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Maximum Works Performed by Signed Partial Transformations of a Finite Set

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Abstract

Let X_n and X_n^* to be the finite sets $\{1, 2, 3, ..., n\}$ and $\{\pm 1, \pm 2, \pm 3, ..., \pm n\}$ respectively. A map $\alpha : X_n \to X_n$ is a transformation on X_n. We call α a signed transformation if $\alpha : X_n \to X_n^*$. Let $\widetilde{P}_n$ be the set of all signed partial transformations on X_n. This set consists of all $\alpha \in \widetilde{P}_n$ for which $dom(\alpha) \subseteq X_n$. The work $w(\alpha)$ performed by a transformation α is defined as the sum of all the distances $|i - i\alpha|$ for each $i \in dom(\alpha)$. In this paper, we characterize all $\alpha \in \widetilde{P}_n$ that attain maximum and minimum works and deduce formulas for the values of these minimum and maximum. We further present a range for the values of $w(\alpha)$ for all $\alpha \in \widetilde{P}_n$.

Keywords: Transformation; Signed Transformation; Signed Partial Transformation; Work

Introduction

Consider the finite sets $X_n = \{1, 2, 3, ..., n\}$ and $X_n^* = \{\pm 1, \pm 2, \pm 3, ..., \pm n\}$. A transformation on X_n is a self-map of X_n where as a map from X_n to X_n^* is called a signed transformation on X_n. The collection of all signed transformation for which $dom(\alpha) \subseteq X_n$ is called a signed partial transformation on X_n denoted by $\widetilde{P}_n$. The set $\widetilde{P}_n$ is the analogue of the famous set of partial transformation of the self-map of X_n usually studied in Semigroup theory.

If we consider $i\alpha$ to be the image of any $i \in dom(\alpha)$ and insist on the elements of X_n as equally spaced points, [1] pointed that any point $i \in X_n$ can be moved a distance of $|i - i\alpha|$ units. If these distances are summed as i varies over the $dom(\alpha)$, the cardinal gives the (total) work performed by α and will be denoted by $w(\alpha)$. He used these ideas to investigate the work performed by some subsemigroups of the partial transformation semigroup.

Before the research of [1], the idea of "work" as coined by [1] has been used in [2]. He chose the term displacement to mean exactly what we now consider as work. His investigation was strictly on the displacement of permutations. As noted by [2], the displacement of any given permutation β is the sum of the distances between all $i \in dom(\beta)$ and their images. As observed in [3], that fast-forwarding, concepts closely related to the displacement studied by Knuth were developed

and studied under various nomenclatures by [4], [5], [6] and [7]. These developments were the motivation for the coinage of "work" as studied by [1].

In a more recent research, [8] studied maximum work performed by all the maps in the set of all full transformations (T_n) and the set of all partial transformations (P_n) on X_n. They described the features of elements of T_n and P_n that attain maximum work and further developed a relation that correctly captures the value of this maximum for any natural number n. In a separate study, [3] undertook an extension of [8] to the set of all signed full transformations $\widetilde{T}_n$. Here too, investigation into the behavior of element of $\widetilde{T}_n$ was carried out in relation to the maximum work its elements attain.

We note that it will not be hard to see that with respect to work, the operation that will qualify $\widetilde{P}_n$ to be a semigroup isn't playing any role. Hence, their research can only be carried out with just the elements of $\widetilde{P}_n$ as the underlying objects of consideration [3]. In this article, we extend [3] to the signed partial transformation on X_n.

Preliminaries

In this section, we recall existing definitions and results from [1] and [3]. We shall find these definitions and results useful throughout this paper.

Definition ([1]): The work performed by a (partial) transformation $\alpha \in P_n$ in moving a point $i \in dom(\alpha)$ is defined to be:

$$w_i(\alpha) = \begin{cases} |i - i\alpha| & if\ i \in dom(\alpha) \\ 0 & otherwise, \end{cases}$$

The (total) work performed by α is given by

$$w(\alpha) = \sum_{i \in X_n} w_i(\alpha)$$

The next two results presents a characterization of the elements of $\widetilde{T}_n$ that attains maximum work and the value of the maximum for any $n \in X_n$ respectively.

Theorem 2.1([3]): Let $\alpha \in \widetilde{T}_n$. α performs the maximum work in $\widetilde{T}_n$ if and only if for each $i \in dom(\alpha)$, $i\alpha = -n$.

Theorem 2.2([3]): Suppose $\alpha \in \widetilde{T}_n$ is such that $i\alpha = -n\ \forall i \in dom(\alpha)$, then $w(\alpha) = \dfrac{3n^2 + n}{2}, n \in X_n$.

Consider $\widetilde{T}_n^- = \{\alpha \in \widetilde{T}_n \mid i\alpha < 0, \forall i \in dom(\alpha), n \in X_n\}$, the collection of all $\alpha \in \widetilde{T}_n$ whose image sets consists only of non-positive numbers.

Now,

Theorem 2.3([3]): Let $\alpha \in \widetilde{T}_n^-$. Such α performs minimum work in $\widetilde{T}_n^-$ if and only if for each $i \in dom(\alpha)$, $i\alpha = -1$.

Theorem 2.4([3]): Suppose $\alpha \in \widetilde{T}_n^-$ is such that $i\alpha = -1$, then for all $i \in dom(\alpha)$, $w(\alpha) = \frac{n^2 + 3n}{2}, n \in X_n.$

Theorem 2.4 presents the value of the work the map in Theorem 2.3 attains.

The range of values of the work performed by all $\alpha \in \widetilde{P}_n$ will be:

$$0 \le w(\alpha) \le \frac{3n^2 + n}{2}.$$

Zero (0) in the above range of values originates from the fact that the identity map is the map that attains minimum work in the set of the signed full transformation.

Main Results

In the first result below, we present a descriptive feature of the map $\alpha \in \widetilde{P}_n$ whose value of work is the maximum compared to all other maps in $\widetilde{P}_n$.

Theorem 3.1: Let $\alpha \in \widetilde{P}_n$. The $w(\alpha)$ in $\widetilde{P}_n$ will be the maximum if and only if for each $i \in dom(\alpha)$, $i\alpha = -n$.

The next result presents the value of the maximum in the above result.

Theorem 3.2: Suppose α in $\widetilde{P}_n$ is such as in Theorem 3.1 above, then $w(\alpha) = \frac{3n^2 + n}{2}$.

We note that the map described in Theorem 3.1 is an element of $\widetilde{P}_n$ that is a signed full transformation on X_n. This follows to say that the results in Theorem 3.1 and 3.2 above coincides with the result of [3] corresponding to Theorems 2.1 and 2.2 above respectively. Their proofs which can be found in [3] is omitted here to avoid unnecessary duplication.

This being so, we shall directly adopt the example of [3] to concretize the idea of these results.

Example 3.1:

$$\text{Let } \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -5 & -5 & -5 & -5 \end{pmatrix} \in \widetilde{P}_5.$$

Now,

$$w(\alpha) = \sum_{i=1}^{n} |i - i\alpha| = |1-(-5)| + |2-(-5)| + |3-(-5)| + |4-(-5)| + |5-(-5)|$$
$$= 6+7+8+9+10 = 40.$$

Since $n = 5$, we can verify Theorem 2.2,

$$w(\alpha) = \frac{3n^2 + n}{2} = \frac{3 \times 5^2 + 5}{2} = \frac{80}{2} = 40.$$

Consider $\widetilde{SP}_n$, a subset of $\widetilde{P}_n$ for which every $\alpha \in \widetilde{SP}_n$ is such that $|dom(\alpha)| \le n-1$.

Theorem 3.3: Let $\alpha \in \widetilde{SP}_n$. α Performs maximum work in $\widetilde{SP}_n$ if and only if $dom(\alpha) = X_n \setminus \{1\}$ and $i\alpha = -n$.

Proof:

Following the proof of Theorem 3.1, any map $\alpha \in \widetilde{SP}_n$ that will attain maximum must be a map for which $|dom(\alpha)| = n-1$. Again, since the map in Theorem 3.1 is one that attain such maximum in $\widetilde{P}_n$, then with respect to $\widetilde{SP}_n$ any map that must attain maximum must be such that $dom(\alpha) = X_n \setminus \{1\}$. The image of such map must therefore be $-n \ \forall\, i \in dom(\alpha)$.

Conversely, if $\alpha \in \widetilde{SP}_n$ is such that $dom(\alpha) = X_n \setminus \{1\}$ and $i\alpha = -n$, then, for each $i \in dom(\alpha)$, $|i - i\alpha|$ will be at maximum. Thus, $w(\alpha)$ will be maximum. And so $\alpha \in \widetilde{SP}_n$ performs maximum work in $\widetilde{SP}_n$.

Next, we deduce a formula that captures the value of the maximum work that the map in Theorem 3.3 attains.

Theorem 3.4: Suppose α in $\widetilde{SP}_n$ is such as in Theorem 3.3, then, $w(\alpha) = \dfrac{(3n+2)(n-1)}{2}$.

Proof:

From the map in Theorem 3.3,

$$
\begin{aligned}
w(\alpha) &= \sum_{i=2}^{n} | i - i\alpha | \\
&= \sum_{i=2}^{n} | i - (-n) | \\
&= \sum_{i=2}^{n} | i + n | = \sum_{i=2}^{n} | n + i | \\
&= (n+2) + (n+3) + (n+4) + \ldots + 2n \\
&= n + n + n + \ldots + 2 + 3 + 4 + 5 + \ldots \\
&= (n^2 - n) + \frac{n(n+1)}{2} - 1 \\
&= \frac{(3n+2)(n-1)}{2}.
\end{aligned}
$$

Example 3.2:

Let $\alpha = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ -7 & -7 & -7 & -7 & -7 & -7 \end{pmatrix} \in \widetilde{SP}_7$.

Now,

$$
\begin{aligned}
w(\alpha) &= \sum_{i=2}^{n} | i - i\alpha | = | 2-(-7) | + | 3-(-7) | + | 4-(-7) | + | 5-(-7) | + | 6-(-7) | + | 7-(-7) | \\
&= 9 + 10 + 11 + 12 + 13 + 14 = 69.
\end{aligned}
$$

Since $n = 7$, we can verify Theorem 3.4,

$$w(\alpha) = \frac{(3n+2)(n-1)}{2} = \frac{(3\times 7+2)(7-1)}{2} = \frac{138}{2} = 69.$$

We conclude this section with a result in the class of maps in $\widetilde{SP}_n$ for which $i\alpha < 0$ $(\forall i \in dom(\alpha))$ and $| dom(\alpha) | = n-1$.

Theorem 3.5: If α in $\widetilde{SP}_n$ is such that $i\alpha < 0$ $\forall i \in dom(\alpha)$ and $| dom(\alpha) | = n-1$, then such α will attain minimum work if $dom(\alpha) = X_n \setminus \{n\}$ and $i\alpha = -1$. Moreover, the value for this minimum will be $\frac{(n-1)(n+2)}{2}$.

Proof:

Given that $\alpha \in \widetilde{SP}_n$ is such that $i\alpha < 0\ \forall i \in dom(\alpha)$ and $|dom(\alpha)| = n-1$, then considering Theorem 3.3, such α can only attain minimum work in $\widetilde{SP}_n$ only if $dom(\alpha) = X_n \setminus \{n\}$. The image of such map must therefore be $-1\ \forall\, i \in dom(\alpha)$.

Conversely, if $\alpha \in \widetilde{SP}_n$ is such that $dom(\alpha) = X_n \setminus \{n\}$ and $i\alpha = -1$, then, for each $i \in dom(\alpha)$, $|i - i\alpha|$ will be at its minimum. Thus, $w(\alpha)$ will be minimum. And so such $\alpha \in \widetilde{SP}_n$ performs minimum work in $\widetilde{SP}_n$.

Moreover, from the map in Theorem 3.5,

$$\begin{aligned}
w(\alpha) &= \sum_{i=1}^{n-1} |i - i\alpha| \\
&= \sum_{i=1}^{n-1} |i - (-1)| \\
&= \sum_{i=1}^{n-1} |i+1| = \sum_{i=2}^{n} |1+i| \\
&= (1+1)+(1+2)+(1+3)+\ldots+(1+(n-1)) \\
&= 1+1+1+\ldots 1+2+3+\ldots+(n-1) \\
&= (n-1) + \frac{n(n+1)}{2} - n \\
&= \frac{n(n+1)}{2} - n + (n-1) \\
&= \frac{(n-1)(n+2)}{2}.
\end{aligned}$$

Example 3.3:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix} \in \widetilde{SP}_6$.

Now,

$$\begin{aligned}
w(\alpha) = \sum_{i=2}^{n} |i - i\alpha| &= |1-(-1)| + |2-(-1)| + |3-(-1)| + |4-(-1)| + |5-(-1)| \\
&= 2+3+4+5+6 = 20.
\end{aligned}$$

Since $n = 6$, we can verify Theorem 3.5,

$$w(\alpha) = \frac{(n-1)(n+2)}{2} = \frac{(6-1)(6+2)}{2} = \frac{40}{2} = 20.$$

Remark 3.1: From Theorem 3.4 and 3.5, the range of values of the work performed by all $\alpha \in S\widetilde{P}_n$ whose domain size is $n-1$ will be:

$$\frac{(n-1)(n+2)}{2} \leq w(\alpha) \leq \frac{(3n+2)(n-1)}{2}.$$

Conclusion

This study of maximum work performed by elements of T_n as studied in [8] and that of $\widetilde{T}_n$ as studied by [3] will all collapse into this present work. It is therefore safe to say that with respect to the maximum work performed by T_n and $\widetilde{T}_n$ as studied independently, this present work generalizes both. This generalization extends even to the range of values for maximum work performed by all $\alpha \in \widetilde{P}_n$. The range presented by [3] to be $0 \leq w(\alpha) \leq \frac{3n^2+n}{2}$ will be the range in $\widetilde{P}_n$. This therefore means that for any natural number n, no map will have a work greater than $\frac{3n^2+n}{2}$.

Competing Interests

The authors declare that there are no competing Interests

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