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A Graph-Theoretic Characterization of Orbits in the Finite Full Transformation Semigroup

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Abstract

This paper investigates the orbit structures of elements in the full transformation semigroup TnT_n through the framework of digraph connectivity. Transformations are characterized based on whether their associated functional digraphs are strongly connected, weakly connected, or unilateral. It is shown that strong connectivity corresponds precisely to transformations whose orbits form a single nn-cycle. In contrast, unilateral connectivity arises when orbits constitute directed paths terminating in a unique cycle, and weak connectivity is identified when all elements belong to a single weakly connected component. Furthermore, the paper provides enumeration results, proving that there are exactly (n−1)!(n - 1)! transformations with strongly connected (cyclic) orbits and n!(n−1)n!(n - 1) transformations with unilateral orbit structures. These findings offer new structural and enumerative insights into the full transformation semigroup by analyzing the connectivity patterns of its orbit representations.

Keywords: Orbit; Functional Digraph; Full Transformation Semigroup; Digraph Connectivity; Transformation Enumeration

Introduction

Let $X_n = \{1, 2, \ldots, n\}$. A function θ for which $\mathrm{dom}(\theta) \subseteq X_n$ and $\mathrm{im}(\theta) \subseteq X_n$ is referred to as a *partial transformation* on X_n. The totality of such mappings forms the set P_n. Since the composition of two partial transformations yields another partial transformation, the set P_n is closed under composition and thereby constitutes a semigroup, commonly known as the *partial transformation semigroup*. When the domain of θ coincides entirely with X_n, the transformation is called a *full transformation*. The collection of all such transformations is denoted by T_n, and under function composition, T_n forms the *full transformation semigroup*. A directed graph, or digraph, is formally described as an ordered pair $\Gamma = (V, A)$, where V is the vertex set and $A \subseteq V \times V$ denotes the set of arcs. For any transformation θ, the corresponding digraph $\Gamma_\theta = (V_\theta, A_\theta)$ is constructed by setting $V_\theta = \mathrm{dom}(\theta)$, and an arc $(u, v) \in A_\theta$ exists precisely when $\theta(u) = v$ for some $u \in V_\theta$. A functional digraph arises when one considers a transformation in T_n, where each vertex has exactly one outgoing arc. Such graphical representations are instrumental in uncovering both semigroup and combinatorial properties of transformations. This framework has been studied in works such as [1], where the graph-theoretic structure of singular transformations—those in $\mathrm{Sing}_n \subseteq \mathrm{T}_n$—was explored. Furthermore, in [2], it was demonstrated that for $n \geq 3$, the semigroup Sing_n

can be generated by idempotents if and only if the corresponding digraph is strongly connected and forms a complete undirected graph. In addition to structural analysis, the digraph associated with a mapping provides insight into its decomposition via orbits. Let $\varphi \in \mathrm{T}_n$. Define a relation $\sim$ on X_n such that for $x, y \in X_n$, we have $x \sim y$ if there exist non-negative integers m and k with $x\varphi^m = y\varphi^k$. This relation is easily verified to be an equivalence relation and partitions X_n into equivalence classes known as *orbits* of φ. Each orbit corresponds to a connected component in the digraph representation of φ. This concept mirrors the decomposition of elements in the symmetric group S_n into disjoint cycles, although the structure in T_n is inherently more complex due to the absence of invertibility. A detailed discussion of this orbit structure and its implications can be found in [7]. This study aims to extend the work of [6] by employing the notion of orbits to investigate the structural and combinatorial behavior of elements in T_n. Through this lens, we seek to provide a refined perspective on functional digraphs, with particular emphasis on how orbit decomposition can be used to describe and analyze transformation semigroups.

Preliminaries

Definition 1. [3] A *semigroup* is a set $S_\downarrow$ equipped with a binary operation (denoted multiplicatively) that is associative; that is, for all $\mu, \nu, \omega \in$ S, we have $(\mu\nu)\omega = \mu(\nu\omega)$. If $S_\downarrow$ contains an identity element e such that $e\lambda = \lambda e = \lambda$ for all $\lambda \in S_\downarrow$, then $S_\downarrow$ is called a *monoid*. Additionally, if there exists an element $z \in S_\downarrow$ satisfying $z\lambda = \lambda z = z$ for all $\lambda \in S_\downarrow$, then z is referred to as a *zero element*, and $S_\downarrow$ is said to be a semigroup with zero.

Definition 2 (Subsemigroup [5]). Let $S_\downarrow$ be a semigroup. A non-empty subset $T \subseteq S_\downarrow$ is called a *sub semigroup* if it is closed under the binary operation of $S_\downarrow$. That is, for any $\xi, \eta \in T$, the product $\xi\eta \in T$. Equivalently, $T^2 \subseteq T$.

Definition 3. [4] The *in-degree* of a vertex y, denoted as $\deg^-(y)$, refers to the number of arcs of the form (x, y), and the *out-degree* of y, denoted as $\deg^+(y)$, refers to the number of arcs of the form (y, x). If $\deg^-(y) = 0$, then y is called a source; if $\deg^+(y) = 0$, then y is called a sink.

Definition 4. [4]A graph is considered connected if a path exists linking every pair of its vertices.

Definition 5. [4]A directed graph (digraph) Δ is termed *strongly connected* or simply *strong* when, for any two vertices x and y in Δ, there is a directed path from x to y and another from y to x. It is called *unilaterally connected* or *unilateral* if at least one directed path exists between x and y in either direction. If a semi-path (a path where the direction of edges is ignored) connects every pair of vertices, the digraph is said to be *weakly connected* or *weak*.

Results

Note: Orbits are the connected components of the associated digraph in a full transformation semigroup. For instance, let $a \in T_n$ be given as;

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14 \end{pmatrix}$$

The associated digraph is;

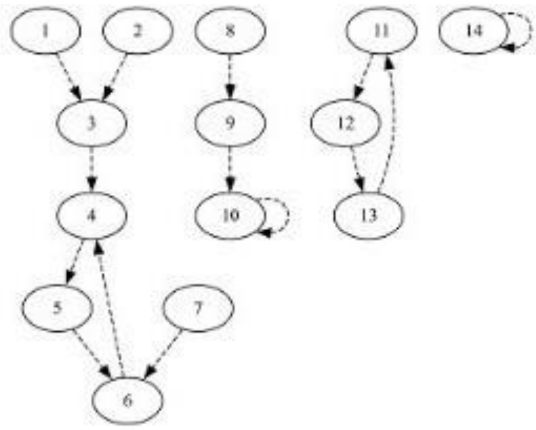

Figure 1. Connected Orbits in T_n

The example illustrates the various types of orbits that can arise. The leftmost orbit is *standard*, followed by an *acyclic* orbit, then a *cyclic* orbit, and finally a *trivial* orbit.

Theorem 6 (Strongly Connected Orbits). *A transformation $\beta \in T_n$ has a strongly connected digraph Δ_β if and only if β consists of a single cyclic orbit covering all n elements.*

proof. ($\Rightarrow$) If Δ_β is strongly connected, then for any $u, v \in X_n$, there exists a directed path from u to v. This implies all elements must lie in a single orbit. Since the digraph is strongly connected, this orbit cannot have any terminal components (like trees attached to cycles), hence must be a single n-cycle.

($\Leftarrow$) If β is an n-cycle, then for any $u, v \in X_n$, there exists $k \geq 0$ such that $\beta^k(u) = v$, making Δ_β strongly connected. □

Theorem 7 (Unilateral Orbit Structure). *Let $\beta \in T_n$. The functional digraph Δ_β is* unilateral *if and only if the orbits of β consist of a collection of directed paths that each terminate in a unique cyclic orbit.*

proof. Suppose first that the digraph Δ_β is unilateral. By definition, for any two vertices $u, v \in X_n$, there exists a directed path from u to v or from v to u. This implies that the reachability relation on the vertices is total and linear. Such a property can only hold if the orbit structure of β is organized around a unique terminal strongly connected component namely, a single cyclic orbit with every other orbit forming a directed path into this cycle. These paths must be disjoint, without any branching or merging, and must terminate within the cycle. The existence of any additional cycles or a branching structure would violate the linear reachability requirement and thus contradict the assumption that the digraph is unilateral.

Conversely, suppose the orbits of β consist of a single cyclic orbit and several directed paths, each leading into the cycle and without any branching or additional cycles. In this case, every vertex either lies in the cycle or maps into it along a unique directed path. For any two vertices $u, v \in X_n$, either both lie on the same path or cycle, in which case one maps to the other directly, or they lie on different paths which both terminate in the same cycle, in which case each can reach a vertex in the cycle and thus there exists a common reachable point. Hence, in all scenarios, either u can reach v or v can reach u, satisfying the definition of a unilateral digraph.

Theorem 8 (Weakly Connected Orbits). *A transformation $\beta \in T_n$ has a weakly connected digraph Δ_β if and only if all elements belong to the same weakly connected orbit component.*

proof. (⇒) Weak connectivity means the underlying undirected graph is connected. Thus, for any u, v, there exists an undirected path between them, which corresponds to the existence of $p, q \geq 0$ with $u\beta^p = v\beta^q$.

(⇐) If all pairs satisfy $u\beta^p = v\beta^q$, then we can construct undirected paths between any vertices, making the digraph weakly connected. □

Theorem 9 (Counting Cyclic and Unilateral Orbits). *The number of transformations in T_n with a single n-cycle orbit is $(n-1)!$, while the number with a unilateral orbit structure (a cycle with one attached path) is $n!(n-1)$.*

proof. We begin with the case of transformations whose functional digraph consists of a single n-cycle. Such transformations correspond precisely to cyclic permutations of n elements. Since any n-cycle in the symmetric group S_n can be uniquely determined by fixing one element and permuting the remaining $n-1$ elements around it, the total number of such transformations is $(n-1)!$.

Now consider transformations whose functional digraphs have a unilateral orbit structure—specifically, a single directed cycle with one directed path attached. Let the length of the cycle be k, where $1 \leq k \leq n-1$. There are $\binom{n}{k}$ ways to choose the k elements that form the cycle. These k elements can be arranged into a directed cycle in $k!$ ways. The remaining $n-k$ elements form a directed path that terminates at a vertex in the cycle. These $n-k$ elements can be arranged in $(n-k)!$ ways. Moreover, the final vertex of the path can be connected to any of the k vertices in the cycle, giving k possible attachments.

Therefore, for each k from 1 to $n-1$, the number of such unilateral structures is given by

$$\binom{n}{k}.k!.(n-k)! = n!$$

Summing over k gives $\sum_{k=1}^{n-1} n! = n!\,(n-1)$.

Hence, the total number of transformations in T_n with a unilateral orbit structure is $n!(n-1)$.☐

Table 1. Number of strong connected orbits in T_n

n	Number of strong orbits
1	0
2	1
3	2
4	6
5	24
6	120
7	720
8	5040
9	40320
10	362880

Table 2. Number of strictly unilateral orbits in T_n

n	Number of strictly unilateral orbits
2	2
3	12
4	72
5	480
6	3600
7	30240
8	282240
9	2903040
10	32659200

Conclusion

We have provided a structural and enumerative analysis of transformations in the full transformation semigroup T_n through the connectivity of their associated functional digraphs. The characterizations revealed that strong connectivity corresponds uniquely to cyclic transformations, unilaterality to path-like orbit structures ending in a single cycle, and weak connectivity to transformations whose entire domain is weakly linked via directed paths. Moreover, the enumeration results offer exact counts for these classes of transformations, specifically identifying $(n-1)!$ strongly connected transformations and $n!(n-1)$ unilateral ones. These findings deepen our understanding of orbit dynamics in transformation semigroups and open up further avenues for research in

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