Mikailalsys Journal of Mathematics and Statistics

e-ISSN : 3030-816X
p-ISSN : 3030-8399

Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJMS.v3i3.6134

On the Connectivity and Eulerian Properties of Cayley Digraphs in Transformation Monoids

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Apr 24, 2025	May 22, 2025	Jun 3, 2025	Jun 8, 2025

Abstract

This paper establishes fundamental connections between the graph-theoretic properties of Cayley digraphs and the algebraic structure of transformation monoids. Our main contributions include a complete characterization of strong connectivity in transformation monoids, proving that for a transformation monoid T acting on a finite set X, the Cayley digraph Cay(T, S) with respect to a generating set $S \subseteq T$ is strongly connected if and only if T contains the full symmetric group S(X); and a classification of Eulerian properties in symmetric groups, demonstrating that for the symmetric group S_n with any generating set S, the Cayley digraph Cay(S_n, S) is Eulerian precisely when it is strongly connected. We provide concrete examples illustrating these theorems, including detailed Cayley graph constructions for S_3 with explicit generating sets. Our results reveal deep connections between monoid theory and graph theory, showing how algebraic properties manifest in combinatorial structures. The proofs employ techniques from semigroup theory, algebraic graph theory, and finite group theory, offering new insights into the representation of transformation monoids through their generator-dependent digraphs. This work contributes to the broader understanding of how algebraic structures can be studied through their associated graphs.

Keywords: Cayley Digraph; Strong Connectivity; Transformation Monoid; Symmetric Group; Algebraic Graph Theory

Introduction

Let S_Q be a semigroup with a subset A. The Cayley graph Cay(S_Q, A) of S_Q using the connection set A is a directed graph where each element of S_Q is a vertex and there is an edge between two vertices x and y if there exists an element a in A such that $y = xa$ [12]. The properties of Cayley graphs for semigroups have been extensively explored by multiple researchers (see [1], [3], [5], [4], [6]). In particular, in their work [4], A. V. Kelarev and C. E. Praeger focused on the vertex-transitive Cayley graphs of semigroups and demonstrated that if S_Q is a semigroup where A is a completely simple subset and $S_Q A = S_Q$, then every connected component of Cay(S_Q, A) is strongly connected. Subsequently, in [7], Y. Lue et al. provided a comprehensive description of all strongly connected bipartite Cayley graphs for completely simple semigroups. More recently, in [8], T. Suksumran and S. Panma investigated the connectedness of Cayley graphs for semigroups and presented necessary and sufficient conditions for a Cayley graph to be strongly connected or weakly connected. The focus of this paper is to explore the connection between the Cayley digraph of a transformation semigroup and its algebraic attributes.

Preliminaries

Basic Concepts

Following Howie (1995), we define a semigroup S_Q as a non-empty set equipped with an associative binary operation, typically represented by juxtaposition. This means for any elements $a, b, c \in S_Q$, the equality $(ab)c = a(bc)$ holds. When S_Q contains an identity element 1 satisfying $x1 = 1x = x$ for all $x \in S_Q$, it is called a monoid. Similarly, if there exists an element $0 \in S_Q$ such that $0x = x0 = 0$ for all $x \in S_Q$, we call 0 a zero element and S_Q a semigroup with zero.

A subsemigroup T of S_Q is a non-empty subset closed under the binary operation, meaning $ab \in T$ for all $a, b \in T$ (or equivalently, $T^2 \subseteq T$). For instance, in a semigroup with zero, the set of all elements whose products equal zero forms a subsemigroup. A transformation semigroup consists of transformations (functions $f: X \to X$) closed under composition, where $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Such a semigroup is called connected if any two transformations can be linked by a sequence of compositions within the semigroup.

In the context of finite sets, we consider $X_n = \{1, 2, ..., n\}$ and define P_n as the semigroup of all partial transformations on X_n, while T_n denotes the full transformation semigroup. Graph-theoretically, a graph $\Gamma = (V, E)$ consists of vertices V and edges E (2-element subsets of V). Important concepts include walks (sequences of alternating vertices and edges), trails (walks with distinct edges), paths (walks with distinct vertices), and cycles (closed paths). For directed graphs (digraphs), edges are ordered pairs, and an Eulerian digraph is one where every arc appears exactly once in a traversal. The Cayley digraph $Cay\ (G_a, S_\varsigma)$ of a group G_a with generating set S_ς has vertices corresponding to group elements and edges representing right multiplication by generators.

Results

Theorem 3.1. *Let T be a transformation monoid acting on a finite set X. The Cayley digraph $\Gamma = \mathrm{Cay}(T, S)$ with respect to a generating set $S \subseteq T$ is strongly connected if and only if T contains the symmetric group $\mathrm{S}(X)$.*

Proof. (⇒) Suppose Cay (T, S) is strongly connected. Then for any $u, v \in X$, there exists a sequence $s_1, \ldots, s_k \in S$ such that $s_k \cdots s_1(u) = v$. This implies T can realize any transposition $(u\ v) \in \mathrm{S}(X)$. Since transpositions generate $\mathrm{S}(X)$, we have $T \supseteq \mathrm{S}(X)$.

(⇐) If $T \supseteq \mathrm{S}(X)$, take any generating set S of T containing generators of $\mathrm{S}(X)$.

For any $u, v \in X$, there exists $\sigma \in S^*$ (a product of generators) with $\sigma(u) = v$, making Cay (T, S) strongly connected. □

Example 3.2. *Consider $X = \{1, 2, 3\}$ and the following transformation monoids:*

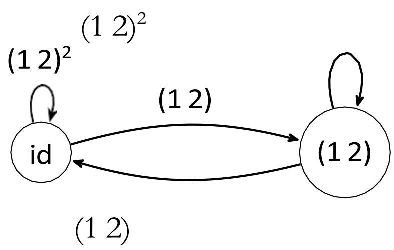
1. ***Full transformation monoid*** T_3*: Contains all transformations* $X \to X$*, including the symmetric group* S_3*. With generating set* $S = \{(1\ 2), (1\ 2\ 3), c_1\}$ *where:*
   $(1\ 2)$ *is a transposition*

   $(1\ 2\ 3)$ *is a 3-cycle*

   c_1 *is the constant map to 1*

   The Cayley digraph $\mathrm{Cay}(\mathrm{T}_3, S)$ *is strongly connected, confirming the "if" direction of Theorem 3.1.*
2. ***Proper submonoid*** $T = \langle (1\ 2), c_1 \rangle$*: Contains only permutations of* $\{1, 2\}$ *and constant maps. The Cayley digraph* $\mathrm{Cay}\ (T, S)$*, where* $S = \{(1\ 2), c_1\}$*, is* not *strongly connected (e.g., no path from 3 to 1), illustrating the" only if" direction since* T *doesn't contain* S_3*.*

Example 3.3. *Consider* $X = \{1, 2\}$ *with* $T = \mathrm{S}_X = \{\mathrm{id}, (1\ 2)\}$ *and generating set* $S = \{(1\ 2)\}$*. The Cayley digraph is:*



This graph is strongly connected since:

From id *we can reach* $(1\ 2)$ *via the generator*

From $(1\ 2)$ *we can return to* id *via the same generator*

Both elements have self-loops via $(1\ 2)^2 = \mathrm{id}$

This illustrates Theorem 3.1 since $T = \mathrm{S}_X$*.*

Theorem 3.4. *Let* S_n *be the symmetric group on n elements. For any generating set* $S \subseteq S_n$*, the Cayley digraph* $D = \mathrm{Cay}(S_n, S)$ *is Eulerian if and only if it is strongly connected.*

Proof. (⇒) Assume D is Eulerian. By definition:

1. The digraph is strongly connected (as Eulerian digraphs must have a single strongly connected component containing all edges)

2. Every vertex has $\deg^+(v) = \deg^-(v)$

   The vertex-transitivity of Cayley digraphs guarantees all vertices have identical degree properties, maintaining strong connectivity throughout.

($\Leftarrow$) Suppose D is strongly connected. Then:

Since S_n is a finite group, D is regular (each vertex has $\deg_+(v) = |S| = \deg_-(v)$)

Strong connectivity ensures the existence of directed trails covering all edges

The regularity condition guarantees the balanced degree condition required for Eulerian cycles

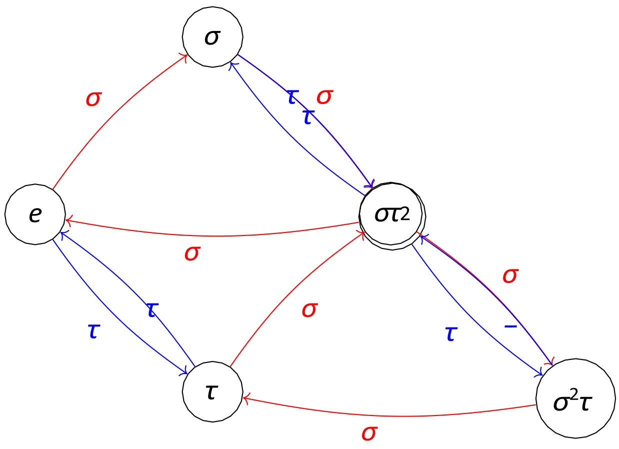
Thus, D satisfies both conditions for being Eulerian. $\square$

Example 3.5. *Consider* $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^2 = e \rangle$ *with:*

$\sigma = (1\ 2\ 3)$ *(3-cycle)*

$\tau = (1\ 2)$ *(transposition)*

The Cayley digraph $\mathrm{Cay}(S_3, \{\sigma, \tau\})$ *is:*



This digraph satisfies both conditions of Theorem 3.4:

Strongly connected: *Any permutation can reach any other via generators (e.g., path* $e \xrightarrow{\sigma} \sigma \xrightarrow{\tau} \sigma\tau$*)*

Eulerian: *Each vertex has* $\deg_-(v) = \deg_+(v) = 2$ *(one σ and one τ edge in each direction)*

Conclusion

In this research, we have demonstrated the important role that the Cayley digraph plays in the structure of a transformation monoid. Our results provide a useful criterion for identifying when a transformation monoid contains the symmetric group and characterize the Eulerian property of the Cayley digraph of the symmetric group. We hope that our work will inspire further investigation into the connection between the Cayley digraph and the algebraic properties of transformation monoids.

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