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A Novel Probability Distribution: Mathematical Derivation and Validation of the Poisson Hamza Model

Bamigbala Olateju Alao[1], Magaji Umar Alhaji[2],
Fadimatu Mohammed Bawuro[3], Gambo Innga Bature[4]
[1,4]Federal University Wukari, Taraba State, Nigeria; [2]Jigawa State College of Education Gumel, Nigeria; [3]Adamawa State Polytechnic, Yola, Nigeria
bamigbala@fuwukari.edu.ng

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Abstract

This study introduces the Poisson Hamza Distribution (PHD), a novel probability distribution developed from the classical Poisson framework to address limitations in modeling count data. While the Poisson distribution is a standard tool for modeling rare events, its inherent assumptions, particularly equidispersion, limit its applicability in complex, real-world contexts. The PHD introduces enhanced modeling flexibility by accommodating overdispersion, thereby extending the utility of Poisson-based models. A comprehensive mathematical formulation of the PHD is presented, along with derivations of its key statistical properties, including moments, variance, standard deviation, skewness, and kurtosis. Theoretical validation is supported by empirical analysis, demonstrating the distribution's robustness and practical relevance. These contributions offer a valuable extension to existing statistical methodologies and provide researchers and practitioners with an alternative model for analyzing overdispersed count data.

Keywords: Poisson Hamza Distribution; Count Data Modeling; Overdispersion; Statistical Properties; Skewness

Introduction

The development of new probability distributions continues to be a vital area of research in statistical theory and applications, particularly for modeling complex real-world phenomena that existing distributions cannot adequately capture. The Poisson distribution expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known constant mean rate and independently of the time since the last event (Wikipedia, 2024). While the classical Poisson distribution has proven invaluable for modeling rare events and count data, its restrictive assumptions often limit its applicability in contemporary statistical modeling scenarios.

Recent advances in probability theory have focused on extending classical distributions to accommodate more flexible modeling requirements. Extensions of distribution families offer more flexible tools for analyzing a wider variety of data due to their theoretical properties (Chaisee, Khamkong & Paksaranuwat, 2024). This trend reflects the growing need for statistical models that can handle overdispersion, zero-inflation, and other complex characteristics commonly observed in modern datasets.

The validity of probability distributions is a critical aspect of statistical modeling, as it determines how well a given distribution can accurately represent real-world phenomena. Validity can be assessed through various methods, including goodness-of-fit tests, graphical analysis, and empirical validation against observed data (Massey, 1951; Kolmogorov, 1933).

The proposed Poisson Hamza distribution represents a novel extension of the classical Poisson framework, designed to address specific limitations in existing count data models. This new distribution aims to provide enhanced flexibility while maintaining mathematical tractability and interpretability. This study contributes to the statistical literature by: (1) providing a rigorous mathematical derivation of the Poisson Hamza distribution, (2) establishing its fundamental statistical properties including moments, variance, standard deviation kurtosis, and skewness.

Derivation of Poisson Hamza Distribution (PHD)

The Probability mass function (pmf) of Poisson distribution is given by:

$$f\left(x/\lambda\right)=\frac{\lambda^{x}e^{-\lambda}}{x!}\qquad x=1,2,3,\ldots\qquad \lambda>0\tag{1}$$

Probability density function of Hamza Distribution proposed by Aijaz, Muzamil, Qurat and Rajnee (2020) is given by:

$$f(x;\alpha,\theta)=\frac{\theta^6}{\alpha\theta^5+120}\left(\alpha+\frac{\theta}{6}x^6\right)e^{-\theta x} \qquad x>0,\quad \theta>0,\ \alpha>0 \tag{2}$$

If the parameter λ follows Hamza Distribution, then,

$$f(\lambda;\alpha,\theta)=\frac{\theta^6}{\alpha\theta^5+120}\left(\alpha+\frac{\theta}{6}\lambda^6\right)e^{-\theta\lambda} \qquad \lambda>0,\quad \theta>0,\ \alpha>0 \tag{3}$$

PHD can be derived as follows

$$f_X(x)=\int_0^\infty f(x/\lambda)f(\lambda;\alpha,\theta)\,d\lambda \tag{4}$$

$$=\int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!}\cdot\frac{\theta^6}{\alpha\theta^5+120}\left(\alpha+\frac{\theta}{6}\lambda^6\right)e^{-\theta\lambda}\,d\lambda$$

$$=\frac{\theta^6}{x!\left(\alpha\theta^5+120\right)}\left(\int_0^\infty \alpha\lambda^x e^{-(1+\theta)\lambda}\,d\lambda+\frac{\theta}{6}\int_0^\infty \lambda^{x+6}e^{-(1+\theta)\lambda}\,d\lambda\right)$$

$$=\frac{\theta^6}{x!\left(\alpha\theta^5+120\right)}\left(\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}}+\frac{\theta\Gamma(x+7)}{6(1+\theta)^{x+7}}\right)$$

Note $x!=\Gamma(x+1)$

$$=\frac{\theta^6}{\Gamma(x+1)\left(\alpha\theta^5+120\right)}\left(\frac{\alpha\Gamma(x+1)}{(1+\theta)^{x+1}}+\frac{\theta(x+6)(x+5)(x+4)(x+3)(x+2)(x+1)\Gamma(x+1)}{6(1+\theta)^{x+7}}\right)$$

$$=\frac{\theta^6\Gamma(x+1)}{\Gamma(x+1)\left(\alpha\theta^5+120\right)}\left(\frac{\alpha}{(1+\theta)^{x+1}}+\frac{\theta(x+6)(x+5)(x+4)(x+3)(x+2)(x+1)}{6(1+\theta)^{x+7}}\right)$$

$$=\frac{\theta^6}{\left(\alpha\theta^5+120\right)}\left(\frac{6\alpha(1+\theta)^6+\theta(x+6)(x+5)(x+4)(x+3)(x+2)(x+1)}{6(1+\theta)^{x+7}}\right)$$

$$f_X(x)=\frac{\theta^6}{\left(\alpha\theta^5+120\right)}\left(\frac{6\alpha(1+\theta)^6+\theta\left(x^6+21x^5+175x^4+735x^3+1624x^2+1764x+720\right)}{6(1+\theta)^{x+7}}\right)$$
(5)

Equation (5) give the pmf of Poisson Hamza Distribution

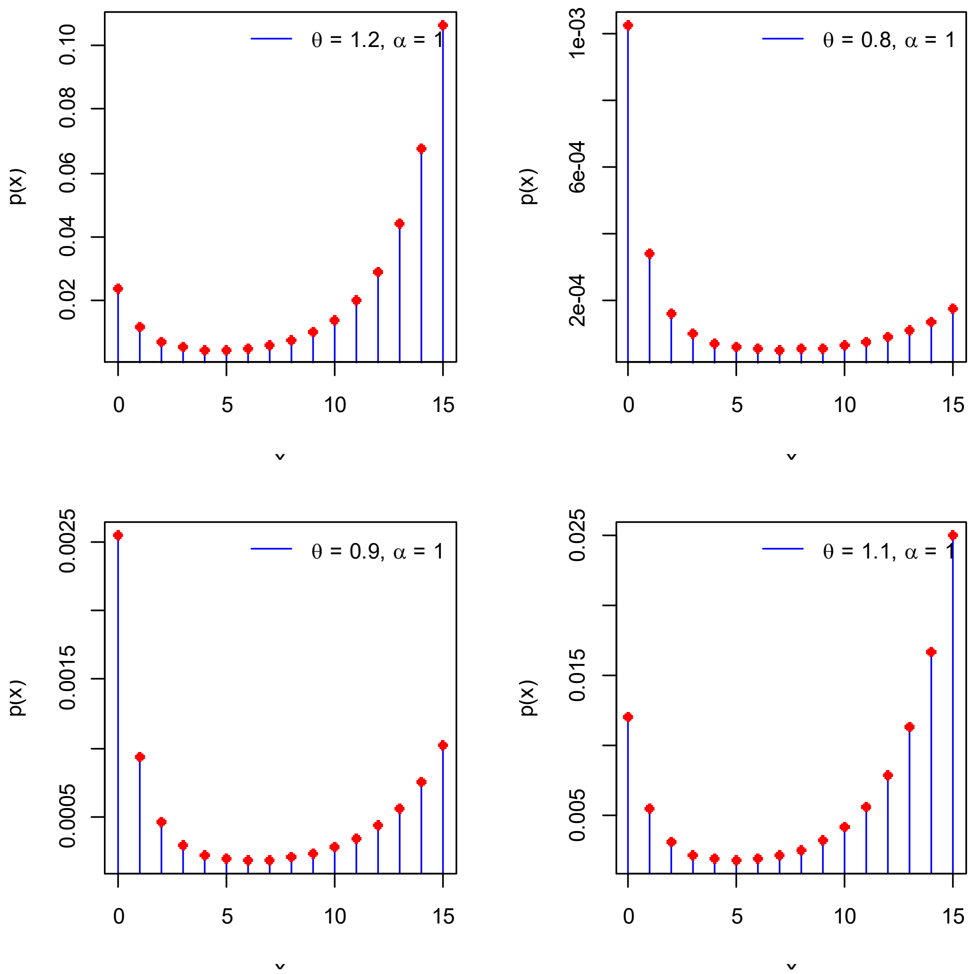

Figure 1: Pmf plot for Poisson Hamza distribution

Figure 1 shows the plots of the probability mass function $p(x)$ for a discrete distribution with parameter a=1 and varying θ. In all cases, the distribution is U-shaped: the probability is highest at the smallest and largest values of x, and lowest in the middle. When $\theta > 1$ (top-left and bottom-right), the right tail (large x) is more pronounced, so higher values of x have greater probability. When $\theta < 1$ (top-right and bottom-left), the left tail (small x) is more pronounced, so lower values of x have greater probability. As θ increases, the probability shifts toward higher xx; as θ decreases, it shifts toward lower x. This demonstrates how θ controls the skewness and spread of the distribution, with the U-shape remaining consistent across all parameter settings

Validity of Poisson Hamza Distribution

To show that

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} \frac{\theta^6}{\left(\alpha\theta^5+120\right)} \left(\frac{6\alpha(1+\theta)^6 + \theta\left(x^6+21x^5+175x^4+735x^3+1624x^2+1764x+720\right)}{6(1+\theta)^{x+7}} \right) = 1$$

Proof

$$= \frac{\theta^6}{6\left(\alpha\theta^5+120\right)(1+\theta)^7} \sum_{x=0}^{\infty} (1+\theta)^{-x} \left(6\alpha(1+\theta)^6 + \theta\left(x^6+21x^5+175x^4+735x^3+1624x^2+1764x+720\right)\right)$$

Split the series

$$= \frac{\theta^6}{6\left(\alpha\theta^5+120\right)(1+\theta)^7} \left(\begin{array}{l} 6\alpha(1+\theta)^6 \sum_{x=0}^{\infty}(1+\theta)^{-x} + \theta\sum_{x=0}^{\infty} x^6 (1+\theta)^{-x} + 21\theta\sum_{x=0}^{\infty} x^5 (1+\theta)^{-x} + \\ 175\theta\sum_{x=0}^{\infty} x^4 (1+\theta)^{-x} + 735\theta\sum_{x=0}^{\infty} x^3 (1+\theta)^{-x} + 1624\theta\sum_{x=0}^{\infty} x^2 (1+\theta)^{-x} + \\ 1764\theta\sum_{x=0}^{\infty} x (1+\theta)^{-x} + 720\theta\sum_{x=0}^{\infty} (1+\theta)^{-x} \end{array} \right)$$

(6)

From eqn (6), solve the summation separately

$$720\theta\sum_{x=0}^{\infty}(1+\theta)^{-x} = 720\theta\sum_{x=0}^{\infty}\frac{1}{(1+\theta)^x} = 720\theta\left(1+\left(\frac{1}{1+\theta}\right)+\left(\frac{1}{1+\theta}\right)^2+\left(\frac{1}{1+\theta}\right)^3+\cdots\right)$$

Using sum of geometric progression to infinity

$$S_\infty = \frac{1}{1-r};\ a=1,\ r=\frac{1}{1+\theta}$$

$$S_\infty = \frac{1}{1-\frac{1}{1+\theta}} = \frac{1+\theta}{\theta}$$

$$720\theta\sum_{x=0}^{\infty}(1+\theta)^{-x} = 720\theta.\frac{1+\theta}{\theta} = 720(1+\theta) \qquad (7)$$

$$6\alpha(1+\theta)^6\sum_{x=0}^{\infty}(1+\theta)^{-x} = 6\alpha(1+\theta)^6.\frac{1+\theta}{\theta} = \frac{6\alpha(1+\theta)^6}{\theta} \qquad (8)$$

$$1764\theta\sum_{x=0}^{\infty}x\left(1+\theta\right)^{-x}=1764\theta\sum_{x=0}^{\infty}x\left(\frac{1}{1+\theta}\right)^{x}$$

Let $Z=\dfrac{1}{1+\theta}$

$$\sum_{x=0}^{\infty}Z^{x}=1+Z+Z^{2}+Z^{3}+\cdots$$

Hence, $a=1,\quad r=Z$

$S_{\infty}=\dfrac{1}{1-Z}$, therefore,

$$\sum_{x=0}^{\infty}Z^{x}=\frac{1}{1+Z}$$

Differentiate both side with respect to Z to get:

$$\sum_{x=0}^{\infty}xZ^{x-1}=\frac{1}{\left(1+Z\right)^{2}}$$

$$\sum_{x=0}^{\infty}\frac{xZ^{x}}{Z}=\frac{1}{\left(1+Z\right)^{2}}$$

Multiply both side by Z

$$\sum_{x=0}^{\infty}xZ^{x}=\frac{Z}{\left(1+Z\right)^{2}} \tag{9}$$

Substitute $Z=\dfrac{1}{1+\theta}$ into equation (9)

$$\sum_{x=0}^{\infty}x\left(1+\theta\right)^{-x}=\frac{1/_{1+\theta}}{\left(1+1/_{1+\theta}\right)^{2}}=\frac{1+\theta}{\theta^{2}}$$

Hence,

$$1764\theta\sum_{x=0}^{\infty}x\left(1+\theta\right)^{-x}=1764\theta.\frac{1+\theta}{\theta^{2}}=\frac{1764\left(1+\theta\right)}{\theta}$$

$$(10)\,1624\theta\sum_{x=0}^{\infty}x^{2}\left(1+\theta\right)^{-x}=\frac{1624\left(2+\theta\right)\left(1+\theta\right)}{\theta^{2}} \tag{11}$$

$$735\theta\sum_{x=0}^{\infty}x^{3}\left(1+\theta\right)^{-x}=\frac{735\left(\theta^{2}+6\theta+6\right)\left(1+\theta\right)}{\theta^{3}} \tag{12}$$

$$175\theta\sum_{x=0}^{\infty}x^{4}\left(1+\theta\right)^{-x}=\frac{175\left(\theta^{3}+14\theta^{2}+36\theta+24\right)\left(1+\theta\right)}{\theta^{4}} \tag{13}$$

$$21\theta\sum_{x=0}^{\infty}x^{5}\left(1+\theta\right)^{-x}=\frac{21\left(\theta^{4}+30\theta^{3}+150\theta^{2}+240\theta+120\right)\left(1+\theta\right)}{\theta^{5}} \tag{14}$$

$$\theta\sum_{x=0}^{\infty}x^{6}\left(1+\theta\right)^{-x}=\frac{\left(1+\theta\right)\left(\theta^{5}+62\theta^{4}+540\theta^{3}+1560\theta^{2}+1800\theta+720\right)}{\theta^{6}} \tag{15}$$

Substitute equation (7), (8), (10), (11), (12), (13), (14), (15) into equation (6)

$$=\frac{\theta^{6}}{6\left(\alpha\theta^{5}+120\right)\left(1+\theta\right)^{7}}\left(\begin{array}{l}720\left(1+\theta\right)+\frac{6\alpha\left(1+\theta\right)^{7}}{\theta}+\frac{1764\left(1+\theta\right)}{\theta}+\frac{1624\left(2+\theta\right)\left(1+\theta\right)}{\theta^{2}}+\\ \frac{735\left(\theta^{2}+6\theta+6\right)\left(1+\theta\right)}{\theta^{3}}+\frac{175\left(\theta^{3}+14\theta^{2}+36\theta+24\right)\left(1+\theta\right)}{\theta^{4}}+\\ \frac{21\left(\theta^{4}+30\theta^{3}+150\theta^{2}+240\theta+120\right)\left(1+\theta\right)}{\theta^{5}}+\\ \frac{\left(1+\theta\right)\left(\theta^{5}+62\theta^{4}+540\theta^{3}+1560\theta^{2}+1800\theta+720\right)}{\theta^{6}}\end{array}\right)$$

Multiply through by $\dfrac{\theta^{6}}{\left(1+\theta\right)^{7}}$

$$=\frac{1}{6\left(\alpha\theta^{5}+120\right)\left(1+\theta\right)^{6}}\left(\begin{array}{l}720\theta^{6}+6\alpha\theta^{5}\left(1+\theta\right)^{6}+1764\theta^{5}+1624\theta^{4}\left(\theta+2\right)+735\theta^{3}\left(\theta^{2}+6\theta+6\right)\\ +175\theta^{2}\left(\theta^{3}+14\theta^{2}+36\theta+24\right)+21\theta\left(\theta^{4}+30\theta^{3}+150\theta^{2}+240\theta+120\right)+\\ \theta^{5}+62\theta^{4}+540\theta^{3}+1560\theta^{2}+1800\theta+720\end{array}\right)$$

$$=\frac{1}{6\left(\alpha\theta^{5}+120\right)\left(1+\theta\right)^{6}}\left(\begin{array}{l}720\theta^{6}+6\alpha\theta^{5}\left(1+\theta\right)^{6}+4320\theta^{5}+10800\theta^{4}\\ +14400\theta^{3}+10800\theta^{2}+4320\theta+720\end{array}\right)$$

$$=\frac{1}{6\left(\alpha\theta^{5}+120\right)\left(1+\theta\right)^{6}}\left(720\left(\theta^{6}+6\theta^{5}+15\theta^{4}+20\theta^{3}+15\theta^{2}+6\theta+1\right)+6\alpha\theta^{5}\left(1+\theta\right)^{6}\right)$$

$$=\frac{1}{6\left(\alpha\theta^{5}+120\right)\left(1+\theta\right)^{6}}\left(720\left(1+\theta\right)^{6}+6\alpha\theta^{5}\left(1+\theta\right)^{6}\right)$$

$$= \frac{1}{6\left(\alpha\theta^5 + 120\right)\left(1+\theta\right)^6}\left(\left(1+\theta\right)^6\left(6\alpha\theta^5 + 720\right)\right)$$

$$= \frac{1}{6\left(\alpha\theta^5 + 120\right)\left(1+\theta\right)^6}\left(\left(1+\theta\right)^6 6\left(\alpha\theta^5 + 120\right)\right)$$

$$f_X\left(x\right) = 1$$

Therefore, Poisson Hamza distribution is a valid pmf

Conclusion

The Poisson Hamza Distribution (PHD) introduces a noteworthy development in the field of probability distributions by offering a versatile and mathematically robust alternative to the traditional Poisson model. This research rigorously derives and validates the PHD, demonstrating its suitability for modeling count data, especially in situations where overdispersion is present.

References

Aijaz, A., Muzamil, J. S., Qurat, U. A. & Rajnee, T. (2020). "The Hamza Distribution With Statistical Properties and Applications". *Asian Journal of Probability and Statistics* 8 (1):28-42. https://doi.org/10.9734/ajpas/2020/v8i130198.

Chaisee, K., Khamkong, M., & Paksaranuwat, P. (2024). A New Extension of the Exponentiated Weibull–Poisson Family Using the Gamma-Exponentiated Weibull Distribution: Development and Applications. *Symmetry*, *16*(7), 780. https://doi.org/10.3390/sym16070780.

Hastie, T., Tibshirani, R., & Friedman, J. (2009). *The elements of statistical learning: Data mining, inference, and prediction* (2nd ed.). Springer.

Kolmogorov, A. N. (1933). Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari*, 4, 83-91.

Massey, F. J. (1951). The Kolmogorov-Smirnov test for goodness of fit. *Journal of the American Statistical Association, 46*(253), 68-78.

Wikipedia Contributors. (2024). Poisson distribution. *Wikipedia*. Retrieved March 24 2025, from https://en.wikipedia.org/wiki/Poisson_distribution