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# Solution of System of Volterra Integral Equations Using the Complex Sadiq Emad Eman Integral Transform

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## Abstract

Various analytical methods have been developed for solving systems of Volterra integral equations of both the first and second kind. In this study, we adopt and apply the complex Sadiq Eman Emad (SEE) integral transform as a novel approach for obtaining solutions to such systems. The complex SEE transform provides an effective framework for simplifying and solving integral equations through the use of operational techniques. This research explores the theoretical formulation of the transform, its properties including convolution and inverse operations and demonstrates its application through illustrative examples. The results confirm that the complex SEE integral transform offers a practical and efficient alternative for solving systems of Volterra integral equations, highlighting its potential for broader use in mathematical and engineering problems involving integral operators.

**Keywords:** System of Volterra integral equations; Complex SEE integral transform; Convolution; Inverse complex SEE.

## Introduction

The Volterra integral equations were formulated by the Italian mathematician named Vito Volterra, in his work dating from 1887. Volterra studied the hereditary influences when he was examining a population growth model. His work resulted in a kind of equation, where both differential and integral operators appeared together in the same equation $f(x) = \lambda \int_0^{(x)} K_i(x,t)\, u(t) dt$ Wazwaz A. M., (1997).

The vast importance of Volterra integral equations encouraged mathematicians to provide as many as possible methods to solve them.

System of Volterra integral equations appear in scientific applications in chemistry, engineering, physics and population growth models. Studies of system of integral equations have attracted much concern in applied sciences. The general ideas and the essential features of these systems are of wide applicability.

The system of Volterra integral equations appear in two kinds. For system of Volterra integral equations of the first kind, the unknown functions appear only under the integral sign while for system of Volterra integral equations of the second kind, the unknown functions appear inside and outside the integral sign. In this research work we will be working with the equation of the first kind in the form:

$$g_1(x) = \int_0^x \left(K_1(x,t)u(t) + \widetilde{K_1}(x,t)v(t) + \cdots\right) dt$$
$$g_2(x) = \int_0^x \left(K_2(x,t)u(t) + \widetilde{K_2}(x,t)v(t) + \cdots\right) dt$$
…(1)

Where $K_1(x,t),\ \widetilde{K_1}(x,t),\ K_2(x,t),\ \widetilde{K_2}(x,t)$ are the kernels, $u(t),\ v(t)$ are the unknown functions and $g_1(x),\ g_2(x)$ are known functions.

The Complex SEE (Complex Sadiq-Emad-Eman) is a new complex integral transform obtained by Eman et al (2021) which has wide application in areas such as calculus, physics, engineering e.t.c.

## Complex SEE Integral Transform

The complex SEE integral transform is denoted by the operator $\varphi[*\,]$ and defined as;

$$\varphi[*\,] = \frac{1}{r^m}\int_0^\infty g(t)e^{-irt}dt\,,\ \ t \geq 0,\ \ b_1 \leq r \leq b_2,\ \ m \in \mathbb{Z} \qquad \ldots(2)$$

Where $(ir)$ is used to factor the variable $t$ in the argument of the function $g(t)$.

**Note:** The sufficient condition for this new complex integral transform to be applied is that the function $g(t)$ must be piecewise continuous and of exponential order.

**Complex Integral Transform of Some Frequently Used Functions**

**(i)** $\varphi[C] = -\frac{iC}{r^{m+1}}$ , where $g(t) = C$ is any constant number.

**(ii)** $\varphi[t] = -\frac{1}{r^{m+2}}$

**(iii)** $\varphi[t^2] = \frac{(2!)i}{r^{m+3}}$

**(iv)** $\varphi[t^3] = \frac{(3!)i}{r^{m+4}}$ , where $g(t) = t, t^2, t^3$ is a polynomial function

**(v)** $\varphi[e^{at}] = \frac{1}{r^m}\left[\frac{a}{a^2+r^2} + i\frac{r}{a^2+r^2}\right]$

**(vi)** $\varphi[\sin(at)] = \frac{-a}{r^m(r^2-a^2)}$ …(3)

**(vii)** $\varphi[\sin(at)] = \frac{-a}{r^m(r^2-a^2)}$

**(viii)** $\varphi[\cos(at)] = \frac{-ir}{r^m(r^2-a^2)}$

**(ix)** $\varphi[\sinh(at)] = \frac{-a}{r^m(r^2+a^2)}$

**(x)** $\varphi[\cosh(at)] = \frac{-ir}{r^m(r^2+a^2)}$

**(xi)** $\varphi\{g_1(x) * g_2(x)\} = r^m G_1(ir).G_2(ir)$ is the convolution property for the complex SEE transform.

**Application**

**Example 1:** Consider the system of Volterra integral equations of the first kind:

$$e^x - 1 = \int_0^x [(x-t)u(t) + (x-t+1)v(t)]dt$$

$$e^{-x} - 1 = \int_0^x [(x-t-1)u(t) + (x-t)v(t)]dt \quad \dots(4)$$

Taking Complex SEE transform of both sides of (4) gives,

$$\varphi\{e^x\} - \varphi\{1\} = \varphi\left\{\int_0^x (x-t)u(t)dt\right\} + \varphi\left\{\int_0^x (x-t+1)v(t)dt\right\}$$

$$\varphi\{e^{-x}\} - \varphi\{1\} = \varphi\left\{\int_0^x (x-t-1)u(t)dt\right\} + \varphi\left\{\int_0^x (x-t)v(t)dt\right\} \quad \dots(5)$$

Applying convolution property of Complex SEE transform on (5) gives,

$$\varphi\{e^x\} - \varphi\{1\} = r^m\varphi\{x\}\varphi\{u(x)\} + r^m\varphi\{1\}\varphi\{x\}\varphi\{v(x)\}$$

$$\varphi\{e^{-x}\} - \varphi\{1\} = r^m\varphi\{-1\}\varphi\{x\}\varphi\{u(x)\} + r^m\varphi\{x\}\varphi\{v(x)\} \qquad \ldots(6)$$

$$\left[\frac{-1}{r^m(1+r^2)} - \frac{-ir}{r^m(1+r^2)}\right] - \left[\frac{-i}{r^{m+1}}\right] = r^m\left[\frac{-1}{r^{m+2}}\right]\varphi\{u(x)\} + r^m\left[\frac{-i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{v(x)\}$$

$$\left[\frac{-1}{r^m(1+r^2)} - \frac{-ir}{r^m(1+r^2)}\right] - \left[\frac{-i}{r^{m+1}}\right] = r^m\left[\frac{i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{u(x)\} + r^m\left[\frac{-1}{r^{m+2}}\right]\varphi\{v(x)\}$$

$$\frac{-r+i}{r^{m+1}(1+r^2)} = -\frac{1}{r^2}\varphi\{u(x)\} + \frac{i}{r^3}\varphi\{v(x)\}$$

$$\frac{r+i}{r^{m+1}(1+r^2)} = \frac{-i}{r^3}\varphi\{u(x)\} - \frac{1}{r^2}\varphi\{v(x)\} \qquad \ldots(7)$$

Solving (7) simultaneously for $\varphi\{u(x)\}$ and $\varphi\{v(x)\}$ we get,

$$\varphi\{u(x)\} = -\frac{1}{r^m}\left[\frac{1+ir}{1+r^2}\right], \qquad \varphi\{v(x)\} = -\frac{1}{r^m}\left[\frac{-1+ir}{1+r^2}\right]$$

Taking the Complex SEE inverse we get,

$$u(x) = \varphi^{-1}\left\{-\frac{1}{r^m}\left[\frac{1+ir}{1+r^2}\right]\right\}, \qquad v(x) = \varphi^{-1}\left\{-\frac{1}{r^m}\left[\frac{-1+ir}{1+r^2}\right]\right\}$$

$$u(x) = e^x, \qquad v(x) = e^{-x}$$

Which are the exact solutions of the equations in (4).

**Example 2:** Consider the system of Volterra integral equations of the first kind

$$\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{12}x^4 = \int_0^x [(x-t-1)u(t) + (x-t+1)v(t)]dt$$

$$\frac{3}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 = \int_0^x [(x-t+1)u(t) + (x-t-1)v(t)]dt$$

…(8)

Taking Complex SEE transform of both sides of (8) gives,

$$\frac{1}{2}\varphi\{x^2\} + \frac{1}{2}\varphi\{x^3\} + \frac{1}{12}\varphi\{x^4\} = \varphi\left\{\int_0^x (x-t-1)u(t)dt\right\} + \varphi\left\{\int_0^x (x-t+1)v(t)dt\right\}$$

$$\frac{3}{2}\varphi\{x^2\} - \frac{1}{6}\varphi\{x^3\} + \frac{1}{12}\varphi\{x^4\} = \varphi\left\{\int_0^x (x-t+1)u(t)dt\right\} + \varphi\left\{\int_0^x (x-t-1)v(t)dt\right\}$$

…(9)

Applying convolution property of Complex SEE transform on (9) gives,

$$\frac{1}{2}\varphi\{x^2\} + \frac{1}{2}\varphi\{x^3\} + \frac{1}{12}\varphi\{x^4\} = r^m\varphi\{-1\}\varphi\{x\}\varphi\{u(x)\} + r^m\varphi\{1\}\varphi\{x\}\varphi\{v(x)\}$$

$$\frac{3}{2}\varphi\{x^2\}-\frac{1}{6}\varphi\{x^3\}+\frac{1}{12}\varphi\{x^4\}=r^m\varphi\{1\}\varphi\{x\}\varphi\{u(x)\}+r^m\varphi\{-1\}\varphi\{x\}\varphi\{v(x)\}$$

…(10)

$$\frac{1}{2}\left[\frac{2!i}{r^{m+3}}\right]+\frac{1}{2}\left[\frac{3!i}{r^{m+4}}\right]+\frac{1}{12}\left[\frac{-4!i}{r^{m+5}}\right]=r^m\left[\frac{i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{u(x)\}+r^m\left[\frac{-i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{v(x)\}$$

$$\frac{3}{2}\left[\frac{2!i}{r^{m+3}}\right]-\frac{1}{6}\left[\frac{3!i}{r^{m+4}}\right]+\frac{1}{12}\left[\frac{-4!i}{r^{m+5}}\right]=r^m\left[\frac{-i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{u(x)\}+r^m\left[\frac{i}{r^{m+1}}\right]\left[\frac{-1}{r^{m+2}}\right]\varphi\{v(x)\}$$

$$\frac{ir^2+3r-2i}{r^{m+5}}=\frac{-i}{r^3}\varphi\{u(x)\}+\frac{i}{r^3}\varphi\{v(x)\}$$

$$\frac{3r^2-r-2i}{r^{m+5}}=\frac{i}{r^3}\varphi\{u(x)\}+\frac{-i}{r^3}\varphi\{v(x)\}$$

…(11)

Solving (11) simultaneously for $\varphi\{u(x)\}$ and $\varphi\{v(x)\}$ gives,

$$\varphi\{u(x)\}=\left[\frac{-i}{r^{m+1}}-\frac{1}{r^{m+2}}\right],\qquad \varphi\{v(x)\}=\left[\frac{-i}{r^{m+1}}-\frac{2i}{r^{m+3}}\right]$$

Taking the inverse Complex SEE we get,

$$u(x)=\varphi^{-1}\left\{\frac{-i}{r^{m+1}}-\frac{1}{r^{m+2}}\right\},\qquad v(x)=\varphi^{-1}\left\{\frac{-i}{r^{m+1}}-\frac{2i}{r^{m+3}}\right\},$$

$$u(x)=1+x,\qquad v(x)=1+x^2$$

Which are the exact solutions of equations (8).

## Conclusion

The complex SEE integral transform serves as a very important method through demonstrating its strength in solving system of Volterra integral equations of first kind. The steps which are straight forward involved in the process includes;

a) Taking the Complex SEE transform of both sides of the given equation.
b) Applying the property of convolution.
c) Taking the inverse Complex SEE transform which in return gives the exact solution to the given equation.

Due to it simplicity in handling, we recommend that it should be used in solving other kinds of integral equations.

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