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Graph-Theoretic Characterization of Quasi-Nilpotent Elements in Finite Semigroups of Full Order-Preserving Transformations

Eze C.[1*], Olaiya O. O.[2], S. Kasim[3]
[1,2]National Mathematical Centre Abuja, Nigeria; [3]Ahmadu Bello University Zaria, Nigeria
ezewisdom8@gmail.com; oolaiya@nmc.edu.ng

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Abstract

This paper investigates the structural behavior of quasi-nilpotent ele- ments within the semigroup O_n of all full order-preserving transformations on a finite chain $X_n = \{1, 2, . . ., n\}$. While quasi-nilpotency has been extensively studied in full and partial transformation semigroups, its characterization in O_n remains largely unex- plored. By employing a graph-theoretic approach, we associate to each transformation $a \in O_n$ a digraph $\Gamma(a)$ and establish necessary and sufficient conditions under which a is quasi-nilpotent. Specifically, we show that a is quasi-nilpotent if and only if $\Gamma(a)$ has a unique sink and all vertices are connected to it via directed paths. This char- acterization is further refined by relating the height of $\Gamma(a)$ to the number of convex blocks in the domain partition of a. Illustrative examples and explicit constructions are provided to validate the theoretical findings. The results offer new insights into the interplay between algebraic properties of transformation semigroups and their combi- natorial representations.

Keywords: Quasi-Nilpotent; Order-Preserving Transformation; Functional Digraph

Introduction

Semigroup theory plays a foundational role in modern algebra, with wide-ranging appli- cations in automata theory, combinatorics, and theoretical computer science. Among the most studied classes of semigroups are transformation semigroups—sets of map- pings closed under composition [3]. A prominent subclass is the full order-preserving transformation semigroup, denoted by O_n, comprising all functions on a finite chain $X_n = \{1, 2, \ldots, n\}$ that preserve the natural order [9]. The rich structural properties of these semigroups have been extensively investigated, particularly through the study of key elements such as idempotents [10], nilpotents [11], and quasi-idempotents [8]. This work focuses on an important but less explored class of elements—*quasi-nilpotent elements*—which generalize nilpotents by requiring that some power of the transforma- tion maps all elements to a constant. Although quasi-nilpotents have been examined in symmetric inverse semigroups, partial transformation semigroups, and full transforma- tion semigroups [11], their presence and behavior within the context of the full order-preserving transformation semigroup O_n remain largely uncharted. To address this gap, we propose a *graph-theoretic approach* to characterizing quasi-nilpotent elements in O_n.

By representing transformations as digraphs and analyzing their structural evolution under iteration, we aim to uncover combinatorial patterns and necessary conditions for quasi-nilpotency. This method not only provides visual and structural insight but also facilitates algorithmic identification of such elements. The graph-theoretic character- ization contributes a novel perspective to the understanding of transformation semi- groups and opens pathways for applications in formal language theory, combinatorial optimization, and the theory of finite dynamical systems.

Preliminaries

In this section, we introduce fundamental definitions and concepts relevant to the study of quasi-nilpotent elements in the semigroup of full order-preserving transformations.

Definition 2.1. [3] A **semigroup** is a non-empty set S equipped with an associative binary operation $\cdot$, that is, for all $a, b, c \in S$, we have:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Definition 2.2. [3] Let $X_n = \{1, 2, \ldots, n\}$ be a finite set. The **full transformation semigroup**, denoted by T_n, is the set of all mappings from X_n to itself under the operation of function composition.

Definition 2.3. [3] A transformation $a : X_n \rightarrow X_n$ is said to be **order-preserving** if for all $a, b \in X_n$ such that $a \leq b$, it follows that $a(a) \leq a(b)$. The set of all order- preserving transformations on X_n forms a semigroup, denoted by O_n.

Definition 2.4. [3] Let S be a semigroup. Two elements $a, \beta \in S$ belong to the same J**-class** if they generate the same principal ideal, i.e., if:

$$SaS = S\beta S.$$

Definition 2.5. [3] Given a transformation $a \in O_n$, an element $x \in X_n$ is called a **fixed point** of a if:

$$a(x) = x.$$

The set of all fixed points of a is denoted by:

$$\text{fix}(a) = \{x \in X_n \mid a(x) = x\}.$$

Definition 2.6. [3] For a transformation $a \in O_n$ and an element $x \in X_n$, the **preim- age** of x under a is defined as:

$$xa^{-1} = \{y \in X_n \mid a(y) = x\}.$$

Definition 2.7. [6] An element $a \in O_n$ is called a **quasi-nilpotent** if there exists a positive integer p such that a^p is a constant mapping, i.e.,

$$a^p(x) = c, \quad \forall x \in X_n,$$

where c is a fixed point.

Definition 2.8. [6] A transformation $a \in O_n$ is called a **contraction mapping** if it is both increasing and decreasing, meaning that there exists a unique fixed point x such that for all $y \in X_n$, there exists an integer $m \geq 0$ satisfying:

$$ya^m = x.$$

Definition 2.9. [1] A transformation ξ can be represented as a **directed graph** $G(\xi)$, where the vertices are the elements of $\{1, 2, . . . , n\}$, and there is a directed edge (i, j) if $\xi(i) = j$. The graph may contain cycles (fixed points) and chains leading to cycles.

Definition 2.10. [1] The **digraph** of a transformation $a \in T_n$ is the functional directed graph $\Gamma = (V, E)$ where $V = X_n$ and $E = \{(x, a(x)) \mid x \in X_n\}$.

Definition 2.11. [2] The *in-degree* of a vertex y, denoted as $\deg^-(y)$, refers to the number of arcs of the form (x, y), and the *out-degree* of y, denoted as $\deg^+(y)$, refers to the number of arcs of the form (y, x). If $\deg^-(y) = 0$, then y is called a source; if $\deg^+(y) = 0$, then y is called a sink.

Results

Theorem 3.1. Let $a \in O_n$ be an order-preserving transformation on $X_n = \{1, 2, . . . , n\}$, and let $\Gamma(a)$ denote its associated digraph with vertex set X_n and edge set $E = \{(x, xa) : x \in X_n\}$. Then, a is quasi-nilpotent if and only if the following equivalent graph-theoretic conditions hold:

- **(i)** $\Gamma(a)$ contains a unique sink vertex $v^* \in X_n$ (i.e., a vertex with no outgoing edges).
- **(ii)** For every vertex $u \in X_n$, there exists a directed path from u to v^*.
- **(iii)** The height of $\Gamma(a)$, defined as the length of the longest directed path, equals the number of blocks in the domain partition of a.

Proof. Assume a is quasi-nilpotent. Then, by definition, there exists a positive integer k such that a^k is a constant transformation; that is, for all $x \in X_n$, $xa^k = c$ for some $c \in X_n$. Let $\Gamma(a)$ be the digraph of a. Each vertex $x \in X_n$ has an edge to xa, and by composing a, we trace the directed path of x as it moves through successive images under iteration. Since a^k is constant, all such paths must terminate at the same vertex c after k steps. This implies that:

c is a **sink** in $\Gamma(a)$ (it has no outgoing edge since $ca = c$),

There is a directed path from each $x \in X_n$ to c,

No other vertex can serve as a terminal point, so c is the **unique sink**.

Hence, conditions **(i)** and **(ii)** hold. Now, consider the block structure of a. Let the domain of a be partitioned into r convex blocks, each mapped to a single point in an order-preserving way. Since each block maps to an element in a higher block in the chain (except possibly the last), the maximum number of iterations before all points reach c is r, giving the maximal path length in $\Gamma(a)$. Thus, the height of the digraph equals the number of blocks, establishing condition **(iii)**.

Conversely, assume conditions **(i)** and **(ii)** hold. Then, all paths in $\Gamma(a)$ terminate at a unique sink vertex v^*. This implies that repeated application of a maps every $x \in X_n$ into v^*. Thus, there exists some positive integer k such that a^k is a constant transformation. Since a is also order-preserving by construction, it follows that a is quasi-nilpotent. Finally, if **(iii)** holds, then the digraph has finite height equal to

the number of blocks in the domain of a. Since each block maps into the next and eventually into v^*, the path lengths are bounded by the block structure. This aligns with the contraction behavior of quasi-nilpotent elements. Hence, all three conditions are equivalent. □

Example 3.2. Let $a \in O_n$ for $n \geq 2$. For $n = 12$ and $r = 5$, consider the following mappings:

Case 1:

$$\alpha_1 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 2 & 3 & 7 & 8 & 9 \end{pmatrix}$$

Since $\text{fix}(a_1) = 2$, it follows that a_1 is **not** a quasi-nilpotent element.

Case 2:

$$\alpha_2 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 4 & 7 & 8 & x & 9 \end{pmatrix}$$

Here, α_2 is also **not** a quasi-nilpotent element since $\alpha_2 \neq 5$.

Case 3:

$$\alpha_3 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

in O_{12}. The transformation α_3 has the fixed block $\{4, 5, 6\}$ and shift blocks:

$$\{1, 2, 3\}, \quad \{7, 8\}, \quad \{9\}, \quad \{10, 11, 12\}.$$

Corollary 3.3. Let $\alpha_1, \alpha_2, \alpha_3 \in O_{12}$ be defined by their respective mappings as in example above. Then;

α_1 is not quasi-nilpotent.

α_2 is not quasi-nilpotent.

α_3 is quasi-nilpotent.

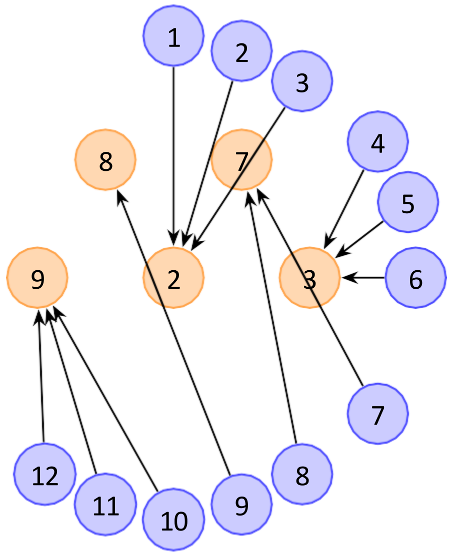
Proof. We analyze each transformation using the digraph $\Gamma(\alpha_i)$ corresponding to α_i for $i = 1, 2, 3$.

Case 1: α_1

$$\alpha_1 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 2 & 3 & 7 & 8 & 9 \end{pmatrix}$$

In $\Gamma(\alpha_1)$, the elements map as follows:

$$1, 2, 3 \to 2; \quad 4, 5, 6 \to 3; \quad 7, 8 \to 7; \quad 9 \to 8; \quad 10, 11, 12 \to 9$$



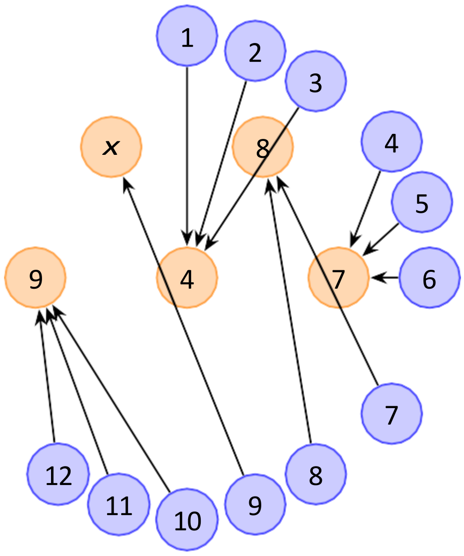
There is no single vertex to which all paths converge. Multiple vertices act as terminal points: 7, 8, and 9. Thus, $\Gamma(a_1)$ has multiple sinks and lacks a unique convergence structure. Therefore, by Theorem 1, a_1 is not quasi-nilpotent.

Case 2: a_2

$$\alpha_2 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 4 & 7 & 8 & x & 9 \end{pmatrix}$$

This mapping structure gives rise to disjoint paths:

$$1, 2, 3 \to 4; \quad 4, 5, 6 \to 7; \quad 7, 8 \to 8; \quad 10, 11, 12 \to 9; \quad 9 \to x$$



Again, there is no unique terminal vertex. Moreover, the image x of 9 is undefined or ambiguous, violating closure and convergence conditions. Thus, $\Gamma(\alpha_2)$ fails to satisfy conditions (i) and (ii) of Theorem 3.1, so α_2 is not quasi-nilpotent.

Case 3: α_3

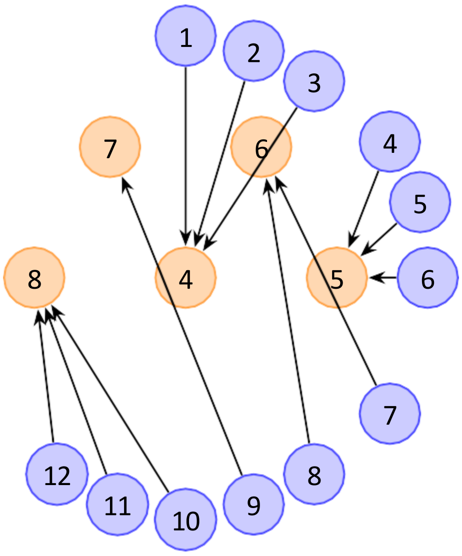
$$\alpha_3 = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8\} & \{9] & \{10,11,12\} \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

This transformation defines a chain-like structure in $\Gamma(\alpha_3)$:

$$1, 2, 3 \rightarrow 4; \quad 4, 5, 6 \rightarrow 5; \quad 7, 8 \rightarrow 6; \quad 9 \rightarrow 7; \quad 10, 11, 12 \rightarrow 8$$

By tracing paths:

$$1 \rightarrow 4 \rightarrow 5 \rightarrow 5; \;\; 9 \rightarrow 7 \rightarrow 6 \rightarrow 5; \;\; 10 \rightarrow 8 \rightarrow 6 \rightarrow 5$$



All paths terminate at vertex 5, which serves as the unique sink. Every vertex has a path to 5, and the height of the digraph equals the number of domain blocks (i.e., 5). Thus, conditions (i), (ii), and (iii) of Theorem 1 are satisfied. Hence, a_3 is quasi-nilpotent.

Conclusion

In this paper, we have presented a graph-theoretic characterization of quasi-nilpotent elements in the semigroup O_n of full order-preserving transformations on a finite chain. By associating each transformation with a directed graph, we established that a trans- formation $a \in O_n$ is quasi-nilpotent if and only if its digraph $\Gamma(a)$ has a unique sink vertex and every other vertex has a directed path leading to it. Additionally, we showed that the height of such a digraph corresponds to the number of domain blocks in the partition structure of a. This novel perspective bridges algebraic properties of trans- formations with combinatorial structures, offering a new tool for analyzing semigroup elements through graphical representations. The findings enhance our understanding of dynamic behaviors under iteration and provide a clear criterion for identifying quasi- nilpotency in O_n. Future work may extend this approach to other transformation semigroups, such as partial order-preserving transformations and monotone mappings, or investigate algorithmic methods for detecting quasi-nilpotency via graph traversal techniques.

Moreover, the interplay between fixed points, contraction mappings, and graph connectivity remains a promising area for deeper exploration.

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