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Fixed Point Results on Generalized Weakly Quasi-Type Contractive Operators

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Abstract

This study introduces and investigates generalized weakly quasi-type contractive operators within the context of *b*-metric-like spaces, aiming to establish rigorous conditions for the existence and uniqueness of fixed points. While weakly contractive mappings have been widely examined in standard metric spaces, their behavior in *b*-metric-like spaces remains underexplored. Addressing this gap, the paper extends existing theoretical frameworks and contributes novel results relevant to this generalized setting. The proposed assertions are substantiated through non-trivial comparative examples, and several corollaries are presented to illustrate how the main findings generalize and unify various established results in fixed point theory. These contributions enhance the understanding of contractive mappings in non-standard metric-like structures and open pathways for further applications.

Keywords: Fixed Point; *b*-Metric-Like Spaces; Weakly Quasi-Type Contractions; Generalized Contractive Operators; Uniqueness Criteria

Introduction

Fixed point theory is a field of mathematics dedicated to finding points that remain unchanged when a function is applied to them. A pivotal result in this area is the Banach [3] fixed point theorem, which guarantees a single, unique fixed point for any contraction mapping operating within a complete metric space. This powerful theorem has been widely applied to solve a variety of problems in diverse fields like differential equations, optimization, and dynamic systems.

Fixed point theory has recently expanded beyond its traditional scope of metric and b-metric spaces. This expansion has been achieved by relaxing some of the original metric space axioms. In 1989, Bakhtin [4] (also in 1993, Czerwik [9]) significantly advanced fixed point theory by introducing generalizations of the well-known Banach fixed point theorem. This important work was conducted within the framework of b-metric spaces, yielding several new and impactful fixed point results. In a similar fashion In 2013, Alghamdi et al. [2] introduced a new concept: b-metric-like space. They swiftly proved the existence and uniqueness of fixed points within this newly defined space and applied these findings to generate fresh results for coupled fixed points and fixed points in other established areas like partial metric spaces, metric-like spaces, and b-metric spaces. Also, some new applications to integral equations and illustrative examples were given. Moreover, Alber et al. [1] pioneered the concept of weak contraction mappings within the framework of Hilbert space, achieving this by introducing an additional algebraic structure to the space. Subsequently, Rhoades [12] demonstrated that these weak contraction mappings possess a unique fixed point in metric spaces. Since then, researchers have continuously explored new generalizations of weak contraction mappings in metric spaces. Notably, Chen and Zhu [6] offered a new generalization by replacing the standard Banach contraction with Chatterjea's [5] contractive mapping. Following this, Choudhury et al. [8] further expanded on the concept, employing an altering distance function to establish that a new class of weakly C-contractive mappings guarantees a unique fixed point in complete metric spaces. In a similar innovative approach, Cho [7] introduced a new generalization of Choudhury et al.'s [8] weak contraction mapping by incorporating a lower semi-continuous function, thereby developing new fixed point results for weakly contractive mappings in metric space.

A thorough review of published literature shows that there's been little to no research on weakly quasi contractive operators within the specific context of b-metric-like spaces. Building on the concepts in [7, 11], we therefore introduce a novel idea on generalized weakly quasi-type contractive operator in b-metric-like spaces. This new framework allows us to rigorously explore the existence and uniqueness of fixed points for such operators. Our contribution also presents new generalizations of several wellknown results in the field. To demonstrate the practical utility of our newly proposed notion and to compare it with existing findings, we provide substantial new examples. Furthermore, we introduce and analyze new corollaries that forge crucial connections between our work and other established concepts in the literature.

The paper is organized as follows: Section 1 presents the introduction and review of the related literature. In Section 2, the fundamental concepts needed in the sequel are collated. The main findings of the paper are discussed in Section 3.

Preliminaries

In this section, we record specific basic concepts that are needed in the sequel.

Definition 2.1. [10] Let X be a non-null set and $s \geq 1$ be a real constant. A mapping

$d_b : X \times X \to \mathbb{R}^+$ is referred to as a b-metric if for all $x, y, z \in X$, it satisfies the following axioms:

(i) $d_b(x, y) \geq 0 \Leftrightarrow x = y;$

(ii) $d_b(x, y) = d_b(y, x);$

(iii) $d_b(x, z) \leq d_b(x, y) + d_b(y, z).$

The pair, (X, d_b) is referred to as a b-metric space.

Definition 2.2. [2] Let X be a non-null set and $s \geq 1$ be a real constant. A mapping

$\sigma_b : X \times X \to \mathbb{R}^+$ is referred to as a b-metric-like if for all $x, y, z \in X$, it satisfies the following axioms:

(i) $\sigma_b(x,y)=0 \Rightarrow x=y;$

(ii) $\sigma_b(x,y)=\sigma_b(y,x);$

(iii) $\sigma_b(x,z)\le \sigma_b(x,y)+\sigma_b(y,z).$

The pair, (X,d_b) is referred to as a b-metric-like space.

Definition 2.3. [2] Let (X,σ_b) be a b-metric-like space and let $\{x_n\}_{n\in N}$ be a sequence of points of X. A point x ∈ X is said to be the limit of the sequence $\{x_n\}_{n\in N}$ if $\sigma_b(x,x)=\lim_{n\to\infty}\sigma_b(x_n,x)$. and we say that the sequence $\{x_n\}_{n\in N}$ is convergent to x and denoted it by $x_n \to x$ as $n \to \infty$ n ⟶ ∞.

Definition 2.4. [2] Let $(X,\ \sigma_b)$ be a b-metric-like space.

(i) A sequence $\{x_n\}_{n\in ¥}\in X$ is said to be a σ-Cauchy sequence if $\lim_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is finite.

(ii) A be a b-metric-like space $(X,\ \sigma_b)$ is said to be complete if, for every Cauchy sequence $\{x_n\}_{n\in ¥}\in X$ so that,

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m)=\lim_{n\to\infty}\sigma_b(x_n,x)$$

Example 2.5 [13] If $X = ¡^+\cup\{0\}$, then $\sigma_b(x,y)=(x+y)^2$ defines a b-metric-like on X with parameter $s=2$.

Definition 2.7. [9] Let $T: X\to X$ be a self mapping on a metric space (X, d). Then, F is called a generalized C´iric´-type contraction mapping if it satisfies the following condition:

$$d(Tx,Ty)\le\lambda(W(x,y,))$$

for all $x,\ y\in X$, where $\lambda\in(0,\ 1)$ and

$$W(x,y)=\max\left\{\begin{matrix} d(x,y), d(x,Tx), d(y,Ty), \\ \frac{1}{2}[d(x,Ty)+d(y,Tx)] \end{matrix}\right\}.$$

Definition 2.7. [12] Let (X, d) be a b-metric space. A mapping $T : X \to X$ is said to be weakly contractive, if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \lambda(d(x, y)),$$

where $\lambda : R^{+} \to R^{+}$ is continuous and non-decreasing function such that $\lambda(0) = 0$ and $\lim_{t\to\infty} \lambda(t) = \infty$.

Definition 2.7. [12] A function $T : X \to [0, \infty)$, where X is a b-metric space, is called lower semi-continuous if, for all $x \in X$ and $\{x_n\}_{n\in N} \subset X$ are b-convergent with

$\lim_{n\to\infty} x_n = x$, we have

$$T(x) \leq \liminf_{n\to\infty} Tx_n.$$

Let $\Psi = \{\psi : [0, \infty) \to [0, \infty) \mid \psi$ is continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$. Also, let

$\Phi = \{\phi : [0, \infty) \dashrightarrow [0, \infty) \mid \phi$ is lower semi continuous and $\phi(t) = 0 \Leftrightarrow t = 0\}$.

Lemma 2.8. [13]. Let (X, σ_b) be b-metric-like space with $s \geq 1$ a real constant and $\{x_n\}_{n\in ¥}$ be a sequence such that $\lim_{n\to\infty} \sigma_b(x_n, x_{n+1}) = 0$. If $\lim_{n,m\to\infty} \sigma_b(x_n, x_m) \neq 0$, then there exist ε>0 and sequences $\lim_{n\to\infty}\{x_{m(k)}\}_{k\in ¥}$ and $\lim_{n\to\infty}\{x_{n(k)}\}_{k\in ¥}$ of positive integers with $n_k > m_k > k$,such that

$\sigma_b(x_{m(k)}, x_{n(k)}) \geq ò, \sigma_b(x_{m(k)}, x_{n(k-1)}) \geq ò, \frac{ò}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k-1)}, x_{n(k-1)}) \leq òs$,and

$$\frac{ò}{s} \leq \limsup_{k\to\infty} d(x_{n(k-1)}, x_{m(k)}) \leq ò, \frac{ò}{s} \leq \limsup_{k\to\infty} d(x_{m(k-1)}, x_{n(k)}) \leq òs^2.$$

Raj et.al [11] obtained the following result in the context of metric space.

Definition 2.9. [11] Let X be a metric space with metric d, let $T : X \to X$, and let $\varphi : X \to [0, \infty)$ be a lower semi-continuous function. Then, T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \psi(M(x, y, \varphi)) - \phi(N(x, y, \varphi)) \tag{2.1}$$

for all $x, y \in X$, where $\psi \in \Psi, \ \phi \in \Phi$ and

$$\mathrm{M}\ (x,y,\varphi) = \max\left\{\begin{array}{c} d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), \\ d(x,Ty)+\varphi(y)+\varphi(Ty), \dfrac{d(x,y)+\varphi(x)+\varphi(y)}{1+d(x,y)+\varphi(x)+\varphi(y)}, \\ \dfrac{1}{2}[d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(y)+\varphi(Tx)] \end{array}\right\},$$

$$\mathrm{N}\ (x,y,\varphi) = \max\left\{\begin{array}{c} d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty), \\ \dfrac{d(x,y)+\varphi(x)+\varphi(y)}{1+d(x,y)+\varphi(x)+\varphi(y)} \end{array}\right\}.$$

The main result of [11] is as follows.

Theorem 2.10. [11]. Let X be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

Main Results

This section introduces the concept of generalized weakly quasi-type contractive operators within the framework of b-metric-like spaces and examines the conditions for the existence of a fixed point for such operators.

In this section, we introduce the concept of generalized weakly quasi-type contractive operator in the context of b-metric-like space and examine the conditions for the existence of a fixed point of such operator.

Definition 3.1. Let (X,σ_b) be a b-metric-like space. A mapping $T : X \to X$ is called a generalized weakly quasi-type contractive operator, if it satisfies the following condition:

$$\psi((s^p\sigma_b(Tx,Tx)+\sigma_b(Tx,Ty)+\varphi(Tx)+\varphi(Ty))) \leq \psi(\mathrm{K}(x,y,\varphi)) - \phi(\mathrm{L}(x,y,\varphi)), \tag{3.1}$$

for all $x, y \in X$, where $\psi \in \Psi, \phi, \varphi \in \Phi$ and

$$\mathrm{K}(x,y,\varphi) = \max \left\{ \begin{array}{c} \sigma_b(x,x) + \sigma_b(x,y) + \varphi(x) + \varphi(y), \\ \frac{1}{2}[\sigma_b(x,Tx) + \varphi(x) + \varphi(Tx) + \sigma_b(y,y) + \sigma_b(y,Ty) + \varphi(y) + \varphi(Ty)], \\ \frac{\sigma_b(x,y) + \varphi(x) + \varphi(y)}{1 + \sigma_b(x,y) + \varphi(x) + \varphi(y)}, \\ \frac{1}{2s}[\sigma_b(x,Ty) + \varphi(x) + \varphi(Ty) + \sigma_b(y,Tx) + \varphi(y) + \varphi(Tx)] \end{array} \right\} \tag{3.2}$$

and

$$\mathrm{L}(x,y,\varphi) = \max \left\{ \begin{array}{c} \sigma_b(x,x) + \sigma_b(x,y) + \varphi(x) + \varphi(y), \\ \sigma_b(y,y) + \sigma_b(y,Ty) + \varphi(y) + \varphi(Ty), \\ \frac{\sigma_b(x,y) + \varphi(x) + \varphi(y)}{1 + \sigma_b(x,y) + \varphi(x) + \varphi(y)} \end{array} \right\}. \tag{3.3}$$

The following is the main result of this paper.

Theorem 3.2. Let (X, σ_b) be a complete b-metric-like space. If T is a generalized weakly quasi-type contractive operator, then there exists a unique $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$. Proof Let $x_0 \in X$ be arbitrary but fixed. We will construct a recursive sequence $\{x_n\}_{n \in ¥}$ in the following manner: $x_n = Tx_{n-1}$, for all $n \in ¥$.

It is observed that if $x_n = x_{n-1} = Tx_{n-1}$, for some $n \in ¥$, then x_n is a fixed point of T and the proof is finished. Hence, we assume that $x_n \neq x_{n-1}$ for all $n \in ¥$. By replacing $x = x_{n-1}$ and $y = x_n$ in equation (3.2), we obtain

$$\mathrm{K}(x_{n-1}, x_n, \varphi) = \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \frac{1}{2}[\sigma_b(x_{n-1}, Tx_{n-1}) + \varphi(x_{n-1}) + \varphi(Tx_{n-1}) \\ + \sigma_b(x_n, x_n) + \sigma_b(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n)], \\ \frac{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}{1 + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}, \\ \frac{1}{2s}[\sigma_b(x_{n-1}, Tx_n) + \varphi(x_{n-1}) + \varphi(Tx_n) \\ + \sigma_b(x_n, Tx_{n-1}) + \varphi(x_n) + \varphi(Tx_{n-1})] \end{array} \right\}$$

$$
= \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \frac{1}{2}[\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) \\ +\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})], \\ \frac{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}{1 + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}, \\ \frac{1}{2s}[\sigma_b(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}) \\ +\sigma_b(x_n, x_n) + \varphi(x_n) + \varphi(x_n)] \end{array} \right\}
$$

$$
\leq \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \frac{1}{2}[\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) \\ +\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})], \\ \frac{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}{1 + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}, \\ \frac{1}{2s}[\sigma_b(x_n, x_n) + \varphi(x_n) + \varphi(x_n) \\ +s\sigma_b(x_{n-1}, x_n) + s\sigma_b(x_{n+1}, x_n) + \varphi(x_{n-1}) + \varphi(x_{n+1})] \end{array} \right\}
$$

$$
\leq \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \frac{1}{2}[\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) \\ +\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})], \\ \frac{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}{1 + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}, \end{array} \right\}
$$

$$
\leq \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \max\{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})\}, \\ \frac{\sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)}{1 + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)} \end{array} \right\}
$$

$$
\leq \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \end{array} \right\}.
$$

Hence, (3.2) becomes

$$
\mathrm{K}(x_{n-1}, x_n, \varphi) = \max \left\{ \begin{array}{c} \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), \\ \sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \end{array} \right\}.
$$

Similarly, we obtain

$$\mathrm{L}(x_{n-1},x_n,\varphi)=\max\left\{\begin{array}{c}\sigma_b(x_{n-1},x_{n-1})+\sigma_b(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),\\ \sigma_b(x_n,x_n)+\sigma_b(x_n,Tx_n)+\varphi(x_n)+\varphi(Tx_n),\\ \dfrac{\sigma_b(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)}{1+\sigma_b(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)}\end{array}\right\}$$

$$=\max\left\{\begin{array}{c}\sigma_b(x_{n-1},x_{n-1})+\sigma_b(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),\\ \sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\end{array}\right\}.$$

Consequently, (3.1) gives

$$\psi(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$

$$\leq\psi(s^p(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})))$$

$$\leq\psi(s^p(\sigma_b(Tx_{n-1},Tx_{n-1})+\sigma_b(Tx_{n-1},Tx_n)+\varphi(Tx_{n-1})+\varphi(Tx_n))$$

$$\leq\psi(\mathrm{K}(x_{n-1},x_n,\varphi))-\phi(\mathrm{L}(x_{n-1},x_n,\varphi)). \tag{3.4}$$

If

$$\sigma_b(x_{n-1},x_{n-1})+\sigma_b(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)$$

$$<\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$$

for some positive integer n, then it follows from (3.4) that

$$\psi(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$

$$\leq\psi(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$

$$-\phi(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$$

which implies that

$$\phi(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))=0.$$

Hence,

$$\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})=0,$$

from which we notice that

$x_n = x_{n+1}, \sigma_b(x_n, x_n) = 0$ and $\varphi(x_n) = \varphi(x_{n+1}) = 0$

which is a contradiction. Thus,

$$\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$$
$$\leq \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n) \tag{3.5}$$

for all $n = 1, 2, 3, \ldots$

Therefore,

$$\mathrm{K}(x_{n-1}, x_n, \varphi) = \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)$$

and

$$\mathrm{L}(x_{n-1}, x_n, \varphi) = \sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n).$$

From (3.1), we have

$$\psi(\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}))$$
$$\leq \psi(\sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n))$$
$$-\phi(\sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)). \tag{3.6}$$

This implies that from (3.5) that the sequence $\{\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})\}$ is bounded below and non-increasing. Therefore,

$\sigma_b(x_n, x_n) + \sigma_b(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \to \eta$ as $n \to \infty$, for some $\eta \geq 0$. Consider $\eta > 0$. Taking limit in (3.6) as $n \to \infty$, applying the continuity of ψ and the lower semi-continuity of ϕ, lead to

$$\psi(\eta) \leq \psi(\eta) - \liminf_{n\to\infty} \phi(\sigma_b(x_{n-1}, x_{n-1}) + \sigma_b(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n))$$
$$\leq \psi(\eta) - \phi(\eta) < \psi(\eta),$$

which is a contradiction. Therefore, $\lim_{n\to\infty}(\sigma_b(x_n,x_n)+\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))=0$, from which we have

$$\lim_{n\to\infty}\sigma_b(x_n,x_n)=\lim_{n\to\infty}\sigma_b(x_n,x_{n+1})=0 \tag{3.7}$$

$$\lim_{n\to\infty}\varphi(x_n)=\lim_{n\to\infty}\varphi(x_{n+1})=0. \tag{3.8}$$

Now, we prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Assume that $\{x_n\}_{n\in\mathbb{N}}$ is not Cauchy, and by Lemma 2.8, there exist $\epsilon>0$ and subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, satisfying $n(k)$ as the smallest index for which $n(k)>m(k)>k$.

From (3.2), we have

$$\mathrm{K}(x_{m(k)},x_{n(k)},\varphi)=\max\left\{\begin{array}{c}\sigma_b(x_{m(k-1)},x_{m(k-1)})+\sigma_b(x_{m(k-1)},x_{n(k-1)})\\ +\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)}),\\ \frac{1}{2}[\sigma_b(x_{m(k-1)},Tx_{m(k-1)})+\varphi(x_{m(k-1)})+\varphi(Tx_{m(k-1)})\\ +\sigma_b(x_{n(k-1)},x_{n(k-1)})+\sigma_b(x_{n(k-1)},Tx_{n(k-1)})\\ +\varphi(x_{n(k-1)})+\varphi(Tx_{n(k-1)})],\\ \dfrac{\sigma_b(x_{m(k-1)},x_{n(k-1)})+\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)})}{1+\sigma_b(x_{m(k-1)},x_{n(k-1)})+\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)})},\\ \frac{1}{2s}[\sigma_b(x_{m(k-1)},Tx_{n(k-1)})+\varphi(x_{m(k-1)})+\varphi(Tx_{n(k-1)})\\ +\sigma_b(x_{n(k-1)},Tx_{m(k-1)})+\varphi(x_{n(k-1)})+\varphi(Tx_{m(k-1)})]\end{array}\right\} \tag{3.9}$$

As $k\to\infty$ in (3.9), applying Lemma 2.8 and using equations (3.7) and (3.8) yield

$$\lim_{k\to\infty}\mathrm{K}(x_{n(k)},x_{m(k)},\varphi)=\epsilon s. \tag{3.10}$$

Following similar steps, (3.3) becomes

$$\mathrm{L}(x_{m(k)},x_{n(k)},\varphi)=\max\left\{\begin{array}{c}\sigma_b(x_{m(k-1)},x_{m(k-1)})+\sigma_b(x_{m(k-1)},x_{n(k-1)})\\ +\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)}),\\ \sigma_b(x_{n(k-1)},x_{n(k-1)})+\sigma_b(x_{n(k-1)},Tx_{n(k-1)})\\ +\varphi(x_{n(k-1)})+\varphi(Tx_{n(k-1)}),\\ \dfrac{\sigma_b(x_{m(k-1)},x_{n(k-1)})+\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)})}{1+\sigma_b(x_{m(k-1)},x_{n(k-1)})+\varphi(x_{m(k-1)})+\varphi(x_{n(k-1)})}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \sigma_b(x_{m(k-1)}, x_{m(k-1)}) + \sigma_b(x_{m(k-1)}, x_{n(k-1)}) \\ +\varphi(x_{m(k-1)}) + \varphi(x_{n(k-1)}), \\ \sigma_b(x_{n(k-1)}, x_{n(k-1)}) + \sigma_b(x_{n(k-1)}, x_{n(k)}) \\ +\varphi(x_{n(k-1)}) + \varphi(x_{n(k)}), \\ \dfrac{\sigma_b(x_{m(k-1)}, x_{n(k-1)}) + \varphi(x_{m(k-1)}) + \varphi(x_{n(k-1)})}{1 + \sigma_b(x_{m(k-1)}, x_{n(k-1)}) + \varphi(x_{m(k-1)}) + \varphi(x_{n(k-1)})} \end{array}\right\}$$

Hence,

$$\frac{ò}{s^2} \leq \lim_{k\to\infty} \mathrm{L}(x_{n(k)}, x_{m(k)}, \varphi) = òs. \tag{3.11}$$

From (3.1), we have

$$\psi(òs) \leq (òs^p)$$

$$\leq \psi(s^p \limsup_{k\to\infty}[\sigma_b(x_{m(k)}, x_{m(k)}) + \sigma_b(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)})])$$

$$\leq \psi(s^p \limsup_{k\to\infty}[\sigma_b(Tx_{m(k-1)}, Tx_{m(k-1)}) + \sigma_b(Tx_{m(k-1)}, Tx_{n(k-1)}) + \varphi(Tx_{m(k-1)}) + \varphi(Tx_{n(k-1)})])$$

$$\leq \psi(\mathrm{K}(x_{m(k)}, x_{n(k)}, \varphi)) - \phi(\mathrm{L}(x_{m(k)}, x_{n(k)}, \varphi)). \tag{3.12}$$

As $k \to \infty$ in (3.12), and making use of Lemma 2.8, the continuity of ψ, the lower semi-continuity of ϕ, and by using equations (3.7), (3.8), (3.10) and (3.11), we obtain

$$\psi(òs) \leq \psi(òs) - \lim_{n,m\to\infty} \phi(\mathrm{L}(x_{m(k)}, x_{n(k)}, \varphi)),$$

from which it follows that $\lim_{n,m\to\infty} \phi(\mathrm{L}(x_{m(k)}, x_{n(k)}, \varphi)) = 0$ a contradiction to (3.11). It follows that, $\{x_n\}_{n\in¥}$ is a Cauchy sequence. The completeness of X implies that there exists $u \in X$ such that

$\lim_{n\to\infty} x_n = u$.Given that ϕ is lower semi-continuous, $\varphi(u) \leq \liminf_{n\to\infty} \varphi(x_n) \leq \lim_{n\to\infty} \varphi(x_n) = 0$, from which it follows that $\varphi(u) = 0$.

We now show that u is the fixed point of T. Using (3.2) with $x = x_n$ and $y = u$, we obtain

$$\mathrm{K}(x_n,u,\varphi) = \max\left\{\begin{array}{c} \sigma_b(x_n,x_n)+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u), \\ \frac{1}{2}[\sigma_b(x_n,Tx_n)+\varphi(x_n)+\varphi(Tx_n) \\ +\sigma_b(u,u)+\sigma_b(u,Tu)+\varphi(u)+\varphi(Tu)], \\ \frac{\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)}{1+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)}, \\ \frac{1}{2s}[\sigma_b(x_n,Tu)+\varphi(x_n)+\varphi(Tu) \\ +\sigma_b(u,Tx_n)+\varphi(u)+\varphi(Tx_n)] \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \sigma_b(x_n,x_n)+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u), \\ \frac{1}{2}[\sigma_b(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) \\ +\sigma_b(u,u)+\sigma_b(u,Tu)+\varphi(u)+\varphi(Tu)], \\ \frac{\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)}{1+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)}, \\ \frac{1}{2s}[\sigma_b(x_n,Tu)+\varphi(x_n)+\varphi(Tu) \\ +\sigma_b(u,x_{n+1})+\varphi(u)+\varphi(x_{n+1})] \end{array}\right\},$$

from which we have

$$\lim_{n\to\infty}\mathrm{K}(x_n,u,\varphi) = \max\left\{\begin{array}{c} \sigma_b(u,u),\sigma_b(u,u)+\sigma_b(u,Tu)+\varphi(Tu), \\ \frac{1}{2s}[\sigma_b(u,Tu)+\varphi(Tu)] \end{array}\right\}$$

$$= \sigma_b(u,u)+\sigma_b(u,Tu)+\varphi(Tu). \tag{3.13}$$

In like manner, we have

$$\lim_{n\to\infty}\mathrm{L}(x_n,u,\varphi) = \lim_{n\to\infty}\max\left\{\begin{array}{c} \sigma_b(x_n,x_n)+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u), \\ \sigma_b(u,u)+\sigma_b(u,Tu)+\varphi(u)+\varphi(Tu), \\ \frac{\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)}{1+\sigma_b(x_n,u)+\varphi(x_n)+\varphi(u)} \end{array}\right\}$$

$$= \max\{\sigma_b(u,u), \sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu)\}$$
$$= \sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu). \tag{3.14}$$

Therefore, from (3.1), we have

$$\psi(\sigma_b(Tx_n,Tx_n) + \sigma_b(Tx_n,Tu) + \varphi(Tx_n) + \varphi(Tu))$$
$$= \psi(s^p(\sigma_b(Tx_n,Tx_n) + \sigma_b(Tx_n,Tu) + \varphi(Tx_n) + \varphi(Tu)))$$
$$\leq \psi(\mathrm{K}(x_n,u,\varphi)) - \phi(\mathrm{L}(x_n,u,\varphi)). \tag{3.15}$$

As $n \to \infty$ in (3.15) and applying the continuity of ψ, the lower continuity of ϕ and making use of equations (3.13) and (3.14), we have

$$\psi(\sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu))$$

$$\leq \psi(\sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu)) - \phi(\sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu)). \tag{3.16}$$

The expression (3.16) implies that $\phi(\sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu)) = 0$ and hence, $\sigma_b(u,u) + \sigma_b(u,Tu) + \varphi(Tu) = 0$. Therefore, $\sigma_b(u,u) = 0$, $u = Tu$ and $\varphi(Tu) = 0$.

To demonstrate the uniqueness of the fixed point, let's assume that p is another fixed point of the mapping, where $x = u$ and $y = p$. Then, $p = Tp$ and $\varphi(p) = 0$. Now, using (3.1), we have

$$\psi(\sigma_b(u,u) + \sigma_b(u,p)) + \varphi(u) + \varphi(p))$$

$$\leq \psi(s^p(\sigma_b(Tu,Tu) + \sigma_b(Tu,Tp) + \varphi(Tu) + \varphi(Tp)))$$

$$\leq \psi(\mathrm{K}(u,p,\varphi)) - \phi(\mathrm{L}(u,p,\varphi))$$

$$= \psi(\sigma_b(u,u) + \sigma_b(p,u)) - \phi(\sigma_b(u,u) + \sigma_b(p,u)).$$

Thus, $u = p$.

We construct the following example to verify the hypotheses of Theorem 3.2.

Example 3.3 Let $X = [0,\infty)$ and $\sigma_b(x,y) = (x+y)^2$ for all $x, y \in X$. Then obviously, σ_b is a b-metric-like on X with constant $s = 2$ and (X, σ_b) is complete. Also, σ_b is not a metric-like or b-metric (and not a metric) on X. Define the self mapping T by $Tx = \frac{x}{2}$ for all $x \in X$. To see that the T is a generalized weakly contractive operator; let $\psi(t) = \frac{5}{2}$, $\phi(t) = t$, $\varphi(t) = t$ for all $t \geq 0$ and $p = 2$. For all $x, y \in X$, if $x \neq y$, then

$$\psi(s^p(\sigma_b(Tx,Tx) + \sigma_b(Tx,Ty) + \varphi(Tx) + \varphi(Ty)))$$

$$= 2^2\left(\frac{x}{2} + \frac{x}{2} + \frac{x}{2} + \frac{y}{2} + x + y\right)$$

$$= 4\left(x^2 + \frac{x^2}{4} + \frac{y^2}{4} + xy + x + y\right)$$

$$= 5x^2 + y^2 + 4xy + x + y$$

$$< 15x^2 + 3y^2 + 6xy + 3x + 3y$$

$$= \frac{1}{2}\left(25x^2 + 5y^2 + 10xy + 5x + 5y - 10x^2 - 2y^2 - 4xy - 2x - 2y\right)$$

$$= \frac{5}{2}(5x^2 + y^2 + 2xy + x + y) - \frac{2}{2}\left(5x^2 + y^2 + 2xy + x + y\right)$$

$$= \frac{5}{2}(\sigma_b(x,x)^2 + \sigma_b(x,y)^2 + \varphi(x) + \varphi(y)) - \frac{1}{2}\left(\sigma_b(x,x)^2 + \sigma_b(x,y)^2 + \varphi(x) + \varphi(y)\right)$$

$$= \psi(\max\{\mathrm{K}(x,y,\varphi)\}) - \phi(\max\{\mathrm{L}(x,y,\varphi)\}).$$

Thus, Theorem 3.2 conditions are satisfied. Therefore, the mapping T has a unique fixed point in X. On the other hand, since σ_b is not a metric on X and hence, the equivalent findings in [11] are not useful in this example to find a fixed point of T.

In what follows, we present some consequences of Theorem 3.2

Corollary 3.4 Let (X, σ) be a σ-complete metric-like space with $s \geq 1$ a real constant. and let T be a given self-mapping. Suppose that $p \geq 2$ is a constant such that

$$\psi(s^p(\sigma_b(Tx,Tx) + \sigma_b(Tx,Ty))) \leq \psi(k_1(x,y) - \phi(L_1(x,y)),$$

for all x, y ò X, where ψ ò Ψ, $\phi \in \Phi$ and

$$K_1(x,y) = \max\left\{\begin{matrix} \sigma_b(x,x)+\sigma_b(x,y), \dfrac{1}{2}[\sigma_b(x,Tx)+\sigma_b(y,y)+\sigma_b(y,Ty)], \\ \dfrac{\sigma_b(x,y)}{1+\sigma_b(x,y)}, \dfrac{1}{2s}[\sigma_b(x,Ty)+\sigma_b(y,Tx)] \end{matrix}\right\},$$

$$L_1(x,y) = \max\left\{\begin{matrix} \sigma_b(x,x)+\sigma_b(x,y), \sigma_b(y,y)+\sigma_b(y,Ty), \\ \dfrac{\sigma_b(x,y)}{1+\sigma_b(x,y)} \end{matrix}\right\}.$$

It follows that there's a unique u ò X such that $u = Tu$.

Proof. By taking $\varphi(x) = 0$, for all $x \in X$, Theorem 3.2 can be applied to find a unique u ò X such that $u = Tu$.

Corollary 3.5 Let (X, σ) be a σ-complete metric-like space with $s \geq 1$ a real constant. and let T be a given self mapping. Suppose that $p \geq 2$ is a constant such that $\psi(s^p(\sigma_b(Tx,Tx)+\sigma_b(Tx,Ty))) \leq \psi(\mathrm{K}(x,y,\varphi))$

for all x, y ò X, where $\mathrm{K}(x,y,\varphi)$ is same as in Theorem 3.2. It follows that there's a unique u ò X such that $u = Tu$.

Proof. By taking $\phi(t) = 0$, for all $t \in R^+$, Theorem 3.2 can be applied to find a unique u ò X such that $u = Tu$.

Corollary 3.6 Let (X, σ) be a σ-complete metric-like space with $s \geq 1$ a real constant. and let T be a given self-mapping. Suppose that $p \geq 2$ is a constant such that

$$\psi(s^p(\sigma_b(Tx,Tx)+\sigma_b(Tx,Ty)+\varphi(Tx)+\varphi(Ty)))$$
$$\leq \psi(\sigma_b(x,x)+\sigma_b(x,y)+\varphi(x)+\varphi(y))$$
$$-\phi(\sigma_b(x,x)+\sigma_b(x,y)+\varphi(x)+\varphi(y)),$$

for all x, y ò X, where ψ ò Ψ, $\phi \in \Phi$. It follows that there's a unique u ò X such that $u = Tu$.

Conclusion

Fixed point theory has experienced significant advancements in recent years, particularly in the examination of self-mappings within metric spaces. A key contribution in this area came from Alghamdi et al. they generalized the concept of b-metric spaces by introducing b-metric-like spaces and subsequently derived relevant fixed point results in this generalized setting. This paper introduces generalized weakly quasi-type contractive operators within the framework of b-metric-like spaces and establishes the conditions for the existence and uniqueness of fixed points for such operators. The theoretical findings were supported by comparative examples. Additionally, corollaries were presented, demonstrating how the results extend and generalize previous concepts in fixed-point theory. This research significantly contributes to the field by expanding the understanding of contractive mappings in broader metric structures.

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