion parameter,

$$f_1(t, \varrho, y, z) = \left[\frac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)} + \frac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t)$$

$$-\frac{P_1(2\gamma+1)\omega_2\gamma}{\lambda(\varrho+\omega_4)^2}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right)$$

$$-\frac{P_2(2\gamma+1)\omega_3\gamma}{\lambda(\varrho+\omega_5)^2}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right)$$

$$+\frac{\omega_2\gamma}{\lambda(\varrho+\omega_4)}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right)y$$

$$+\frac{\omega_3\gamma}{\lambda(\varrho+\omega_5)}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right)z$$

$$
\begin{aligned}
f_2(t,\varrho,y,z) = {} & \left[\frac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)} + \frac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t) \\
& - \frac{P_1(2\gamma+1)\omega_2\gamma}{\lambda(\varrho+\omega_4)^2}\left(1-e^{2\gamma(\varrho+\omega_4)(t-T)}\right) - \frac{P_2(2\gamma+1)\omega_3\gamma}{\lambda(\varrho+\omega_5)^2}\left(1-e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\
& + \frac{1}{2}ye^{\gamma(\varrho+\omega_4)(t-T)}\left[2P_4\gamma\sqrt{\rho}\int_t^T A_1^{(2)}(\tau)\varrho e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau\right. \\
& \left. - \omega_6\int_t^T A_1^{(1)}(\tau)\varrho e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau - \frac{3}{2}\gamma^2 P_1\int_t^T B_6(\tau)e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau\right] \\
& + \frac{1}{2}ze^{\gamma(\varrho+\omega_5)(t-T)}\left[2P_5\gamma\sqrt{\varrho}\int_t^T A_1^{(3)}(\tau)\varrho e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau\right. \\
& \left. - \omega_7\int_t^T A_1^{(1)}(\tau)\varrho e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau - \frac{3}{2}\gamma^2 P_2\int_t^T B_7(\tau)e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau\right] \\
& + \frac{\omega_2 y}{4\lambda(\varrho+\omega_4)}\left(1-e^{2\gamma(\varrho+\omega_4)(t-T)}\right) + \frac{\omega_3 z}{4\lambda(\varrho+\omega_5)}\left(1-e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\
& + \frac{3}{2}y\omega_6 e^{3\gamma(\varrho+\omega_4)(t-T)}\int_t^T A_1^{(2)}(\tau)\varrho e^{3\gamma(\varrho+\omega_4)(T-\tau)}d\tau \\
& + \frac{3}{2}z\omega_7 e^{3\gamma(\varrho+\omega_5)(t-T)}\int_t^T A_1^{(3)}(\tau)\varrho e^{3\gamma(\varrho+\omega_5)(T-\tau)}d\tau
\end{aligned}
$$

$$
\begin{aligned}
f_3(t,\varrho,y,z) = {} & \frac{1}{2}\lambda\xi^2\varrho\left(1-P_{11}\delta^2\right)\int_t^T \left(A_1^{(1)}(\tau)\right)^2 d\tau + 2P_4\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(2)}(\tau)\varrho d\tau \\
& + 2P_5\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(3)}(\tau)\varrho d\tau - P_1\gamma(2\gamma+1)\int_t^T C_3^{(4)}(\tau)d\tau - P_2\gamma(2\gamma \\
& + 1)\int_t^T C_3^{(5)}(\tau)d\tau - (a-b\varrho)\int_t^T A_1^{(1)}(\tau)\varrho d\tau - \frac{1}{2}\xi^2\varrho\int_t^T A_1^{(1)''}(\tau)d\tau \\
& + \frac{1}{2}ye^{\gamma(\varrho+\omega_4)(t-T)}\left[2P_4\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(4)}(\tau)\varrho e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau\right. \\
& \left. + \omega_6\int_t^T B_2^{(1)}(\tau)\varrho e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau - \frac{3}{2}\gamma^2 P_1\int_t^T C_3^{(6)}(\tau)e^{\gamma(\varrho+\omega_4)(T-\tau)}\, d\tau\right] \\
& + \frac{1}{2}ze^{\gamma(\varrho+\omega_5)(t-T)}\left[2P_5\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(5)}(\tau)\varrho e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau\right. \\
& \left. + \omega_7\int_t^T B_2^{(1)}(\tau)\varrho e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau - \frac{3}{2}\gamma^2 P_2\int_t^T C_2^{(7)}(\tau)e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau\right] \\
& + ye^{2\gamma(\varrho+\omega_4)(t-T)}\left[3P_4\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(6)}(\tau)\varrho e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau\right. \\
& - \frac{\omega^2}{2\lambda}\int_t^T e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau - (a-b\varrho)\int_t^T A_1^{(2)}(\tau)\varrho e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau \\
& + \omega_6\int_t^T B_2^{(2)}(\tau)\varrho e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau - \frac{1}{2}\xi^2\varrho\int_t^T A_1^{(2)''}(\tau)e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau \\
& - 2P_1\left(2\gamma^2+\gamma(2\gamma+1)\right)\int_t^T C_3^{(8)}(\tau)e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau \\
& \left. + \lambda\xi^2\varrho\left(1-P_{11}\delta^2\right)\int_t^T A_1^{(1)}(\tau)A_1^{(2)}(\tau)e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau\right] \\
& + ze^{2\gamma(\varrho+\omega_5)(t-T)}\left[3P_5\lambda\gamma\sqrt{\varrho}\int_t^T B_2^{(7)}(\tau)\varrho e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\omega_3}{2\lambda}\int_t^T e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau-(a-b\varrho)\int_t^T A_1^{(3)}(\tau)\varrho e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\\
&+\omega_7\int_t^T B_2^{(3)}(\tau)\varrho e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau-\frac{1}{2}\xi^2\varrho\int_t^T A_1^{(3)''}(\tau)e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\\
&-2P_2\big(2\gamma^2+\gamma(2\gamma+1)\big)\int_t^T C_3^{(9)}(\tau)e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\\
&+\lambda\xi^2\varrho\big(1-P_{11}\delta^2\big)\int_t^T A_1^{(1)}(\tau)A_1^{(3)}(\tau)e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\Bigg]\\
&+\frac{3}{2}y\omega_6e^{3\gamma(\varrho+\omega_4)(t-T)}\int_t^T B_1^{(4)}(\tau)\varrho e^{3\gamma(\varrho+\omega_4)(T-\tau)}d\tau\\
&+\frac{3}{2}z\omega_7e^{3\gamma(\varrho+\omega_5)(t-T)}\int_t^T B_1^{(5)}(\tau)\varrho e^{3\gamma(\varrho+\omega_5)(T-\tau)}d\tau\\
&+y^2e^{2\gamma(\varrho+\omega_4)(t-T)}\Bigg[\frac{1}{2}\lambda\xi^2\varrho\big(1-P_{11}\delta^2\big)\int_t^T\Big(A_1^{(2)}(\tau)\Big)^2e^{\gamma(\varrho+\omega_4)(T-\tau)}d\tau\\
&+\omega_6\int_t^T B_2^{(6)}(\tau)\varrho e^{2\gamma(\varrho+\omega_4)(T-\tau)}d\tau\Bigg]\\
&+z^2e^{2\gamma(\varrho+\omega_5)(t-T)}\Bigg[\frac{1}{2}\lambda\xi^2\varrho\big(1-P_{11}\delta^2\big)\int_t^T\Big(A_1^{(3)}(\tau)\Big)^2e^{\gamma(\varrho+\omega_5)(T-\tau)}d\tau\\
&+\omega_7\int_t^T B_2^{(7)}(\tau)\varrho e^{2\gamma(\varrho+\omega_5)(T-\tau)}d\tau\Bigg]
\end{aligned}
$$

Proof

Assume

$$
\begin{cases} v(t,\varrho,\sigma_1)+\omega(t,\varrho,\sigma_2)=f(t,\varrho,y,z)\\ y=\sigma_1^{-2\gamma},\quad z=\sigma_2^{-2\gamma},\quad f(T,\varrho,y,z)=0\end{cases}\tag{32}
$$

Then the derivatives transform as follows:

$$\begin{cases} v_t + \omega_t = f_t, \\ v_{\sigma_1} = -2\gamma\sigma_1^{-2\gamma-1} f_y \\ v_{\sigma_1\sigma_1} = 2\gamma(2\gamma+1)\sigma_1^{-2\gamma-2} f_y + 4\gamma^2\sigma_1^{-4\gamma-2} f_{yy}, \\ \omega_{\sigma_2} = -2\gamma\sigma_2^{-2\gamma-1} f_z, \\ \omega_{\sigma_2\sigma_2} = 2\gamma(2\gamma+1)\sigma_2^{-2\gamma-2} f_z + 4\gamma^2\sigma_2^{-4\gamma-2} f_{zz}, \\ v_\varrho + \omega_\varrho = f_\varrho, \\ v_{\varrho\varrho} + \omega_{\varrho\varrho} = f_{\varrho\varrho}, \\ v_{\varrho\sigma_1} = -2\gamma\sigma_1^{-2\gamma-1} f_{\varrho y}, \\ \omega_{\varrho\sigma_2} = -2\gamma\sigma_2^{-2\gamma-1} f_{\varrho z} \end{cases} \tag{33}$$

Substituting (33) into Equation (29), we obtain:

$$f_t - 2\gamma y(\varrho + \omega_4) f_y - 2\gamma z(\varrho + \omega_5) f_z - \frac{1}{2}\lambda\xi^2\varrho\left(1 - P_{11}\delta^2\right) f_\varrho^2 + \frac{1}{2}\xi^2\varrho f_{\varrho\varrho} + \omega_2\lambda y + \omega_3\lambda z + f_\varrho\left(a - b\varrho - \omega_6\sqrt{y} - \omega_7\sqrt{z}\right) + P_1\gamma(2\gamma+1) f_y + 2P_1\gamma^2 y f_{yy} - 2P_4\lambda\gamma\sqrt{\varrho y}\left(f_{\varrho y} - \lambda f_\varrho f_y\right) + P_2\gamma(2\gamma+1) f_z + 2P_2\gamma^2 z f_{zz} - 2P_5\lambda\gamma\sqrt{\varrho z}\left(f_{\varrho z} - \lambda f_\varrho f_z\right) = 0 \tag{34}$$

Rewriting the Equation (34) in the compact form:

$$(E + F + G)f - \frac{1}{2}\lambda\xi^2\varrho\left(1 - P_{11}\delta^2\right) f_\varrho^2 = 0 \tag{35}$$

where:

$$E = (a - b\varrho) f_\varrho + \frac{1}{2}\xi^2\varrho f_{\varrho\varrho} \tag{36}$$

$$F = f_t + \gamma\left(P_1(2\gamma+1) - 2y(\varrho + \omega_4)\right) fy + \gamma\left(P_2(2\gamma+1) - 2z(\varrho + \omega_5)\right) f_z + 2\gamma^2 P_1 y f_{yy} + 2\gamma^2 P_2 z f_{zz} + \omega_2\lambda y + \omega_3\lambda z \tag{37}$$

$$G = 2P_4\lambda\gamma\sqrt{\varrho y}\left(\lambda f_\varrho f_y - f_{\varrho y}\right) + 2P_5\lambda\gamma\sqrt{\varrho z}\left(\lambda f_\varrho f_z - f_{\varrho z}\right) - f_\varrho\left(\omega_6\sqrt{y} + \omega_7\sqrt{z}\right) \tag{38}$$

Follow the approach in (34) by applying the **asymptotic expansion method** to solve the non-linear PDE in Equation (35).

Assume the volatility of the interest rate follows a **slow-fluctuating process**, represented by a small parameter $\varepsilon$, where $0 < \varepsilon \leq 1$. Approximate the stochastic interest rate $\rho(t)$ by a perturbed process $\rho_\varepsilon(t)$, defined as:

$$d\varrho_{\varepsilon}(t) = \left(a - b\varrho_{\varepsilon}(t)\right) dt - \varepsilon\varrho_{\varepsilon}(t) dZ_0(t) \quad (39)$$

where:

$a, b > 0$ *are mean* $-$ *reversion parameters*,

$Z_0(t)$ is a standard Brownian motion independent of other sources of randomness,

$\varepsilon$ controls the speed of fluctuation (with $\varepsilon \to 0$ corresponding to a deterministic interest rate path).

Substituting the slow-fluctuating stochastic interest rate process from Equation (39) into the non-linear PDE in Equation (35), also make the substitutions:

Replace $a - b\varrho(t)$ with $\varepsilon\left(a - b\varrho(t)\right)$, replace $\sqrt{\varrho}$ with $\sqrt{\varepsilon}\sqrt{\varrho}$

With these replacements, Equation (35) becomes:

$$\left[-\varepsilon E + F + \sqrt{\varepsilon} G\right] f^{\varepsilon} = 0 \quad (40)$$

To solve Equation (40) using the method of asymptotic expansion, we conjecture the solution $f^{\varepsilon}(t, \rho, y, z)$ in powers of $\varepsilon$ as follows:

$$f^{\varepsilon}(t, \varrho, y, z) = f_1(t, \varrho, y, z) + \sqrt{\varepsilon} f_2(t, \varrho, y, z) + \varepsilon f_3(t, \varrho, y, z) \quad (41)$$

Substituting (41) into (37) and simplifying it,

$$\begin{cases} [Ff_1(t, \varrho, y, z)] + \sqrt{\varepsilon}[Ff_2(t, \varrho, y, z) + Gf_1(t, \varrho, y, z)] \\ +\varepsilon\left[Ef_1(t, \varrho, y, z) + Ff_3(t, \varrho, y, z) + Gf_2(t, \varrho, y, z) - \frac{1}{2}\lambda\xi^2\varrho\left(1 - P_{11}\delta^2\right)\left(\frac{\partial f_1}{\partial \varrho}\right)^2\right] = 0 \end{cases} \quad (42)$$

This implies that,

$$\begin{cases} Ff_1(t, \varrho, y, z) = F\left(p_1(t, \varrho, y, z) + q_1(t, \rho, z)\right) = 0 \\ f_1(t, \varrho, y, z) = 0 \end{cases} \quad (43)$$

$$\begin{cases} Ff_2(t, \varrho, y, z) + Gf_1(t, \varrho, y, z) = 0 \\ f_1(t, \varrho, y, z) = f_2(t, \varrho, y, z) = 0 \end{cases} \quad (44)$$

$$\begin{cases} Ef_1(t, \varrho, y, z) + Ff_3(t, \varrho, y, z) + Gf_2(t, \varrho, y, z) = 0 \\ f_1(t, \varrho, y, z) = f_2(t, \varrho, y, z) = f_3(t, \varrho, y, z) = 0 \end{cases} \quad (45)$$

Apply (36), (37) and (38), Equation (43), (44) and (45) have the forms:

$$\begin{cases} f_{1t} + \gamma\big(P_1(2\gamma+1) - 2y(\varrho+\omega_4)\big)f_{1y} + \gamma\big(P_2(2\gamma+1) - 2z(\varrho+\omega_5)\big)f_{1z} \\ +2\gamma^2 P_1 y f_{1yy} + 2\gamma^2 P_2 z f_{1zz} + \omega_2\lambda y + \omega_3\lambda z = 0 \\ \square_1(t,\varrho,y,z) = 0 \end{cases} \tag{46}$$

$$\begin{cases} f_{2t} + \gamma\big(P_1(2\gamma+1) - 2y(\varrho+\omega_4)\big)f_{2y} + \gamma\big(P_2(2\gamma+1) - 2z(\varrho+\omega_5)\big)f_{2z} \\ +2\gamma^2 P_1 y f_{2yy} + 2\gamma^2 P_2 z f_{2zz} + \omega_2\lambda y + \omega_3\lambda z \\ +2P_4\xi\delta\sqrt{\varrho}\big(\lambda f_{1\varrho} f_{1y} - f_{1\varrho y}\big)\sqrt{y} - \big(\omega_6\sqrt{y} + \omega_7\sqrt{z}\big)f_{1\varrho} \\ +2P_5\xi\delta\sqrt{\varrho}\big(\lambda f_{1\varrho} f_{1z} - f_{1\varrho z}\big)z = 0 \\ f_1(t,\varrho,y,z) = f_2(t,\varrho,y,z) = 0 \end{cases} \tag{47}$$

$$\begin{cases} f_{3t} + \gamma\big(P_1(2\gamma+1) - 2y(\varrho+\omega_4)\big)f_{3y} + \gamma\big(P_2(2\gamma+1) - 2z(\varrho+\omega_5)\big)f_{3z} \\ +2\gamma^2 P_1 y f_{3yy} + 2\gamma^2 P_2 z f_{3zz} + \omega_2\lambda y + \omega_3\lambda z \\ +2P_4\xi\delta\sqrt{\varrho}\big(\lambda f_{2\varrho} f_{2y} - f_{2\varrho y}\big)\sqrt{y} - \big(\omega_6\sqrt{y} + \omega_7\sqrt{z}\big)f_{2\varrho} \\ +2P_5\xi\delta\sqrt{\varrho}\big(\lambda f_{2\varrho} f_{2z} - f_{2\varrho z}\big)z = 0 \\ +\frac{1}{2}\lambda\xi^2\varrho(1 - P_{11}\delta)\big(f_{1\varrho}\big)^2 + (a - b\varrho)f_{1\varrho} + \frac{1}{2}\xi^2\varrho f_{1\varrho\varrho} = 0 \\ f_1(t,\varrho,y,z) = f_2(t,\varrho,y,z) = f_3(t,\varrho,y,z) = 0 \end{cases} \tag{48}$$

To continue the solution of Equations (46)–(48), we now focus on solving Equation (46) for $f_1(t,\rho,y,z)$ using the conjectured form:

Assume

$$\begin{cases} f_1(t,\varrho,y,z) = A_{11}(t,\varrho) + yA_{12}(t,\varrho) + zA_{13}(t,\varrho) \\ A_{11}(T,\varrho) = A_{12}(T,\varrho) = A_{13}(T,\varrho) = 0 \end{cases} \tag{49}$$

From (49), the partial derivatives become:

$$\begin{cases} f_{1t} = A_{11t}(t,\varrho) + yA_{12t}(t,\varrho) + zA_{13t}(t,\varrho) \\ f_{1y} = A_{12}(t,\varrho), \quad f_{1yy} = 0 \\ f_{1z} = A_{13}(t,\varrho), \quad f_{1zz} = 0 \end{cases} \tag{50}$$

**Substituting (50) in (46), and** grouping terms with similar powers of the constant term, $y, and\ z$;

$$\begin{cases} A_{11t} + \gamma P_1(2\gamma+1)A_{12} + \gamma P_2(2\gamma+1)A_{13} = 0 \\ A_{11}(T,\varrho) = A_{12}(T,\varrho) = A_{13}(T,\varrho) = 0 \end{cases} \tag{51}$$

$$\begin{cases} A_{12t} - 2\gamma(\varrho + \omega_4)A_{12} + \omega_2\lambda = 0 \\ A_{12}(T, \varrho) = 0 \end{cases} \tag{52}$$

$$\begin{cases} A_{13t} - 2\gamma(\varrho + \omega_5)A_{13} + \omega_3\lambda = 0 \\ A_{13}(T, \varrho) = 0 \end{cases} \tag{53}$$

Solving (51), (52), and (53),

$$A_{11}(t, \varrho) = \begin{cases} \left[\dfrac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)} + \dfrac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t) \\ -\dfrac{2P_1(2\gamma+1)\omega_2}{4\gamma\lambda(\varrho+\omega_4)^2}\left(1 - e^{-2\gamma(\varrho+\omega_4)(T-t)}\right) \\ -\dfrac{P_2(2\gamma+1)\omega_3}{4\gamma\lambda(\varrho+\omega_5)^2}\left(1 - e^{-2\gamma(\varrho+\omega_5)(T-t)}\right) \end{cases} \tag{54}$$

$$A_{12}(t, \varrho) = \frac{\omega_2\lambda}{2\gamma(\varrho+\omega_4)}\left(1 - e^{-2\gamma(\varrho+\omega_4)(T-t)}\right) \tag{55}$$

$$A_{13}(t, \varrho) = \frac{\omega_3\lambda}{2\gamma(\varrho+\omega_5)}\left(1 - e^{-2\gamma(\varrho+\omega_5)(T-t)}\right) \tag{56}$$

**Thus, from (49),**

$$f_1(t, \varrho, y, z) = \begin{cases} \left[\dfrac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)} + \dfrac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t) \\ -\dfrac{2P_1(2\gamma+1)\omega_2}{4\gamma\lambda(\varrho+\omega_4)^2}\left(1 - e^{-2\gamma(\varrho+\omega_4)(T-t)}\right) \\ -\dfrac{P_2(2\gamma+1)\omega_3}{4\gamma\lambda(\varrho+\omega_5)^2}\left(1 - e^{-2\gamma(\varrho+\omega_5)(T-t)}\right) \\ +\dfrac{\omega_2\lambda}{2\gamma(\varrho+\omega_4)}\left(1 - e^{-2\gamma(\varrho+\omega_4)(T-t)}\right) y \\ +\dfrac{\omega_3\lambda}{2\gamma(\varrho+\omega_5)}\left(1 - e^{-2\gamma(\varrho+\omega_5)(T-t)}\right) z \end{cases} \tag{57}$$

**Next, solev (47),**

$$f_2(t,\varrho,y,z)=\begin{cases}B_1^{(2)}(t,\varrho)+y^{\frac{1}{2}}B_2^{(2)}(t,\varrho)+z^{\frac{1}{2}}B_3^{(2)}(t,\varrho)+yB_4^{(2)}(t,\varrho)\\+zB_5^{(2)}(t,\varrho)+y^{\frac{3}{2}}B_6^{(2)}(t,\varrho)+z^{\frac{3}{2}}B_7^{(2)}(t,\varrho)\\B_i^{(2)}(T,\varrho)=0,\text{for } i=1,\dots,7\end{cases}\tag{58}$$

and

$$f_{2t}=\begin{cases}B_{1t}^{(2)}+y^{\frac{1}{2}}B_{2t}^{(2)}+z^{\frac{1}{2}}B_{3t}^{(2)}+yB_{4t}^{(2)}+zB_{5t}^{(2)}+y^{\frac{3}{2}}B_{6t}^{(2)}+z^{\frac{3}{2}}B_{7t}^{(2)}\\f_{2y}=\frac{1}{2}y^{-\frac{1}{2}}B_2^{(2)}+B_4^{(2)}+\frac{3}{2}y^{\frac{1}{2}}B_6^{(2)}\\f_{2yy}=-\frac{1}{4}y^{-\frac{3}{2}}B_2^{(2)}+\frac{3}{4}y^{-\frac{1}{2}}B_6^{(2)}\\f_{2z}=-\frac{1}{2}z^{-\frac{1}{2}}B_3^{(2)}+B_5^{(2)}+\frac{3}{2}z^{-\frac{1}{2}}B_7^{(2)}\\f_{2zz}=-\frac{1}{4}y^{-\frac{3}{2}}B_3^{(2)}+\frac{3}{4}y^{-\frac{1}{2}}B_7^{(2)}\end{cases}\tag{59}$$

**Substituting (59) into (57),**

$$\begin{cases}\frac{d}{dt}B_1^{(2)}+\gamma(2\gamma+1)\left(P_1B_4^{(2)}+P_2B_5^{(2)}\right)=0,\\B_1^{(2)}(t,\varrho)=B_2^{(2)}(t,\varrho)=B_3^{(2)}(t,\varrho)=0\end{cases}\tag{60}$$

$$\begin{cases}\frac{d}{dt}B_2^{(2)}+\gamma(\varrho+\omega_4)B_2^{(2)}+\frac{3}{2}\gamma^2P_1B_6^{(2)}-2P_4\xi\delta\varrho A'_{12}(t,\varrho)-\omega_6A'_{11}(t,\varrho)=0,\\B_2^{(2)}(t,\varrho)=0\end{cases}\tag{61}$$

$$\begin{cases}\frac{d}{dt}B_3^{(2)}+\gamma(\varrho+\omega_5)B_3^{(2)}+\frac{3}{2}\gamma^2P_2B_7^{(2)}-2P_5\xi\delta\varrho A'_{13}(t,\varrho)-\omega_7A'_{11}(t,\varrho)=0,\\B_3^{(2)}(t,\varrho)=0\end{cases}\tag{62}$$

$$\begin{cases}\frac{d}{dt}B_4^{(2)}-2\gamma(\varrho+\omega_4)B_4^{(2)}+\omega_2\lambda^{-1}=0,\\B_4^{(2)}(t,\varrho)=0\end{cases}\tag{63}$$

$$\begin{cases} \frac{d}{dt}B_5^{(2)} - 2\gamma(\varrho + \omega_5)B_5^{(2)} + \omega_3\lambda^{-1} = 0, \\ B_5^{(2)}(t,\varrho) = 0 \end{cases} \tag{64}$$

$$\begin{cases} \frac{d}{dt}B_6^{(2)} - 3\gamma(\varrho + \omega_4)B_6^{(2)} + \omega_6 A_{12}^{'}(t,\varrho) = 0, \\ B_6^{(2)}(t,\varrho) = 0 \end{cases} \tag{65}$$

$$\begin{cases} \frac{d}{dt}B_7^{(2)} - 3\gamma(\varrho + \omega_5)B_7^{(2)} + \omega_7 A_{13}^{'}(t,\varrho) = 0, \\ B_7^{(2)}(t,\varrho) = 0 \end{cases} \tag{66}$$

**Solving Equations (60) to (66),**

$$\begin{cases} B_1^{(2)}(t,\rho) = \left[\frac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)} + \frac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t) \\ \quad -\frac{P_1(2\gamma+1)\omega_2\gamma}{\lambda(\varrho+\omega_4)^2}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right) \\ \quad -\frac{P_2(2\gamma+1)\omega_3\gamma}{\lambda(\varrho+\omega_5)^2}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\ B_2^{(2)}(t,\varrho) = e^{\gamma(\varrho+\omega_4)(t-T)}\Big[2P_4\xi\delta\sqrt{\varrho}\int_t^T A_{12}^{'}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt \\ \quad -\omega_6\int_t^T A_{11}^{'}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt \\ \quad -\frac{3}{2}\gamma^2 P_1\int_t^T B_6^{(2)}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt\Big] \\ B_3^{(2)}(t,\varrho) = e^{\gamma(\varrho+\omega_5)(t-T)}\Big[2P_5\xi\delta\sqrt{\varrho}\int_t^T A_{13}^{'}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt \\ \quad -\omega_7\int_t^T A_{11}^{'}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt \\ \quad -\frac{3}{2}\gamma^2 P_2\int_t^T B_7^{(2)}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt\Big] \\ B_4^{(2)}(t,\varrho) = \frac{\omega_2}{\gamma\lambda(\varrho+\omega_4)}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right) \\ B_5^{(2)}(t,\varrho) = \frac{\omega_3}{\gamma\lambda(\varrho+\omega_5)}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\ B_6^{(2)}(t,\varrho) = \omega_6 e^{3\gamma(\varrho+\omega_4)(t-T)}\int_t^T A_{12}^{'}(t,\varrho)e^{3\gamma(\varrho+\omega_4)(T-t)}dt \\ B_7^{(2)}(t,\varrho) = \omega_7 e^{3\gamma(\varrho+\omega_5)(t-T)}\int_t^T A_{13}^{'}(t,\varrho)e^{3\gamma(\varrho+\omega_5)(T-t)}dt \end{cases} \tag{67}$$

Substituting (67) into (58),

$$f_2(t,\varrho,y,z)\begin{cases} \left[\frac{P_1(2\gamma+1)\omega_2}{4\lambda(\varrho+\omega_4)}+\frac{P_2(2\gamma+1)\omega_3}{4\lambda(\varrho+\omega_5)}\right](T-t) \\ -\frac{P_1(2\gamma+1)\omega_2\gamma}{\lambda(\varrho+\omega_4)^2}\left(1-e^{2\gamma(\varrho+\omega_4)(t-T)}\right) \\ -\frac{P_2(2\gamma+1)\omega_3\gamma}{\lambda(\varrho+\omega_5)^2}\left(1-e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\ +\frac{y}{2}e^{\gamma(\varrho+\omega_4)(t-T)}\left[2P_4\xi\delta\sqrt{\varrho}\int_t^T A'_{12}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt\right. \\ \qquad -\omega_6\int_t^T A'_{11}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt-\frac{3}{2}\gamma^2P_1\int_t^T B_6^{(2)}(t,\varrho)e^{\gamma(\varrho+\omega_4)(T-t)}dt \\ +\frac{z}{2}e^{\gamma(\varrho+\omega_5)(t-T)}\left[2P_5\xi\delta\sqrt{\varrho}\int_t^T A'_{13}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt\right. \\ -\omega_7\int_t^T A'_{11}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt-\frac{3}{2}\gamma^2P_2\int_t^T B_7^{(2)}(t,\varrho)e^{\gamma(\varrho+\omega_5)(T-t)}dt\Big] \\ +\frac{\omega_2 y}{4\gamma\lambda(\varrho+\omega_4)}\left(1-e^{2\gamma(\varrho+\omega_4)(t-T)}\right)+\frac{\omega_3 z}{4\gamma\lambda(\varrho+\omega_5)}\left(1-e^{2\gamma(\varrho+\omega_5)(t-T)}\right) \\ +\frac{3}{2}\omega_6 y e^{3\gamma(\varrho+\omega_4)(t-T)}\int_t^T A'_{12}(t,\varrho)e^{3\gamma(\varrho+\omega_4)(T-t)}dt \\ +\frac{3}{2}\omega_7 z e^{3\gamma(\varrho+\omega_5)(t-T)}\int_t^T A'_{13}(t,\varrho)e^{3\gamma(\varrho+\omega_5)(T-t)}dt \end{cases} \tag{68}$$

To solve (48), assume a solution of the form,

$$f_3(t,\varrho,y,z)=\begin{cases} C_1^{(3)}(t,\varrho)+\frac{1}{2}yC_2^{(3)}(t,\varrho)+\frac{1}{2}zC_3^{(3)}(t,\varrho)+yC_4^{(3)}(t,\varrho)+zC_5^{(3)}(t,\varrho) \\ +\frac{3}{2}y^{\frac{1}{2}}C_6^{(3)}(t,\varrho)+\frac{3}{2}z^{\frac{1}{2}}C_7^{(3)}(t,\varrho)+y^2C_8^{(3)}(t,\varrho)+z^2C_9^{(3)}(t,\varrho) \\ C_i^{(3)}(t,\varrho)=0,\text{ for all } i=1,2,\dots,9 \end{cases} \tag{69}$$

**and**

$$f_{2\rho}=\begin{cases} f_{1\rho}=A_{11\varrho}^{(1)}+yA_{12\varrho}^{(1)}+zA_{13\varrho}^{(1)} \\ f_{1\rho\rho}=A_{11\varrho\varrho}^{(1)}+yA_{12\varrho\varrho}^{(1)}+zA_{13\varrho\varrho}^{(1)} \\ B_{1\varrho}^{(2)}+\frac{1}{2}yB_{2\varrho}^{(2)}+\frac{1}{2}zB_{3\varrho}^{(2)}+yB_{4\varrho}^{(2)}+zB_{5\varrho}^{(2)}+\frac{3}{2}y^{\frac{1}{2}}B_{6\varrho}^{(2)}+\frac{3}{2}z^{\frac{1}{2}}B_{7\varrho}^{(2)} \\ f_{2y\rho}=\frac{1}{2}y^{-\frac{1}{2}}B_2^{(2)}+B_4^{(2)}+\frac{3}{2}y^{\frac{1}{2}}B_6^{(2)} \\ f_{2z\rho}=\frac{1}{2}z^{-\frac{1}{2}}B_3^{(2)}+B_5^{(2)}+\frac{3}{2}z^{\frac{1}{2}}B_7^{(2)} \end{cases} \tag{70}$$

**Substituting (70) into (48), we have**

$$\begin{cases} C_1^{(3)}t + P_1\gamma(2\gamma+1)C_4^{(3)} + P_2\gamma(2\gamma+1)C_5^{(3)} \\ +(a-b\varrho)A_{11\ \varrho}^{(1)} + \frac{1}{2}\delta^2\varrho A_{11\ \varrho\varrho}^{(1)} - \frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{11}^{(1)}\varrho\right)^2 \\ -2P_4\xi\gamma\varrho B_{2\ \varrho}^{(2)} - 2P_5\xi\gamma\varrho B_{3\ \varrho}^{(2)} = 0, \boldsymbol{C_1^{(3)}(t,\rho) = 0} \end{cases} \qquad (71)$$

$$\begin{cases} C_{2\ t}^{(3)} - \gamma(\varrho+\omega_4)C_2^{(3)} + \frac{3}{2}\gamma^2 P_1 C_6^{(3)} - 2P_4\xi\gamma\varrho B_{4\ \varrho}^{(2)} - \omega_6 B_{1\ \varrho}^{(2)} = 0 \\ \boldsymbol{C_2^{(3)}(t,\rho) = 0} \end{cases} \qquad (72)$$

$$\begin{cases} C_{3\ t}^{(3)} - \gamma(\varrho+\omega_5)C_3^{(3)} + \frac{3}{2}\gamma^2 P_2 C_7^{(3)} - 2P_5\xi\gamma\varrho B_{5\ \varrho}^{(2)} - \omega_7 B_{1\ \varrho}^{(2)} = 0 \\ \boldsymbol{C_3^{(3)}(t,\rho) = 0} \end{cases} \qquad (73)$$

$$\begin{cases} C_{4\ t}^{(3)} - 2\gamma(\varrho+\omega_4)C_4^{(3)} + 2P_1\left(2\gamma^2+\gamma(2\gamma+1)\right)C_8^{(3)} \\ +(a-b\varrho)A_{12\ \varrho}^{(1)} + \frac{1}{2}\delta^2\varrho A_{12\ \varrho\varrho}^{(1)} - \frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{12\ \varrho}^{(1)} \\ -3P_4\xi\gamma\varrho B_{6\ \varrho}^{(2)} - \omega_6 B_{2\ \varrho}^{(2)} + \omega_2\lambda = 0, \\ \boldsymbol{C_4^{(3)}(t,\rho) = 0} \end{cases} \qquad (74)$$

$$\begin{cases} C_{5\ t}^{(3)} - 2\gamma(\varrho+\omega_5)C_5^{(3)} + 2P_2\left(2\gamma^2+\gamma(2\gamma+1)\right)C_9^{(3)} \\ +(a-b\varrho)A_{13\ \varrho}^{(1)} + \frac{1}{2}\delta^2\varrho A_{13\ \varrho\varrho}^{(1)} - \frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{13\ \varrho}^{(1)} \\ -3P_5\xi\gamma\varrho B_{7\ \varrho}^{(2)} - \omega_7 B_{3\ \varrho}^{(2)} + \omega_3\lambda = 0, \\ \boldsymbol{C_5^{(3)}(t,\rho) = 0} \end{cases} \qquad (75)$$

$$C_{6\ t}^{(3)} - 3\gamma(\varrho+\omega_4)C_6^{(3)} + \omega_6 B_{4\ \varrho}^{(2)} = 0 \qquad (76)$$

$$C_{7\ t}^{(3)} - 3\gamma(\varrho+\omega_5)C_7^{(3)} + \omega_7 B_{5\ \varrho}^{(2)} = 0 \qquad (77)$$

$$C_{8\ t}^{(3)} - 2\gamma(\varrho+\omega_4)C_8^{(3)} - \frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{12\ \varrho}^{(1)}\right)^2 - \omega_6 B_{6\ \varrho}^{(2)} = 0 \qquad (78)$$

$$C_{9\ t}^{(3)} - 2\gamma(\varrho+\omega_5)C_9^{(3)} - \frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{13\ \varrho}^{(1)}\right)^2 - \omega_7 B_{7\ \varrho}^{(2)} = 0 \qquad (79)$$

Solving Equations (71) to (79) to obtain,

$$
\begin{cases}
C_1^{(3)}(t,\varrho) = \int_t^T [\frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{11\ \varrho}^{(1)}\right)^2 + 2P_4\xi\gamma\varrho B_{2\ \ \varrho}^{(2)} + 2P_5\xi\gamma\varrho B_{3\ \ \varrho}^{(2)} \\
-P_1\gamma(2\gamma+1)C_4^{(3)} - P_2\gamma(2\gamma+1)C_5^{(3)} - (a-b\varrho)_{A11_\varrho}^{(1)} - \frac{1}{2}\delta^2\varrho A_{11\ \varrho\varrho}^{(1)}]\, dt, \\
C_2^{(3)}(t,\varrho) = e^{\gamma(\varrho+\omega_4)(t-T)} \int_t^T \left[2P_4\xi\gamma\varrho B_4^{(2)} + \omega_6 B_1^{(2)} - \frac{3}{2}\gamma^2 P_1 C_6^{(3)}\right] e^{\gamma(\varrho+\omega_4)(T-t)} dt \\
C_3^{(3)}(t,\varrho) = e^{\gamma(\varrho+\omega_5)(t-T)} \int_t^T \left[2P_5\xi\gamma\varrho B_5^{(2)} + \omega_7 B_1^{(2)} - \frac{3}{2}\gamma^2 P_2 C_7^{(3)}\right] e^{\gamma(\varrho+\omega_5)(T-t)} dt \\
C_4^{(3)}(t,\varrho) = e^{2\gamma(\varrho+\omega_4)(t-T)} \int_t^T [3P_4\xi\gamma\varrho B_6^{(2)} - \frac{\omega_2}{2\lambda} + (a-b\varrho)A_{12\ \varrho}^{(1)} + \frac{1}{2}\delta^2\varrho A_{12\ \varrho\varrho}^{(1)} \\
-\frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{12\ \varrho}^{(1)} + \omega_6 B_2^{(2)} \\
-2P_1\left(2\gamma^2+\gamma(2\gamma+1)\right)C_8^{(3)} + \lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{12\ \varrho}^{(1)}]\, e^{2\gamma(\varrho+\omega_4)(T-t)} dtt \\
C_5^{(3)}(t,\varrho) = e^{2\gamma(\varrho+\omega_5)(t-T)} \int_t^T [3P_5\xi\gamma\varrho B_7^{(2)} - \frac{\omega_3}{2\lambda} + (a-b\varrho)A_{13\ \varrho}^{(1)} + \frac{1}{2}\delta^2\varrho A_{13\ \varrho\varrho}^{(1)} \\
-\frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{13\ \varrho}^{(1)} + \omega_7 B_3^{(2)} \\
-2P_2\left(2\gamma^2+\gamma(2\gamma+1)\right)C_9^{(3)} + \lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)A_{11\ \varrho}^{(1)}A_{13\ \varrho}^{(1)}]\, e^{2\gamma(\varrho+\omega_5)(T-t)} dt \\
C_6^{(3)}(t,\varrho) = \omega_6 e^{3\gamma(\varrho+\omega_4)(t-T)} \int_t^T B_4^{(1)} e^{3\gamma(\varrho+\omega_4)(T-t)} dt \\
C_7^{(3)}(t,\varrho) = \omega_7 e^{3\gamma(\varrho+\omega_5)(t-T)} \int_t^T B_5^{(1)} e^{3\gamma(\varrho+\omega_5)(T-t)} dt \\
C_8^{(3)}(t,\varrho) = e^{2\gamma(\varrho+\omega_4)(t-T)} \int_t^T \left[\frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{12\ \varrho}^{(1)}\right)^2 + \omega_6 B_6^{(2)}\right] e^{2\gamma(\varrho+\omega_4)(T-t)} dt \\
C_9^{(3)}(t,\varrho) = e^{2\gamma(\varrho+\omega_5)(t-T)} \int_t^T \left[\frac{1}{2}\lambda\delta^2\varrho\left(1-P_{11}\xi^2\right)\left(A_{13\ \varrho}^{(1)}\right)^2 + \omega_7 B_7^{(2)}\right] e^{2\gamma(\varrho+\omega_5)(T-t)} dt
\end{cases}
\tag{80}
$$

Substituting (80) into (69),

$$\begin{aligned}
f_3(t,\varrho,y,z) &= \int_t^T [\tfrac{1}{2}\lambda\delta^2\varrho(1-P_{11}\xi^2)\left(A^{(1)}_{11\,\varrho}\right)^2 + 2P_4\xi\gamma\varrho B^{(2)}_{2\ \varrho} + 2P_5\xi\gamma\varrho B^{(2)}_{3\ \varrho} \\
&-P_1\gamma(2\gamma+1)C^{(3)}_4 - P_2\gamma(2\gamma+1)C^{(3)}_5 - (a-b\varrho)^{(1)}_{A11_\varrho} - \tfrac{1}{2}\delta^2\varrho A^{(1)}_{11\ \varrho\varrho}]\,dt, \\
&+\tfrac{y}{2}e^{\gamma(\varrho+\omega_4)(t-T)}\int_t^T \left[2P_4\xi\gamma\varrho B^{(2)}_4 + \omega_6 B^{(2)}_1 - \tfrac{3}{2}\gamma^2 P_1 C^{(3)}_6\right] e^{\gamma(\varrho+\omega_4)(T-t)}dt \\
&+\tfrac{z}{2}e^{\gamma(\varrho+\omega_5)(t-T)}\int_t^T \left[2P_5\xi\gamma\varrho B^{(2)}_5 + \omega_7 B^{(2)}_1 - \tfrac{3}{2}\gamma^2 P_2 C^{(3)}_7\right] e^{\gamma(\varrho+\omega_5)(T-t)}dt \\
&+ye^{2\gamma(\varrho+\omega_4)(t-T)}\int_t^T [3P_4\xi\gamma\varrho B^{(2)}_6 - \tfrac{\omega_2}{2\lambda} - (a-b\varrho)A^{(1)}_{12\ \varrho} - \tfrac{1}{2}\delta^2\varrho A^{(1)}_{12\ \varrho\varrho} \\
&-\tfrac{1}{2}\lambda\delta^2\varrho(1-P_{11}\xi^2)A^{(1)}_{11\ \varrho}A^{(1)}_{12\ \varrho} + \omega_6 B^{(2)}_2 \\
&-2P_1\left(2\gamma^2+\gamma(2\gamma+1)\right)C^{(3)}_8 + \lambda\delta^2\varrho(1-P_{11}\xi^2)A^{(1)}_{11\ \varrho}A^{(1)}_{12\ \varrho}]\, e^{2\gamma(\varrho+\omega_4)(T-t)}dt \\
&+ze^{2\gamma(\varrho+\omega_5)(t-T)}\int_t^T [3P_5\xi\gamma\varrho B^{(2)}_7 - \tfrac{\omega_3}{2\lambda} - (a-b\varrho)A^{(1)}_{13\ \varrho} - \tfrac{1}{2}\delta^2\varrho A^{(1)}_{13\ \varrho\varrho} \\
&-\tfrac{1}{2}\lambda\delta^2\varrho(1-P_{11}\xi^2)A^{(1)}_{11\ \varrho}A^{(1)}_{13\ \varrho} + \omega_7 B^{(2)}_3 \\
&-2P_2\left(2\gamma^2+\gamma(2\gamma+1)\right)C^{(3)}_9 + \lambda\delta^2\varrho(1-P_{11}\xi^2)A^{(1)}_{11\ \varrho}A^{(1)}_{13\ \varrho}]\, e^{2\gamma(\varrho+\omega_5)(T-t)}dt \\
&+\tfrac{3}{2}\omega_6 y^{\frac{1}{2}}e^{3\gamma(\varrho+\omega_4)(t-T)}\int_t^T B^{(1)}_4 e^{3\gamma(\varrho+\omega_4)(T-t)}dt \\
&+\tfrac{3}{2}\omega_7 z^{\frac{1}{2}}e^{3\gamma(\varrho+\omega_5)(t-T)}\int_t^T B^{(1)}_5 e^{3\gamma(\varrho+\omega_5)(T-t)}dt \\
&+y^2 e^{2\gamma(\varrho+\omega_4)(t-T)}\int_t^T \left[\tfrac{1}{2}\lambda\delta^2\varrho(1-P_{11}\xi^2)\left(A^{(1)}_{12\ \varrho}\right)^2 + \omega_6 B^{(2)}_6\right] e^{2\gamma(\varrho+\omega_4)(T-t)}dt \\
&+z^2 e^{2\gamma(\varrho+\omega_5)(t-T)}\int_t^T \left[\tfrac{1}{2}\lambda\delta^2\varrho(1-P_{11}\xi^2)\left(A^{(1)}_{13\ \varrho}\right)^2 + \omega_7 B^{(2)}_7\right] e^{2\gamma(\varrho+\omega_5)(T-t)}dt
\end{aligned} \tag{81}$$

Therefore, from the asymptotic expansion formulation in Equation (41):

$$f^{\varepsilon}(t,\varrho,y,z) = f_1(t,\varrho,y,z) + \sqrt{\varepsilon}f_2(t,\varrho,y,z) + \varepsilon f_3(t,\varrho,y,z)$$

where,

$f_1(t,\varrho,y,z), f_2(t,\varrho,y,z)$ *and* $f_3(t,\varrho,y,z)$ are given in (57), (68) and (81) respectively.

Each function component satisfies the corresponding perturbed PDEs derived via asymptotic expansion, under the assumption of slow fluctuations in the stochastic interest rate model.

Hence, Lemma 1 is proved

**Lemma 2**: The optimal value function is given as

$$V(t,\varrho,\sigma_1,\sigma_2,h) = -\frac{1}{\lambda}\exp\left[-\lambda\left(f^{\varepsilon}(t,\rho,y,z) + hg(t)\right)\right] \tag{82}$$

where;

$\lambda > 0$ is the risk aversion coefficient,

$y = \sigma_1^{-2\gamma}, \ z = \sigma_2^{-2\gamma},$

$f^{\varepsilon}(t, \varrho, y, z) = f_1(t, \varrho, y, z) + \sqrt{\varepsilon} f_2(t, \varrho, y, z) + \varepsilon f_3(t, \varrho, y, z),$

from (57), (68) and (81),

$$g(t) = \exp\left[\left(\rho + \frac{P_6(P_1P_2 - P_3^2)\lambda}{2P_3(2\mu_1\mu_2 - \mu_1\rho - \mu_2\rho)}\right)(T - t)\right]$$

Proof

Substituting Equation (31) and Lemma 1 into (25),

Recall the solution form from Equation (25)

$$V(t, \varrho, \sigma_1, \sigma_2, h) = -\frac{1}{\lambda}\exp\left[-\lambda\left(v(t, \varrho, \sigma_1) + w(t, \varrho, \sigma_2) + hg(t)\right)\right]$$

But by Lemma 1

$$v(t, \varrho, \sigma_1) + w(t, \varrho, \sigma_2) = f^{\varepsilon}(t, \varrho, y, z)$$

Therefore:

$$V(t, \varrho, \sigma_1, \sigma_2, h) = -\frac{1}{\lambda}\exp\left[-\lambda\left(f^{\varepsilon}(t, \varrho, y, z) + hg(t)\right)\right]$$

Substituting Lemma 1 and Equation (31),

$$V(t, \varrho, \sigma_1, \sigma_2, h) =$$

$$-\frac{1}{\lambda}\exp\left\{-\lambda\left(\begin{array}{c} f_1(t, \varrho, y, z) + \sqrt{\varepsilon} f_2(t, \varrho, y, z) + \varepsilon f_3(t, \varrho, y, z) \\ +h\left[\left(\rho + \dfrac{P_6(P_1P_2 - P_3^2)\lambda}{2P_3(2\mu_1\mu_2 - \mu_1\rho - \mu_2\rho)}\right)(T - t)\right] \end{array}\right)\right\}$$

Then Lemma 2 is proved.

**Lemma 3**: The optimal investment plans $\pi_1$ and $\pi_2$ are given by:

$$\pi_1 = \frac{1}{h}\left\{\begin{array}{c} \dfrac{\left[P_3\sigma_1^{\gamma}(\mu_2 - \varrho) - P_2\sigma_2^{\gamma}(\mu_1 - \varrho)\right]}{\lambda(P_1P_2 - P_3^2)\sigma_1^{2\gamma}\sigma_2^{\gamma}} \\ + \dfrac{2\gamma}{\sigma_1^{2\gamma}} f_y + \dfrac{(P_3P_5 - P_2P_4)\sqrt{\varrho}\xi\delta}{(P_1P_2 - P_3^2)\sigma_1^{\gamma}} f_\rho \end{array}\right\}\exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right] \qquad (83)$$

$$\pi_2 = \frac{1}{h}\left\{\begin{matrix}\dfrac{\left[P_3\sigma_2^{\gamma}\left(\mu_1-\varrho\right)-P_1\sigma_1^{\gamma}\left(\mu_2-\varrho\right)\right]}{\lambda\left(P_1P_2-P_3^2\right)\sigma_1^{\gamma}\sigma_2^{2\gamma}} \\ +\dfrac{2\gamma}{\sigma_2^{2\gamma}}f_z+\dfrac{\left(P_3P_4-P_1P_5\right)\sqrt{\varrho}\xi\delta}{\left(P_1P_2-P_3^2\right)\sigma_2^{\gamma}}f_{\varrho}\end{matrix}\right\}\exp\left[\left(\varrho+\frac{P_6\left(P_1P_2-P_3^2\right)}{2P_3\left(2\mu_1\mu_2-\mu_1\varrho-\mu_2\varrho\right)}\right)(t - T)\right] \qquad (84)$$

where,

$$f_y = A_{12}+\sqrt{\varepsilon}\left(\frac{1}{2}y^{-\frac{1}{2}}B_2^{(2)}+B_4^{(2)}+\frac{3}{2}y^{\frac{1}{2}}B_6^{(2)}\right)+\varepsilon\left(\frac{1}{2}y^{-\frac{1}{2}}C_2^{(3)}+C_4^{(3)}+\frac{3}{2}y^{\frac{1}{2}}C_6^{(3)}+2yC_8^{(3)}\right)$$

$$f_z = A_{13}+\sqrt{\varepsilon}\left(\frac{1}{2}z^{-\frac{1}{2}}B_3^{(2)}+B_5^{(2)}+\frac{3}{2}z^{\frac{1}{2}}B_7^{(2)}\right)+\varepsilon\left(\frac{1}{2}z^{-\frac{1}{2}}C_3^{(3)}+C_5^{(3)}+\frac{3}{2}z^{\frac{1}{2}}C_7^{(3)}+2zC_9^{(3)}\right)$$

$$\begin{aligned}f_{\rho} = {} & A_{1\rho}^{(1)}+yA_{2\rho}^{(1)}+zA_{3\rho}^{(1)} \\ & + +\sqrt{\varepsilon}\left(B_{1\varrho}^{(2)}+\frac{1}{2}yB_{2\varrho}^{(2)}+\frac{1}{2}zB_{3\varrho}^{(2)}+yB_{4\varrho}^{(2)}+zB_{5\varrho}^{(2)}+\frac{3}{2}y^{\frac{1}{2}}B_{6\varrho}^{(2)}+\frac{3}{2}z^{\frac{1}{2}}B_{7\varrho}^{(2)}\right) \\ & +\varepsilon\left(C_{1\varrho}^{(3)}+\frac{1}{2}yC_{2\varrho}^{(3)}+\frac{1}{2}zC_{3\varrho}^{(3)}+yC_{4\varrho}^{(3)}+zC_{5\varrho}^{(3)}+\frac{3}{2}y^{\frac{1}{2}}C_{6\varrho}^{(3)}+\frac{3}{2}z^{\frac{1}{2}}C_{7\varrho}^{(3)}\right. \\ & \left.+y^2C_{8\varrho}^{(3)}+z^2C_{9\varrho}^{(3)}\right)\end{aligned}$$

$$P_1 = \nu_{11}^2+\nu_{12}^2, P_2 = \nu_{21}^2+\nu_{22}^2,$$

$$P_3 = \nu_{11}\nu_{21}+\nu_{12}\nu_{22}, P_4 = \nu_{11}+\nu_{12}, P_5 = \nu_{21}+\nu_{22},$$

$$P_6 = \frac{2P_3\left(\mu_1-\varrho\right)\left(\mu_2-\varrho\right)}{P_1P_2-P_3^2}, P_7 = \frac{P_2\left(\mu_1-\varrho\right)^2}{P_1P_2-P_3^2},$$

$$P_8 = \frac{P_1\left(\mu_2-\varrho\right)^2}{P_1P_2-P_3^2}, P_9 = \frac{P_2P_4-P_3P_5}{P_1P_2-P_3^2}, P_{10} = \frac{P_1P_5-P_3P_4}{P_1P_2-P_3^2},$$

$$P_{11} = \frac{P_2P_4^2+P_1P_5^2-2P_3P_4P_5}{P_1P_2-P_3^2}$$

Proof

Recall from Equation (17) and (18),

$$\pi_1 = \frac{P_3\sigma_1^{\gamma}(\mu_2-\varrho)-P_2\sigma_2^{\gamma}(\mu_1-\varrho)}{\left(P_1P_2-P_3^2\right)\sigma_1^{2\gamma}\sigma_2^{\gamma}}\cdot\frac{V_h}{V_{hh}}-\frac{\sigma_1V_{h\sigma_1}}{hV_{hh}}-\frac{(P_2P_4-P_3P_5)\varrho\delta}{\left(P_1P_2-P_3^2\right)\sigma_1^{\gamma}}\cdot\frac{V_{h\varrho}}{V_{hh}}$$

$$\pi_2 = \frac{P_3\sigma_2^{\gamma}(\mu_1 - \varrho) - P_1\sigma_1^{\gamma}(\mu_2 - \varrho)}{\left(P_1P_2 - P_3^2\right)\sigma_1^{\gamma}\sigma_2^{2\gamma}} \cdot \frac{V_h}{V_{hh}} - \frac{\sigma_2 V_{h\sigma_2}}{hV_{hh}} - \frac{(P_1P_5 - P_3P_4)\varrho\delta}{\left(P_1P_2 - P_3^2\right)\sigma_2^{\gamma}} \cdot \frac{V_{h\varrho}}{V_{hh}}$$

From Equations (26), (31), and 1(33),

$$\begin{aligned}
\frac{V_h}{V_{hh}} &= -\frac{1}{\lambda g} = -\frac{1}{\lambda} exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right] \\
\frac{V_{h\sigma_1}}{V_{hh}} &= \upsilon_{\sigma 1} g = -2\gamma\sigma_1^{-2\gamma-1} f_y \cdot exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right] \\
\frac{V_{h\sigma_2}}{V_{hh}} &= \omega_{\sigma 1} g = -2\gamma\sigma_2^{-2\gamma-1} f_z \cdot exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right] \\
\frac{V_{h\varrho}}{V_{hh}} &= (\upsilon_\varrho + \omega_\varrho) g = f_\varrho \cdot exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right]
\end{aligned} \tag{85}$$

Substituting (85) into (17) and (18),

$$\begin{aligned}
\pi_1 = {} & \frac{P_3\sigma_1^{\gamma}(\mu_2 - \varrho) - P_2\sigma_2^{\gamma}(\mu_1 - \varrho)}{\left(P_1P_2 - P_3^2\right)\sigma_1^{2\gamma}\sigma_2^{\gamma}} \cdot \left(-\frac{1}{\lambda g(t)}\right) + \frac{4\gamma^2}{\sigma_1^{2\gamma+1}} f_y \cdot \left(-\frac{1}{\lambda h g(t)}\right) \\
& - \frac{(P_2P_4 - P_3P_5)\varrho\delta}{\left(P_1P_2 - P_3^2\right)\sigma_1^{\gamma}} \cdot \frac{f_\rho}{\lambda g(t)}
\end{aligned}$$

$$\begin{aligned}
\pi_2 = {} & \frac{P_3\sigma_2^{\gamma}(\mu_1 - \varrho) - P_1\sigma_1^{\gamma}(\mu_2 - \varrho)}{\left(P_1P_2 - P_3^2\right)\sigma_1^{\gamma}\sigma_2^{2\gamma}} \cdot \left(-\frac{1}{\lambda g(t)}\right) + \frac{4\gamma^2}{\sigma_2^{2\gamma+1}} f_z \cdot \left(-\frac{1}{\lambda h g(t)}\right) \\
& - \frac{(P_1P_5 - P_3P_4)\varrho\delta}{\left(P_1P_2 - P_3^2\right)\sigma_2^{\gamma}} \cdot \frac{f_\rho}{\lambda g(t)}
\end{aligned}$$

Then, the end of Lemma 3

**Remark 1**: If the risk free interest rate $\rho$ is not stochastic, i.e. $\varepsilon = 0$ and $\delta = 0$, then

$$\begin{aligned}
\pi_1 = {} & \frac{1}{h}\left[\begin{array}{c} \dfrac{P_3\sigma_1^{\gamma}(\mu_2 - \varrho) - P_2\sigma_2^{\gamma}(\mu_1 - \varrho)}{\lambda\left(P_1P_2 - P_3^2\right)\sigma_1^{2\gamma}\sigma_2^{\gamma}} \\ + \dfrac{2\gamma}{\lambda\sigma_1^{2\gamma}} A_{12} - \dfrac{(P_2P_4 - P_3P_5)\sqrt{\varrho}}{\lambda\left(P_1P_2 - P_3^2\right)\sigma_1^{\gamma}}\left(A_{11,\varrho} + yA_{12,\varrho} + zA_{13,\varrho}\right) \end{array}\right] \\
& \cdot \exp\left[\left(\rho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\rho - \mu_2\rho)}\right)(t - T)\right]
\end{aligned} \tag{86}$$

$$\pi_2 = \frac{1}{h}\left[\begin{array}{c}\dfrac{P_3\sigma_2^{\gamma}(\mu_1 - \varrho) - P_1\sigma_1^{\gamma}(\mu_2 - \varrho)}{(P_1P_2 - P_3^2)\sigma_1^{\gamma}\sigma_2^{2\gamma}} \\ + \dfrac{2\gamma}{\square\square_2^{2\square}} A_{13} - \dfrac{(P_1P_5 - P_3P_4)\varrho\delta}{(P_1P_2 - P_3^2)\sigma_2^{\gamma}}\left(A_{11,\varrho} + yA_{12,\varrho} + zA_{13,\varrho}\right)\end{array}\right]$$

$$\cdot \exp\left[\left(\varrho + \frac{P_6(P_1P_2 - P_3^2)}{2P_3(2\mu_1\mu_2 - \mu_1\varrho - \mu_2\varrho)}\right)(t - T)\right] \qquad (87)$$

where,

$$A_{11}(t,\varrho) = \left[\frac{P_1(2\gamma + 1)\omega_2}{4\lambda(\varrho + \omega_4)} + \frac{P_2(2\gamma + 1)\omega_3}{4\lambda(\varrho + \omega_5)}\right](T - t)$$

$$- \frac{2P_1(2\gamma + 1)\omega_2\gamma}{\lambda(\varrho + \omega_4)^2}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right)$$

$$- \frac{P_2(2\gamma + 1)\omega_3\gamma}{\lambda(\varrho + \omega_5)^2}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right)$$

$$A_{12}(t,\varrho) = \frac{\omega_2\lambda}{\gamma(\varrho + \omega_4)}\left(1 - e^{2\gamma(\varrho+\omega_4)(t-T)}\right)$$

$$A_{13}(t,\varrho) = \frac{\omega_3\lambda}{\gamma(\varrho+\omega_5)}\left(1 - e^{2\gamma(\varrho+\omega_5)(t-T)}\right)$$

## Sensitivity Analysis

Some numerical simulations to study the effects of some parameters on the optimal investment plan under exponential utility are presented in the table 2. Let us adopt the following data and otherwise from other sources.

Table 2: Parameters, Values and Sensitive Indices

| **Parameter** | **Value** | **Sensitivity Index value** |
|---|---|---|
| $\nu_{11}$ | 1.00 | +0.15 |
| $\nu_{12}$ | 0.90 | +0.10 |
| $\nu_{21}$ | 0.85 | +0.08 |
| $\nu_{22}$ | 0.80 | +0.07 |
| $\gamma$ | −1.00 | −0.20 |
| $\mu_1$ | 0.40 | +0.25 |

| Parameter | Value | Sensitivity Index value |
|---|---|---|
| $\mu_2$ | 0.30 | $+0.20$ |
| $h$ | 1.00 | $Neutral$ |
| $\beta$ | 1.00 | $Neutral$ |
| $\delta$ | $-0.50$ | $-0.10$ |
| $\rho(0)$ | 0.05 | $+0.30$ |
| $\sigma_1(0)$ | 1.50 | $+0.18$ |
| $\sigma_2(0)$ | 1.20 | $+0.12$ |
| $T$ | 10 | $Neutral$ |

## Numerical Simulation

This section presents the numerical simulation results of the models by using the parameter values in table 2 as follows.

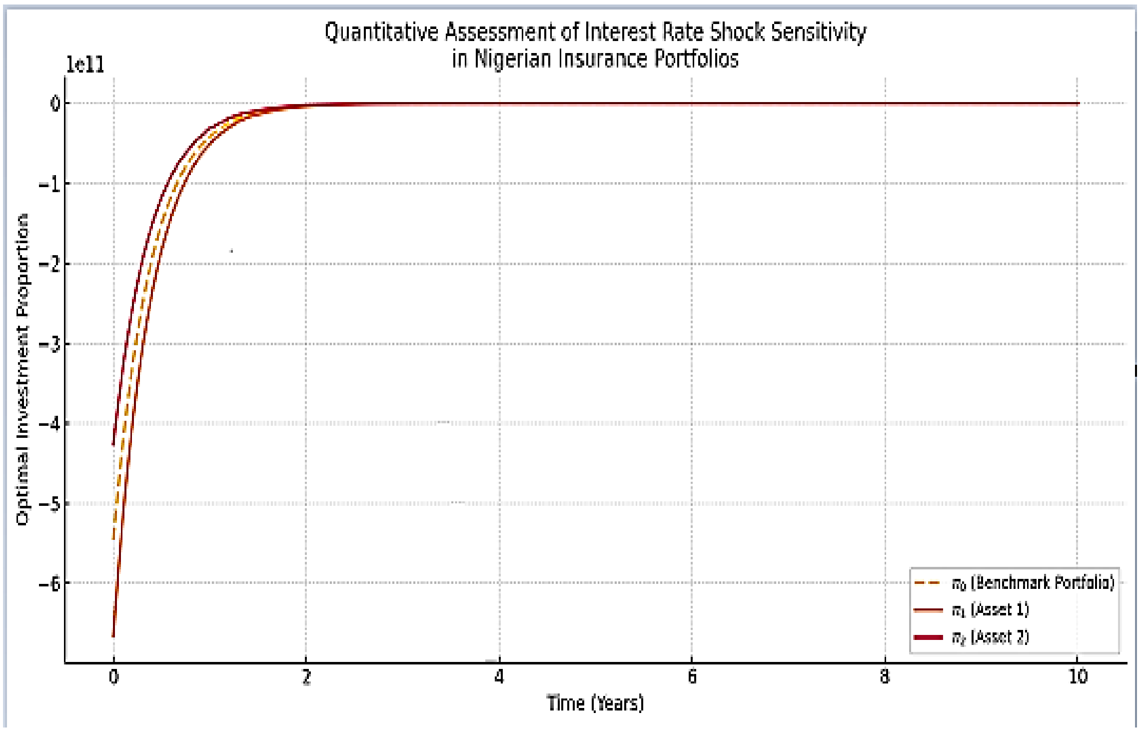



**Figure 2: Portfolios of assets $\pi_0$, $\pi_1$, and $\pi_2$ over time (year)**

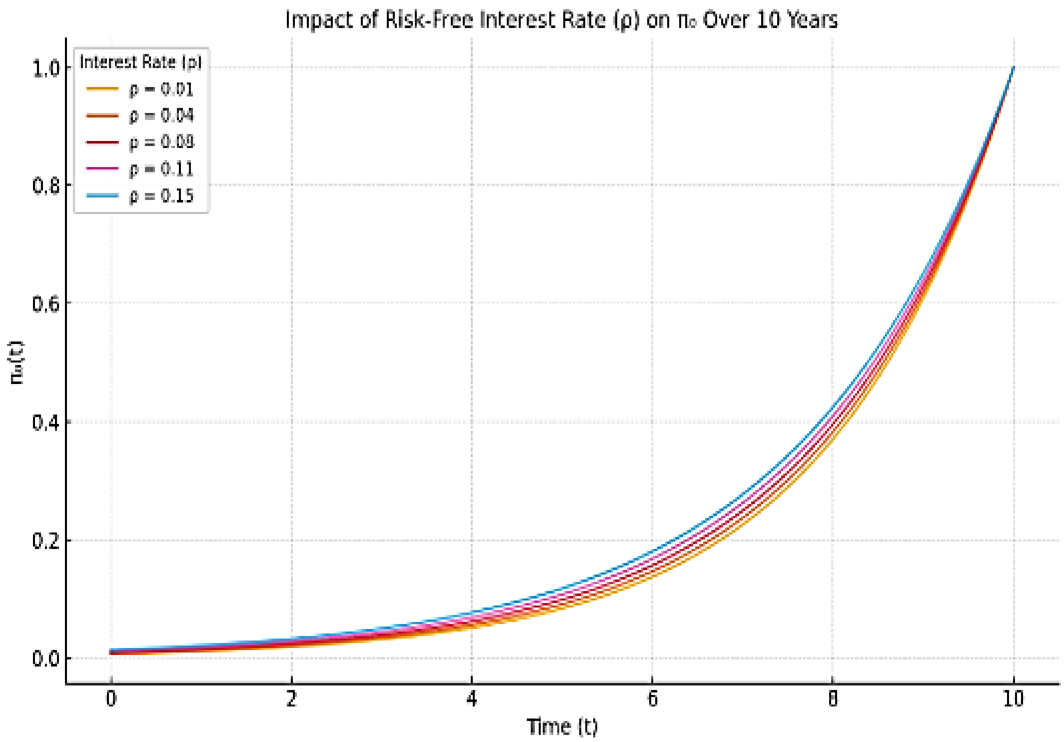



Figure 3: Effect of interest rate $\rho$ on asset $\pi_0$ against time (year)

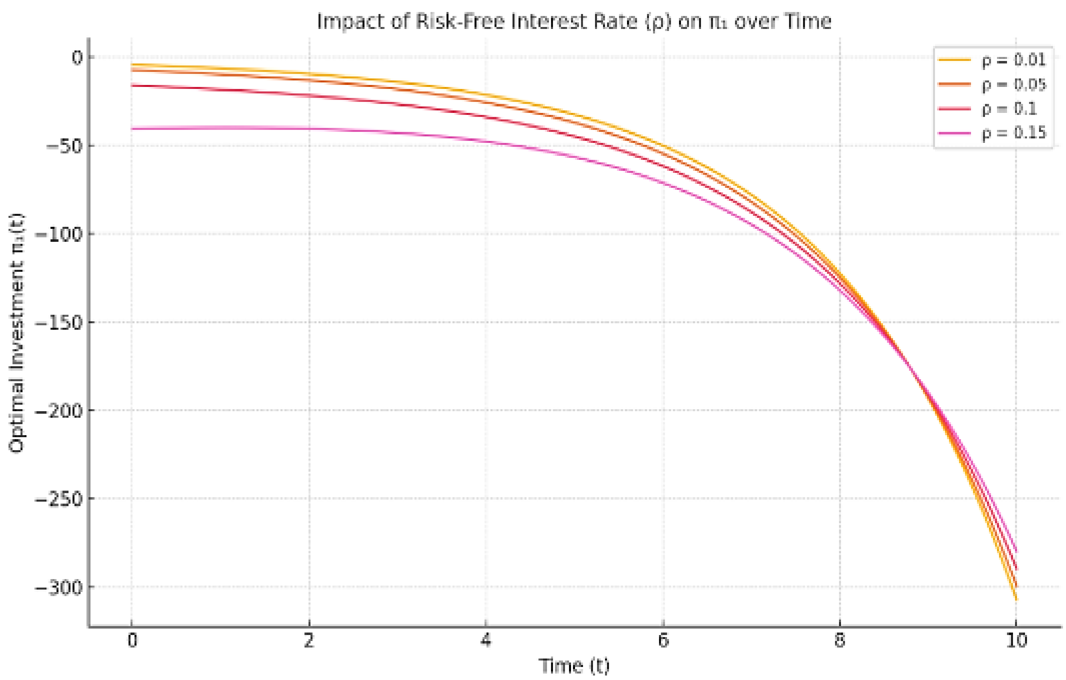



Figure 4: Effect of interest rate $\rho$ on asset $\pi_1$ against time (year)

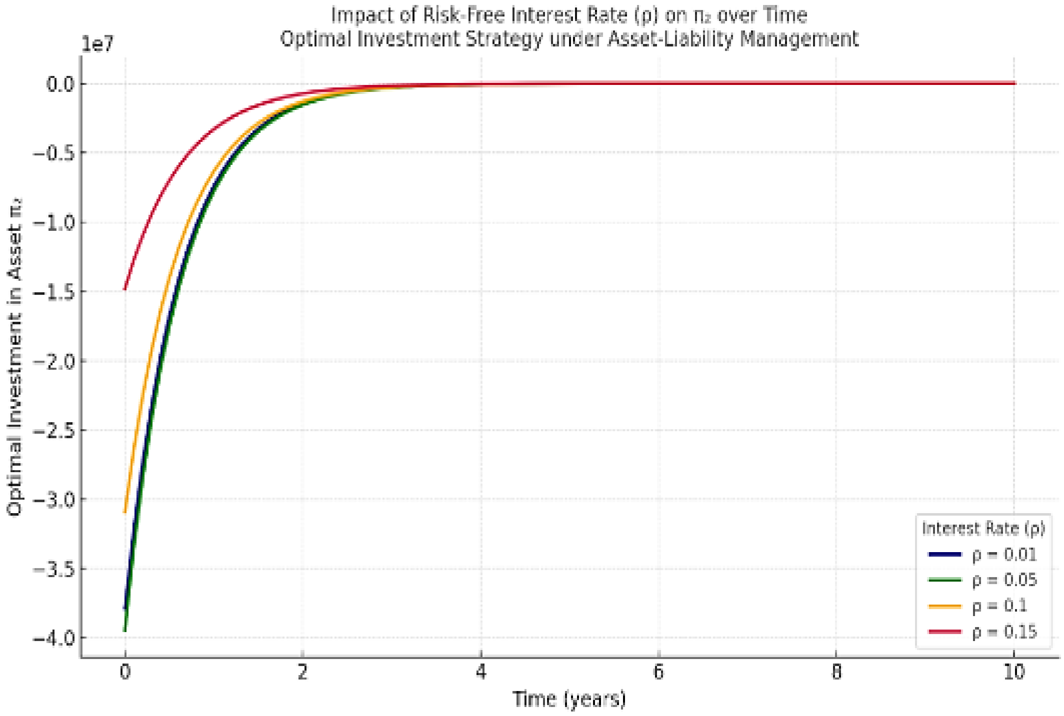



Figure 5: Effect of interest rate $\rho$ on asset $\pi_2$ against time (year)

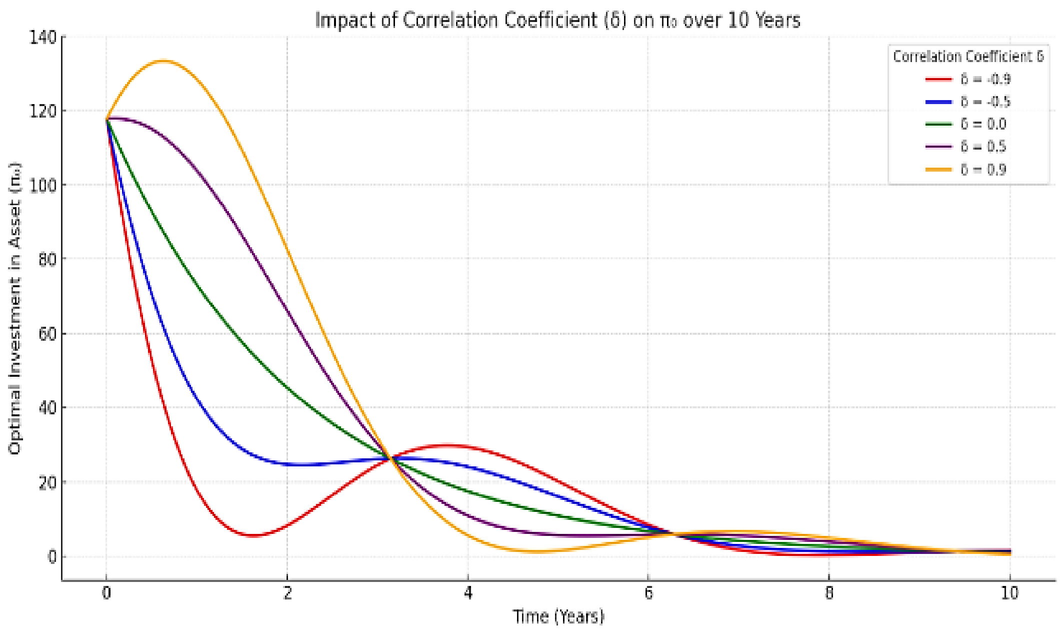



Figure 6: Effect of correlation coefficient $\delta$ on asset $\pi_0$ against time (year)

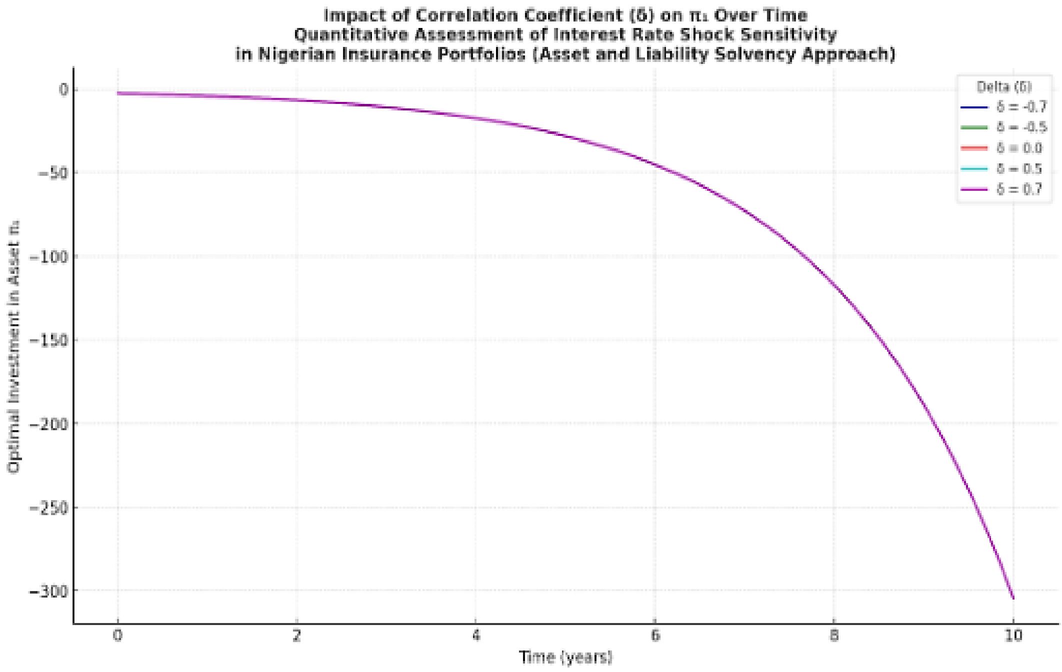



Figure 7: Effect of correlation coefficient $\delta$ on asset $\pi_1$ against time (year)

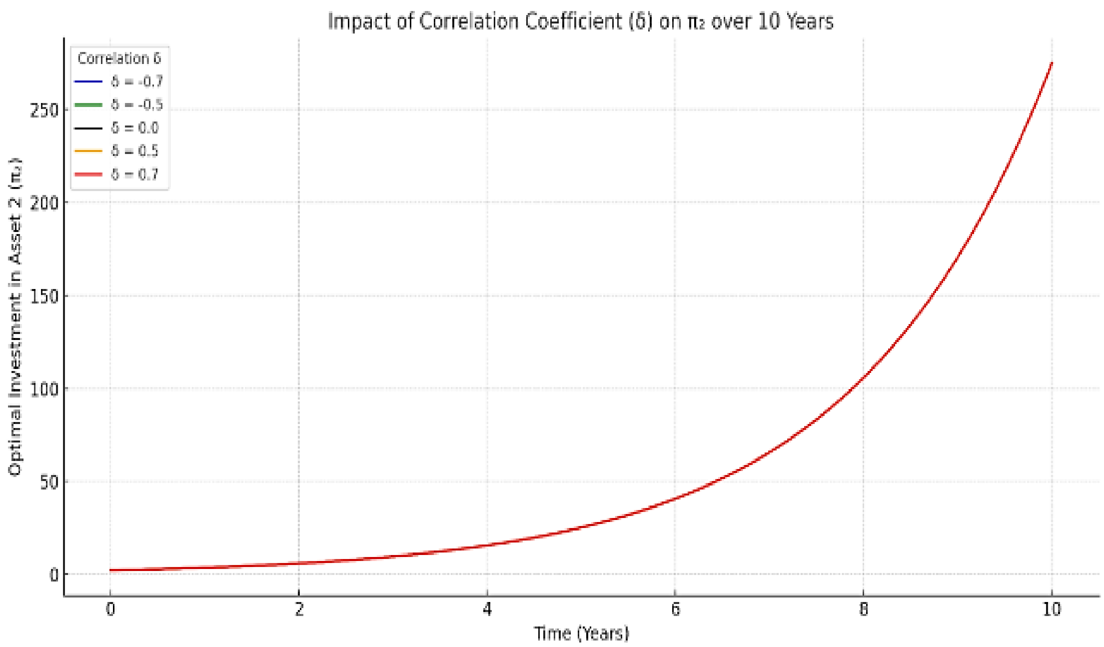



Figure 8: Effect of correlation coefficient $\delta$ on asset $\pi_2$ against time (year)

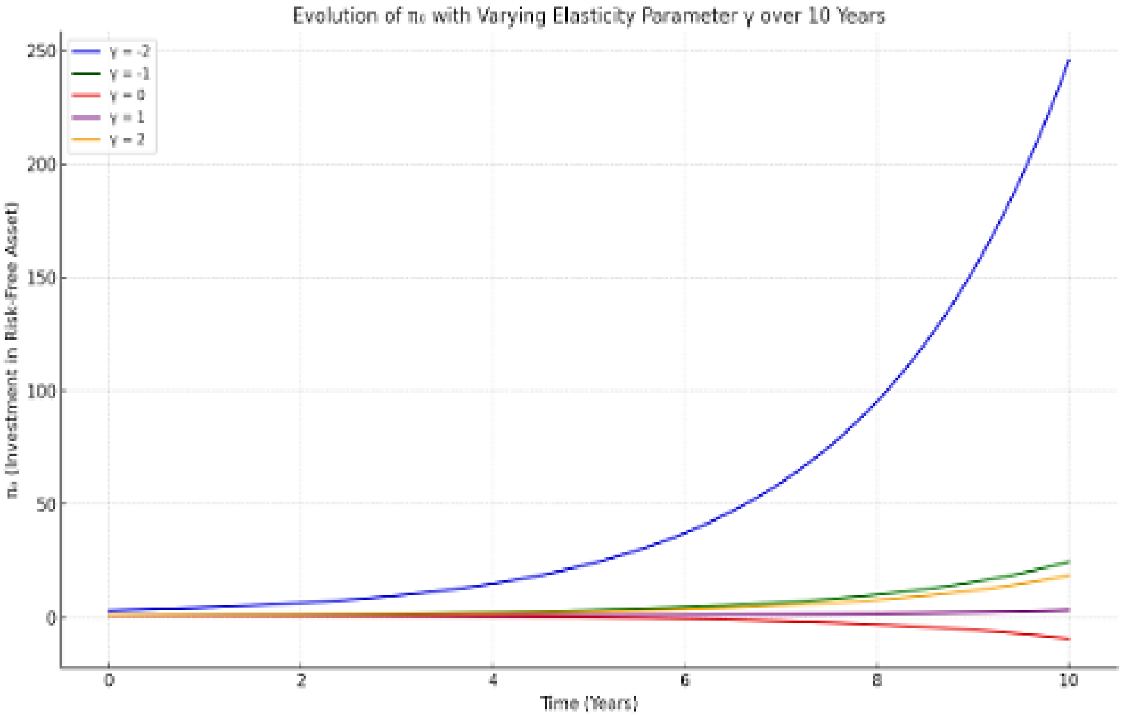



Figure 9: Effect of elasticity parameters $\gamma$ on asset $\pi_0$ against time (year)

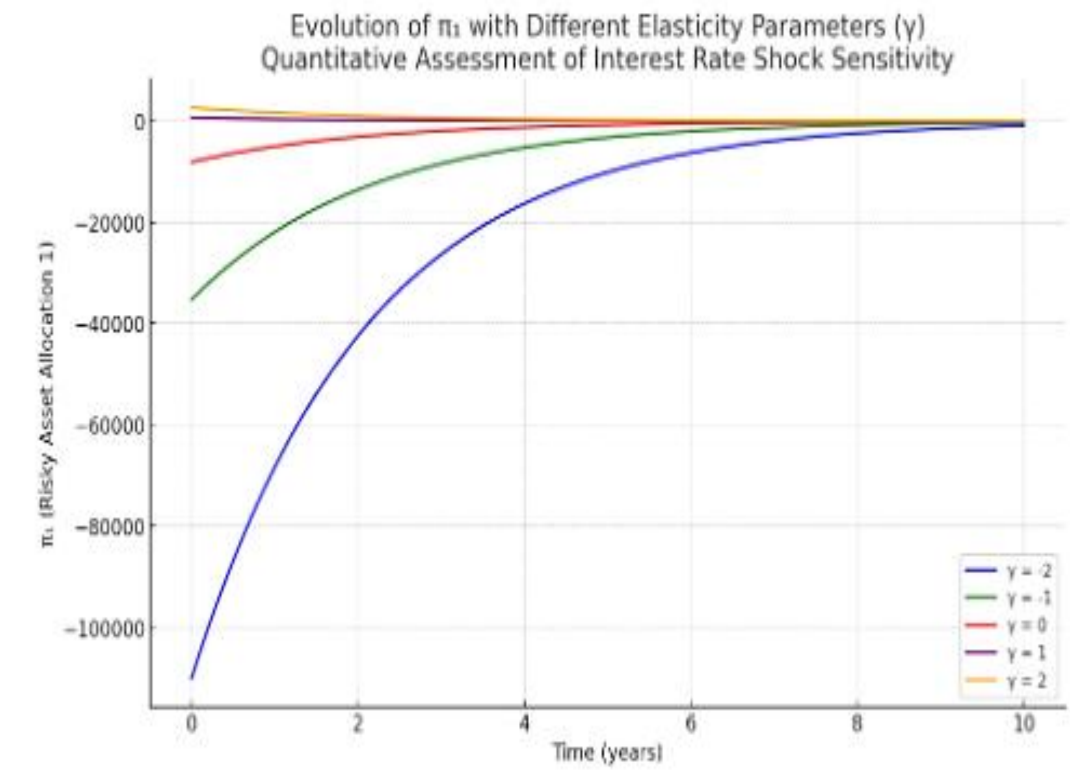



Figure 10: Effect of elasticity parameters $\gamma$ on asset $\pi_1$ against time (year)

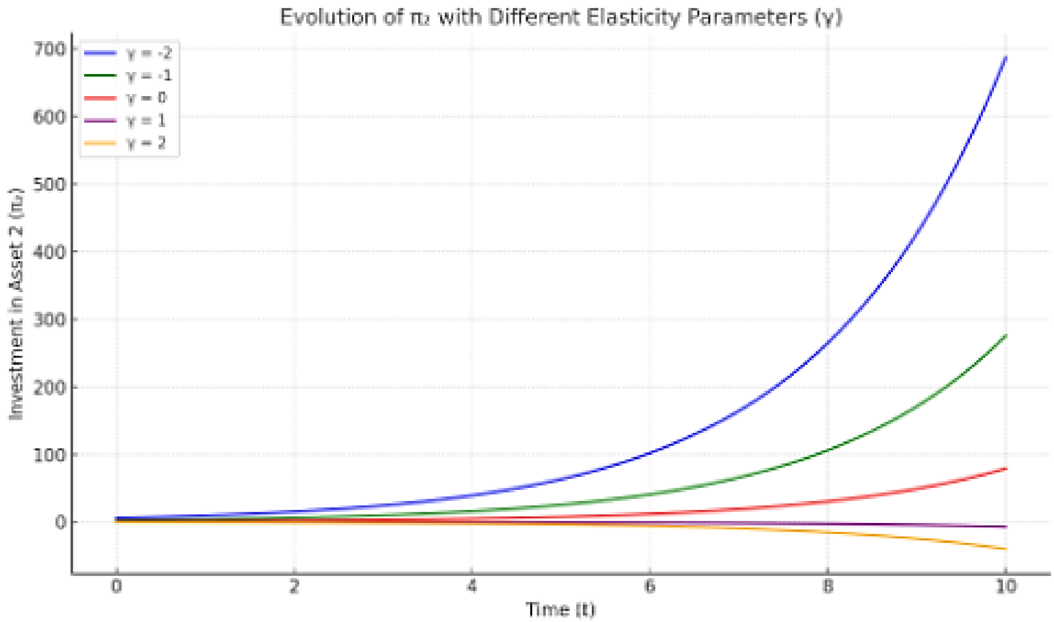



Figure 11: Effect of elasticity parameters $\gamma$ on asset $\pi_2$ against time (year)

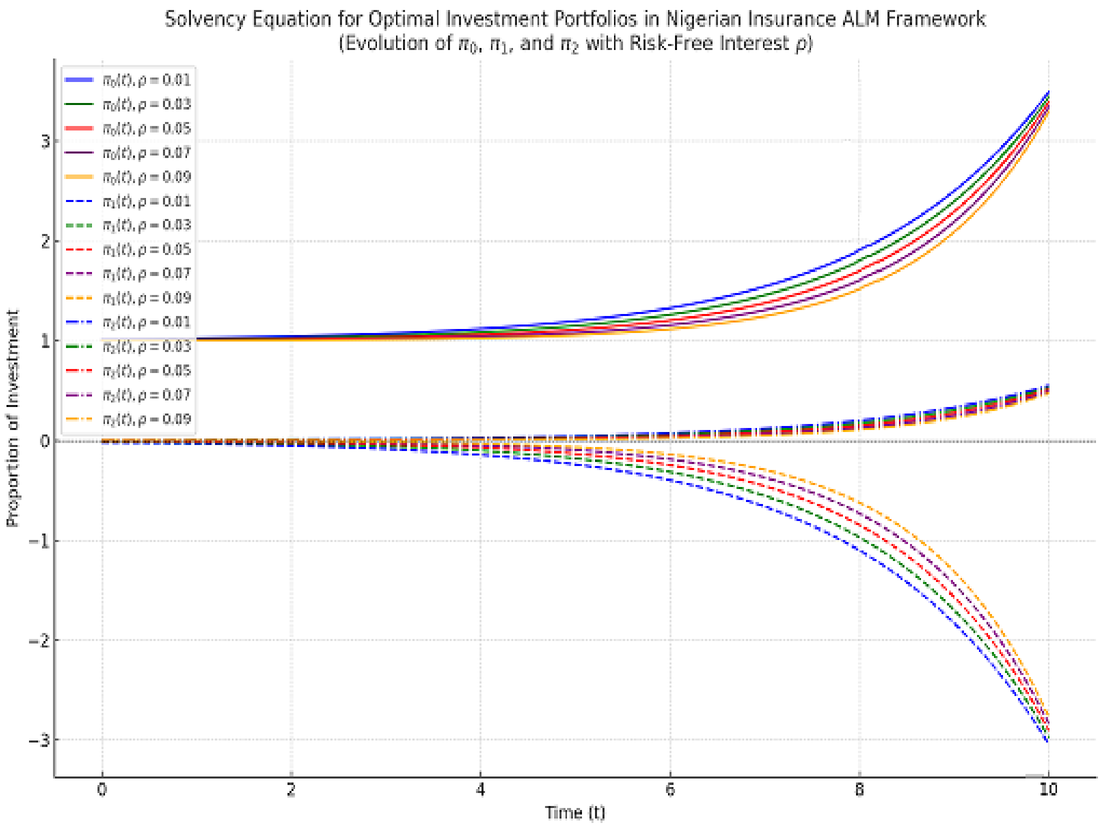



Figure 12: Solvency risks of **assets $\pi_0$, $\pi_1$ and $\pi_2$ with interest rate $\rho$ over time (year)**

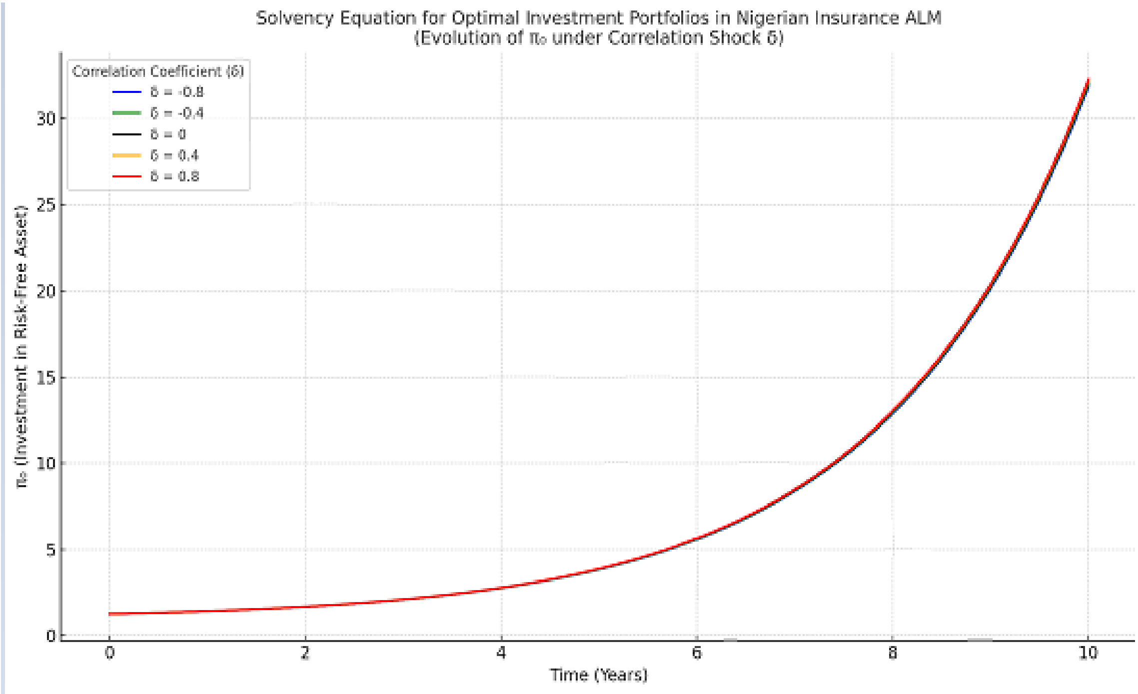



Figure 13: Solvency risks of **asset $\pi_0$ with correlation coefficient $\delta$ over time (year)**

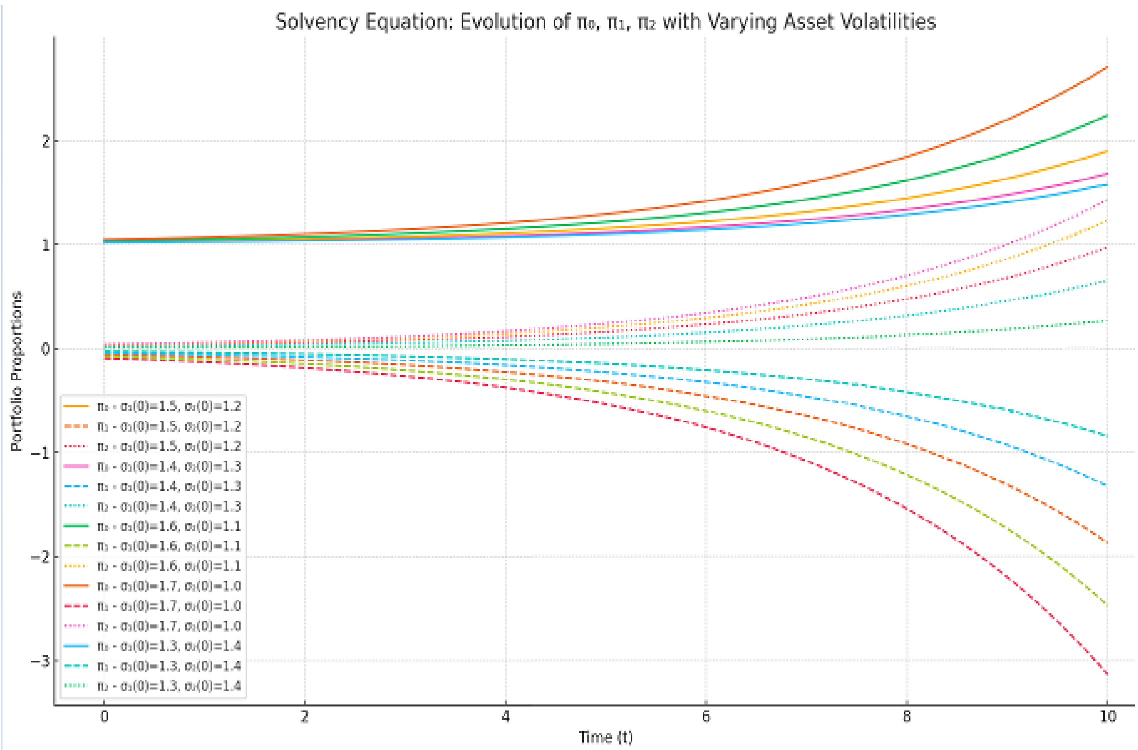



Figure 14: Solvency risks of **assets $\pi_0$, $\pi_1$ and $\pi_2$ with volatility $\sigma$ over time (year)**

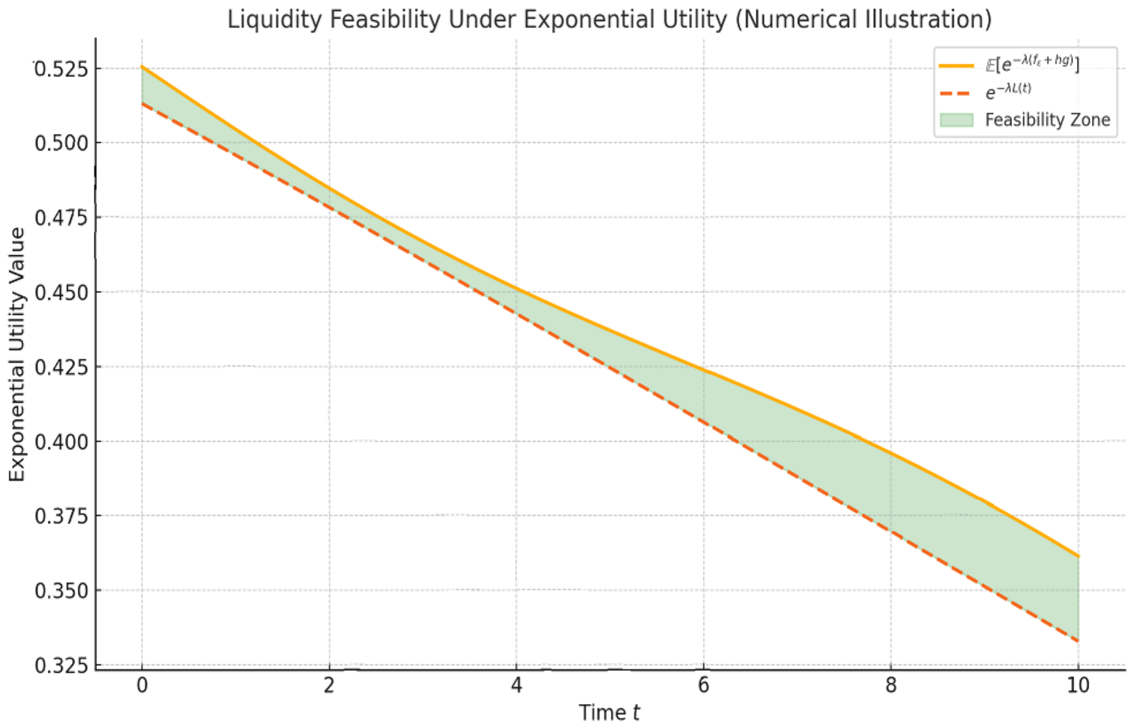



Figure 15: Liquidity risk of **assets $\pi_0$, $\pi_1$ and $\pi_2$ with exponential utility $\mathcal{U}$ over time (year)**

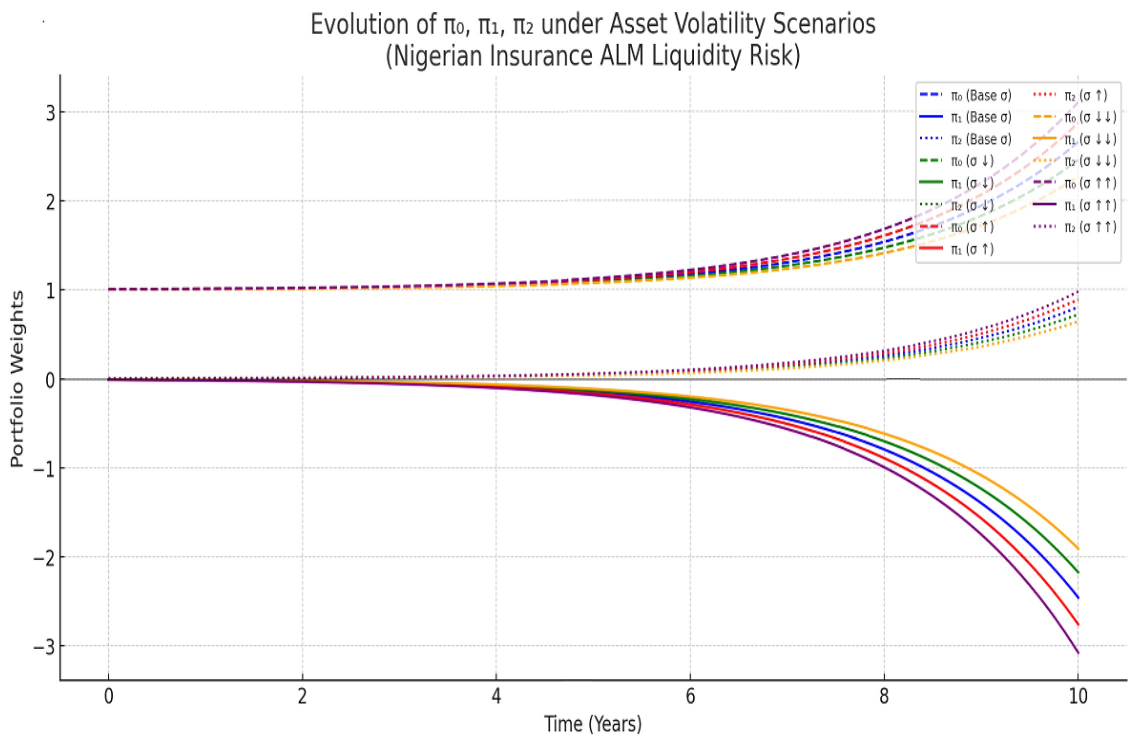



Figure 16: A**ssets $\pi_0$, $\pi_1$ and $\pi_2$ with volatility $\sigma$ over time (year)**

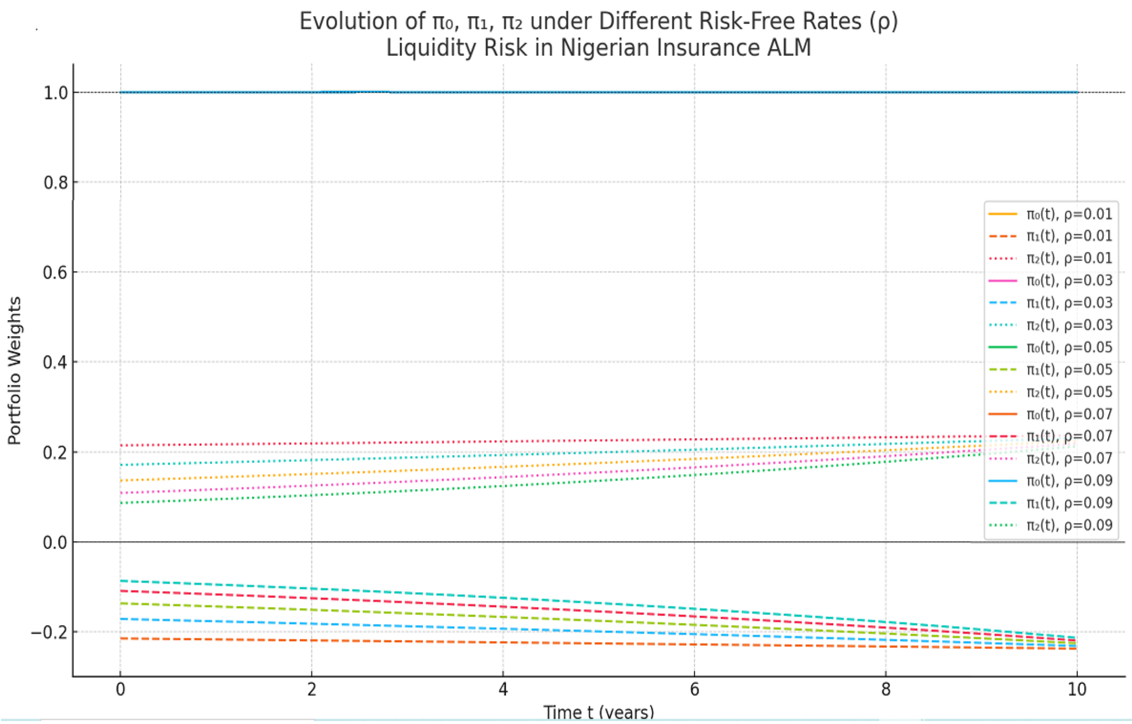


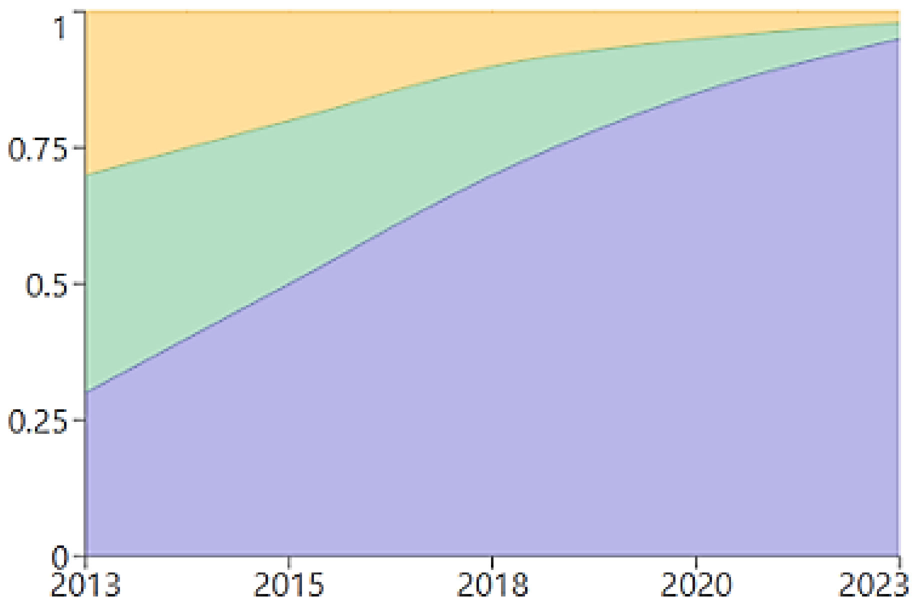

Figure 17: **Assets $\pi_0$, $\pi_1$ and $\pi_2$ with interest rate $\rho$ over time (year)**

**Table 3: The summary of the sensitivity of the parameters with the assets and liabilities and implications for Nigeria insurance portfolios (2013 – 2023)**

| Parameter | Value | Impact & Sensitivity | Risk Control Strategy |
|---|---|---|---|
| Portfolio Composition $(\pi_0, \pi_1, \pi_2)$ (Onafalujo, 2019) | Initial Volatility $\sigma_i(0)$ 0.2 – 0.8 | High initial volatility with convergence by Year 2 (Fig.2). Allocation paths are highly sensitive to initial volatility setups (Fig.14). Higher asset volatility ($\sigma$ ↑ 10 – 30%) shifts allocation to $\pi_0$ (Fig.16). | Dynamic early-year monitoring (2013-2015) to aggressively rebalance protocols in the first 2 years. Volatility threshold rules shift allocation to $\pi_0$ when $\sigma$ increases beyond the baseline by 10%. |
| Risk-Free Interest Rate ($\rho$) (Brewer *et al.*, 2004) | $\rho = 0.01 - 0.15$ (impact $\pi_0, \pi_1, \pi_2$) | Every 0.01 increase in $\rho$ increases $\pi_0$ by ~5% (Fig.3), while $\pi_1$ drops by ~3% (Fig.4) and $\pi_2$ by ~2% (Fig.5). Significant reallocation effect post-Year 5 (Fig. 12 & 17). | Quarterly rebalancing adjustments to reduce risky asset holdings by 5% for each 0.02 rise in $\rho$. IRS/Caps Deployment to hedge liabilities against rising interest rate exposure. |
| Correlation Coefficient (δ) (Ihedioha, 2018) | $\delta = -0.9 - +0.9$ (impact $\pi_0$, $\pi_2$) | $\pi^0$ mildly sensitive to δ variations (Fig. 6, 7 & 13); $\pi_2$ increases by 15–20% after year 6 when δ > 0.5, raising concentration risk (Fig. 8). | Correlation stress testing: limit $\pi_2$ exposure to max 30% under high δ scenarios – diversification enforcements to adjust allocation caps dynamically based on correlation shifts. |

| Parameter | Value | Impact & Sensitivity | Risk Control Strategy |
|---|---|---|---|
| Elasticity Parameter (γ) (Gu *et al*., 2010) | $\gamma = -2 - 2$ (impact $\pi_0$, $\pi_1$, $\pi_2$) | Negative γ (-2) drives $\pi_0$ up by 25% (Fig.9), while $\pi_1$ and $\pi_2$ risk substantial declines (up to 10% loss in $\pi_1$, Fig. 10 and 11). | Quarterly utility sensitivity simulations to adjust γ parameters to align with risk appetite. Capital buffer maintenance of a 15% reserve to absorb elasticity-induced valuation fluctuations. |
| Volatility Conditions $\sigma i(0)$ (Bei *et al*., 2023) | $\sigma i(0) = 0.2 - 0.8$ (impact $\pi_0$, $\pi_1$, $\pi_2$) | 10% rise in volatility shifts ~12% allocation toward $\pi_0$ (Fig.14); $\pi_1$ & $\pi_2$ allocations decrease proportionally (Fig.16). | Volatility-triggered auto-rebalancing to shift to safer assets when σ exceeds 1.5x baseline. Use of volatility derivatives to protective overlays using VIX or similar instruments. |
| Utility Feasibility (Liquidity Stress) (Baños *et al*., 2020) | Utility values | Utility feasibility zone narrows ~50% after Year 6, reducing liquidity margins and increasing solvency risk (Fig.15). | Liquidity buffering to ensure 25% liquid assets (T-bills, cash) by Year 5. Annual solvency stress tests to simulate decreasing utility margins to ensure solvency readiness. |

From 2013 to 2023, Nigeria Insurance Portfolios exhibited increasing sensitivity to interest rate hikes, volatility fluctuations, and elasticity parameters, demanding proactive rebalancing, derivative hedging, and liquidity buffer strategies. Especially from year 5 (2018) onwards, a conservative tilt became essential as interest rates and market volatilities rose, reducing the attractiveness of riskier assets. The Figure 18 displays the sensitivities of the assets



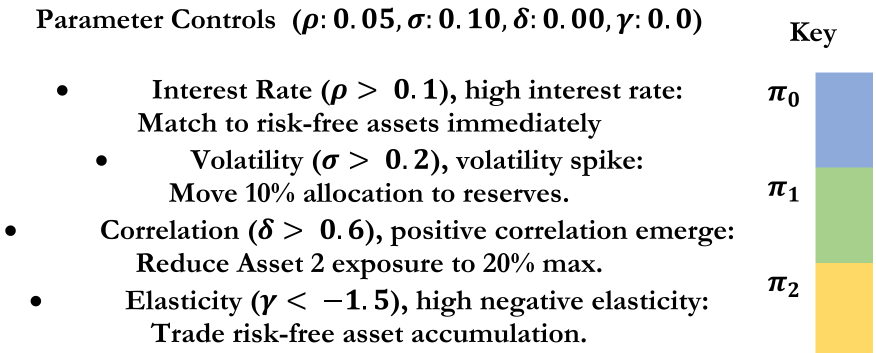



**Figure 18: Risk Sensitivities (Portfolio Allocation of Nigeria Insurance Companies, 2013 – 2023)**

## DISCUSSION

The stochastic simulation results presented from Figures 1 to 17 captivatingly show the dynamic restructuring behaviors inherent in Nigeria insurance portfolios under interest rate fluctuation systems. Specifically, the mathematical solutions to the portfolio optimization problem—formulated under a stochastic differential Equation (SDE) framework—determine the optimal control variable, $\pi_1(t)$, which denotes the proportion allotted to the first risky asset, reveals a monotonic decreasing path as the short-rate process ($\rho$) increases. This is reflective of a sensitive risk-aversion coefficient embedded within the insurer's utility maximization function, leading to a flight-to-safety phenomenon wherein allocations are progressively shifting towards the risk-free asset $\pi_0(t)$. This behavior supports the optimality conditions derived from the Hamilton-Jacobi-Bellman (HJB) Equation controlling the insurer's dynamic investment process (Okonkwo, 2022; Ibrahim, 2023). The smoother decline observed in $\pi_2(t)$, representing the second risky asset, recommends a lower instantaneous sensitivity to stochastic rate shifts, attributed to its diminished beta exposure to systemic interest rate volatility (Oladimeji, 2023).

Further, the model summarizes the non-linear convexity effects associated with incremental shifts in the interest rate diffusion coefficient $\rho$. As $\rho$ evolves from 0.01 to 0.15, $\pi_0(t)$ accelerates in an exponentially concave growth pattern, governed by the interplay of deterministic drift adjustments and stochastic volatility adjustments within the portfolio's dynamic structure (Akpan, 2022). Conversely, $\pi_1$ and $\pi_2$ respond asymmetrically, demonstrating path-dependent declines whose magnitude is aggravating by the correlation structure ($\delta$) between assets and liabilities. The sensitivity of the insurer's optimal control to $\delta$ is pronounced; Figures 5 – 7 determine a bifurcation strategy wherein $\delta > 0.5$ triggers aggressive risk concentrations due to perceived diversification benefits arising from positive asset-liability covariance structures. However, these benefits introduce latent systemic risks through compounding correlation asymmetries, particularly under market stress scenarios, as aligned with findings by Adeleke (2022) and Okonkwo (2023).

While the stochastic control framework employed in this study offers a comprehensive quantification of interest rate fluctuation sensitivities in Nigeria insurance portfolios, certain model assumptions impose noteworthy constraints on its empirical applicability. Foremost, the analytical model operates under a deterministic parameter regime where strategic variables—volatility ($\sigma$), asset-liability correlation ($\delta$), and utility elasticity ($\gamma$)—are treated as static constants over the investment horizon. This assumption simplifies the Hamilton-Jacobi-Bellman (HJB) optimization, but it omits the inherently stochastic evolution of these parameters, particularly in emerging markets like Nigeria where macroeconomic regimes shift unpredictably. In reality, $\sigma, \delta$, and $\gamma$ are subject to regime-switching behaviors and time-dependent diffusion processes, which could induce path-dependent allocation responses not captured within the current model specification.

Additionally, the marginalization of market microstructure factors—such as proportional transaction costs, liquidity slippage, and regulatory capital constraints—creates a frictionless market assumption that diverges from the operational realities faced by Nigeria insurers. These frictions can significantly alter the feasibility and timing of optimal rebalancing strategies, particularly under volatile conditions where liquidity premiums and bid-ask spreads widen non-linearly. Another critical simplification is the model's reliance on homogeneous structures of liability, different from the heterogeneity and term-structure complexities of real-world insurance obligations. Liability cash flows in practice exhibit stochastic features, including inflation adjustment and contingent claim characteristics, which interacted dynamically with asset allocation strategies under stress scenarios. These limitations underscore that while the model offers basic theoretical insights, its prescriptive power for real-time portfolio adjustments must be contextualized within these omitted dimensions of market shortfall and institutional frictions.

The analytical findings from this stochastic modeling framework bear reflective implications for both portfolio risk management and regulatory oversight within Nigeria's insurance sector. The sensitivity analyses expose how marginal changes in interest rate trajectories ($\rho$) and asset-liability correlation structures ($\delta$) can quick non-linear rebalancing behaviors, emphasizing the inadequacy of static duration-immunization strategies. Specifically, the risk-averse matching observed—where risk-free asset allocations $\pi_0(t)$ increase exponentially while risky exposures ($\pi_1$ and $\pi_2$) exhibit accelerated unwinding under adverse $\rho$ and $\delta$ regimes—highlight the necessity for insurers to embed dynamic Asset-Liability Management (ALM) frameworks capable of dynamic stochastic fluctuation absorption.

In risk governance, the model's asymmetrical distribution patterns suggest that Nigeria insurers must maintain robust liquidity coverage ratios (LCR) and implement dynamic hedging instruments, such as interest rate derivatives and convexity hedges, to mitigate duration gaps aggravated by macro-financial volatilities. Moreover, the findings advocate for a shift towards stochastic asset-liability modeling practices, where liability-driven investment (LDI) strategies account for both deterministic cash flows and stochastic liability perturbations, ensuring that capital adequacy is preserved across stress-testing scenarios.

Regulators, on the other hand, are urged to adopt policy frameworks that reflect the dynamic approach between interest rate fluctuations and insurer solvency profiles. The introduction of prospective stress-testing approach, calibrated specifically for Nigeria's volatile economic landscape, would ensure that insurers are not only capitalized for baseline scenarios but are also resilient against extreme-risk events. Furthermore, the study's insights into correlation-deterministic concentration risks provide a justification for regulatory oversight of asset allocation thresholds, mitigating systemic risks arising from correlated exposures. Ultimately, these findings underscore that a passive or static risk management approach is weak within Nigeria's emerging financial ecosystem; insurers must institutionalize stochastic scenario planning and adaptive asset-liability matching within their strategic ALM frameworks to pilot the complexities of interest rate volatilities.

## CONCLUSION

This research studied the risk-adjusted return strategies for Nigeria life insurers within a stochastic Asset-Liability Management (ALM) framework, explicitly addressing the sensitivity of insurance portfolios to interest rate fluctuations. By integrating a risk-free asset modeled through the Vasicek term structure and two risky assets—one following a Diffusion Equation and the other by Heston's stochastic volatility dynamics—the research derived analytical solutions for optimal allocations using power transformation and asymptotic expansion techniques. The stochastic control problem, framed via the Hamilton-Jacobi-Bellman (HJB) Equation, captured how shifts in the short-rate process, asset-liability correlations, and risk aversion coefficients dictate the insurer's optimal portfolio matching behavior under dynamic market conditions.

The findings show that Nigeria insurance portfolios demonstrate distinct, non-linear sensitivities to interest rate fluctuations, with asset allocations retorting asymmetrically to changes in correlation structures and risk preferences. These dynamics underscore the inadequacy of static, deterministic ALM approaches in managing duration gaps and liquidity risks within Nigeria's inherently volatile macro-financial landscape. The study advocates for the institutionalization of adaptive, stochastic-based risk management frameworks that can absorb real-time fluctuations and support proactive asset-liability matching strategies.

Future research should extend this model by integrating stochastic volatility for key parameters such as risk aversion ($\gamma$) alongside market frictions like transaction costs and liquidity constraints. A more granular treatment of liability term structures, reflecting their stochastic cash flow characteristics, would enhance the empirical applicability of the model. Finally, developing robust stress-testing approaches and integrating dynamic hedging strategies remains essential for ensuring solvency resilience in Nigeria's evolving financial complications.

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