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A Class of One-Sixth Hybrid Methods for Direct Solution of Third Order Ordinary Differential Equations

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Abstract

In this paper, a class of Hybrid methods for solving third order ordinary differential equations directly is developed. These methods were derived using interpolation and collocation techniques. The methods were analyzed based on the properties of linear multistep methods and were found to be zero-stable, consistent and convergent with good region of absolute stability. The proposed methods were implemented on higher order ordinary differential initial value problems. The superirity of the proposed methods over existing ones was demonstrated through some numerical examples.

Keywords: Block Method, Hybrid method, Initial Value Problems, One - sixth Step, Order Four, Third Order

INTRODUCTION

The uncertainty property in stock price call for concern on the part of investors and financial managers since the change in stock price occurs frequently and in a random pattern. This fluctuation of stock price is known as random walk movement in finance.

Researchers are therefore derived to look into the unstable and unpredictable market price in order to advise investors and financial managers who are looking for convenient ways to raise money or make profit by issuing shares of stocks.

Two common approaches to predicting stock prices are those based on the theory of technical analysis and those based on the theory of fundamental analysis (Fama, 1995). Technical theorists assume that history repeats itself, that is, past patterns of price behaviors tend to recur in the future. The fundamental analysis approach assumes that at any point in time an individual security has an intrinsic value that depends on the earning potential of the security, meaning some stocks are overpriced or underpriced (Fama, 1995). Random walk and the concept of Martingale tends to contradict these theories.

Changes in stock prices has the same distribution and are independent of each other. Therefore, it assumes that past movements or trends cannot be used to predict the future, if this is not the case, the speculators can beat the market by using the technical analysis of historical series of events. Research focus has mostly been on forecasting stock returns and the predictive accuracy of the geometric Brownian motion. This research work aims to examine whether stock prices contain martingale properties within the geometric Brownian motion framework, with the objectives to examine whether stock prices exhibit randomness on its path making it unpredictable and whether the expected mean and the expected volatility of stock price is an increasing function of time.

The random walk theory is the idea that stocks take a random and unpredictable path, making it near impossible to outperform the market without assuming additional risk. This theory casts serious doubts on the other methods of describing and predicting stock price behaviors. The geometric Brownian motion model incorporates this idea of random walks in stock prices (Reddy and Clinton, 2016). Brewer et at., (2012) describe the uncertain component to the geometric Brownian motion model as the product of the stock's volatility and a stochastic process called Weiner process, which incorporates random volatility and a time interval.

Martingale is a special case of Brownian motion and its interpretation is easy enough. The conditional expected value is influenced neither from the past value nor from any current information, but depends only on the current price. This is equal to assume that the market is efficient such that any available information on the past, present and future is incorporated quickly in the current price. As such, the past prices do not have provisional value in predicting the future price. Modeling stock price changes with Stochastic Differential Equation (SDE) leads to Geometric Brownian Motion model (Samuelson, 1965).

The bedrock of this work is in the study and observation of the Scottish botanist Robert Brown (Bachelier, 1964), who was first to observe and describe the motion of small particles suspended in a liquid as a result of successive and casual impacts of the near particles so as to note if its variance is an increasing function of time (Rossano, 2006). The fundamental discovery of Bachelier in 1900 that prices of risky assets (stock indices, exchange rates and share prices) can be well described by Brownian motion gave birth to a new area of the applications of stochastic processes. Adeosun and Ugbebor (2021) examined the suitability of three stock price models which are; Geometric Brownian motion, symmetric and asymmetric jump-diffusion models on the empirical log-returns of the Nigerian All-Share Index. The suitability analysis results showed that the symmetric jump-diffusion model fits the market indices better. Adolphus and Samuel (2021) in their study used Geometric Brownian Motion to model and simulate the trends and behavioral patterns in the Nigerian Stock Market and hence predict the future stock prices using daily prices for a period of four years (2015-2018). The results showed that in the simulation, there are some actual stock prices located outside trajectory realization that may be from Geometric Brownian Motion model. Thus, the model did not predict accurately the price behavior of some of the listed stocks. It also showed that the predictive power of the model is declining towards the longer the evaluated time frame proven by the higher value of the mean absolute percentage error. Islam and Nguyet (2020) used three different methods, namely autoregressive integrated moving average, artificial neural network, and stochastic process geometric Brownian motion in his study. Each of the methods was used to build predictive models using historical stock data collected from Yahoo Finance and the output from each of the models was compared to the actual stock price. Empirical results showed that the conventional statistical model and the stochastic model provided better approximation for next-day stock price prediction compared to the neural network model. Imoni and Muhammad (2020) applied Geometric Brownian Motion (GBM) model for the prediction of stock prices in the Nigeria Stock Market. They analyzed and examined the appropriateness of the Geometric Brownian Model in Stock price prediction using daily stock price data from Nigeria Stock Exchange (NSE). It was observed that within a short period of time, the Geometric Brownian Motion is effective in precisely predicting the daily stock prices. Rahul and Bidyadhara (2020) forecasted short term return distribution stock index using Geometric Brownian Motion. The Mean Absolute Percentage Error (MAPE) showed that Geometric Brownian Motion model is highly accurate and an appropriate model for forecasting stock price from the Bombay Stock Exchange.

From literatures reviewed, researches focused mostly on forecasting and the predictive accuracy of the geometric Brownian motion. We will go further to test the martingale properties within the geometric Brownian motion framework. The next section is the set up. Section 3 is the methodology; section 3 contains results and discussion. This paper ends with conclusion and recommendations.

MATERIALS AND METHODS

Historical stock price data from the official website of the Nigeria Stock Exchange. The sample period covers a period of five years (2015-2019) collected on a monthly bases. In this work, we chose the All-share Index data as a true picture of the movement of price in the Nigerian stock market, because, it gives the general behavior of all the shares traded on the Nigerian Stock Exchange.

The statistical model used is Geometric Brownian Motion, sometimes referred to as the Exponential Brownian Motion. Given that Brownian motion follows a martingale, its properties would be examined in stock prices using the geometric Brownian motion.

Geometric Brownian Motion

Suppose X is a continuous random variable which follows a lognormal distribution, then $v = \ln X$ is a random variable which is normally distributed with mean μ and variance σ^2 .(Rahul and Bidyadhara, 2020).

Symbolically,

$$v = \ln X \sim N(\mu, \sigma^2) \tag{1}$$

The probability density function for the variable v is given as;

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-1}{2}\left(\frac{v-\mu}{\sigma}\right)^2\right] for -\infty < v, \mu < \infty \text{ and } \sigma > 0 \tag{2}$$

$$for v = \ln X$$

Where $\mu$and σ^2 represents mean and variance of lognormal variable X.

$$E(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-1}{2}\left(\frac{ln\,x-\mu}{\sigma}\right)^2\right] dx \tag{3}$$

If $\quad y = ln\,x - \mu, dy = \frac{1}{x} dx$ and equation (3) becomes

$$E(e^{y+\mu}) = \int_{-\infty}^{\infty} \frac{1}{\sigma x\sqrt{2\pi}} exp(y+\mu)\, exp\left[\frac{-1}{2}\left(\frac{ln\,x-\mu}{\sigma}\right)\right] dy \tag{4}$$

$$= e^{\mu} e^{\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-1}{2}\left(\frac{y-\sigma^2}{\sigma}\right)^2\right] dy$$

If $\quad z = \frac{y-\sigma^2}{\sigma}, dz = \frac{1}{\sigma} dy$ and equation (4) becomes

$$E\left[e^{z\sigma+\sigma^2}\right] = exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left[\frac{-z^2}{2}\right] dz \tag{5}$$

The integral is a probability density function of standard normal distribution subject to the conditions $\int ln\,x = -\infty \Rightarrow y = -\infty$ and $ln\,x = +\infty \Rightarrow y = +\infty$ in equation (4) and $\int y = -\infty \Rightarrow z = -\infty$and $y = +\infty \Rightarrow z = +\infty$ in equation (5).

The expected stock price is given by;

$$S_t = S_0\, exp\left[\left(\mu + \frac{1}{2}\sigma^2\right)\right] t + \sigma B_t \tag{6}$$

is called the Geometric Brownian Motion with drift. Where;

S_0 is the Actual beginning stock price.

μ is Mean of lognormal distribution.

σ^2 is Variance of lognormal distribution.

$B_t\; is$ Brownian motion at time 't' with $\mu = 0$ and defined as,

$$B_t = \mu_t + \sigma W_t \tag{7}$$

Where, W_t is Wiener process at time 't'.

Now $\quad E(B_t) = \mu_t + E(\sigma W_t) = \mu_t$ as $E(W_t) = 0$ and

$$Var(B_t) = E(B_t)^2 - [E(B_t)^2] = {\sigma_t}^2 \tag{8}$$

Hence, Brownian Motion is normally distributed with mean μ_t and variance ${\sigma_t}^2$. Symbolically;

$$B_t \tilde{} N(\mu_t, {\sigma_t}^2)$$

Thus, $ln\, S_t \tilde{} N\left(ln\, S_{t-1} + \left(\mu - \frac{1}{2}\sigma^2\right)t, {\sigma_t}^2\right)$

Hence, expected stock price at time 't' is

$$E(S_t) = S_0\, exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t\right] \text{ and} \quad (9)$$

$$Var(S_t) = {S_0}^2\, exp(2\mu + \sigma t^2)\,[exp(\sigma t^2) - 1] \quad (10)$$

With 95% confidence interval, S_t becomes;

$$exp\left[ln\, S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t - 1.96\sigma\sqrt{t}\right] \leq S_t \leq exp\left[ln\, S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + 1.96\sigma\sqrt{t}\right]$$

Estimation of Volatility and Drift

In developing the random walk algorithm, the volatility (σ) and drift (μ) of the historical stock price has to be estimated.

The volatility is given as (Adeosun *et al.*, 2015)

$$\mu_i = ln\left(\frac{S_i}{S_{i-1}}\right) \quad (11)$$

$$\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i \quad (12)$$

$$v = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \quad (13)$$

Where;

$S_i$is the stock price at the end of ith trading period, $u_i$is the logarithm of the monthly return on the stock over a time interval,$\bar{u}$is the unbiased estimator of the log returns (u_i), $v$is the standard deviation, and σ is the volatility of the monthly stock return.

The drift (μ) is given by (Adeosun et al., 2015);

$$\bar{u} = \left(\mu - \frac{1}{2}\sigma^2\right)\tau \Rightarrow \mu = \bar{u} + \frac{1}{2}\sigma^2 \quad (14)$$

Measure of Accuracy

The Mean Absolute Percentage Error (MAPE) is one of the most popular measures of the estimation accuracy due to its advantages of scale-independency and interpretability (Hanke and Reitsch, 1995). This measure is generally only used when quantity of interest is strictly positive and it is given as (Rahul and Bidyadhara, 2020);

$$\mathrm{M} = \frac{1}{N}\sum_{t=1}^{n}\left(\frac{\mathrm{A}_t - F_t}{\mathrm{A}_t}\right) \tag{15}$$

Where, A_t and F_t denotes the actual and estimated value at specified time 't' respectively, and N is the number of data points.

Table 1: A scale of judgment

MAPE	Prediction Accuracy
< 10%	Highly accurate
11% - 20%	Good prediction
21% - 50%	Reasonable Estimation
> 51%	Inaccurate Estimation

Source: Abidin and Jaffar (2014)

RESULTS AND DISCUSSION

Exploratory Data Analysis

To explore and gain basic understanding of the stock price data, the summary statistics and visualization plots and normality assumption test has been presented below.

Table 2. Log Stock Price Summary Statistics

Statistics	Value
Mean	9.326188708
Standard Error	0.009693697
Q3	9.472145236
Median	9.345295391
Mode	#N/A
Stand. Deviation	0.075087057

Sample Variance	0.005638066
Kurtosis	- 1.016743133
Skewness	0.010408839
Range	0.286137581
Q1	9.180240154
Minimum	9.015146995
Maximum	9.672284576

From Table 2, the mean and median (9.3262 and 9.3453 respectively) of the log of stock price for the period of 60 months (5 years) from 2015 to 2019 shows slight variations while the minimum and maximum values are 9.0151 and 9.6723 respectively. The standard deviation obtained as 0.0751 shows a little spread in the data from the mean. A value of skewedness approximately zero shows a distribution is approximately symmetric, hence, our skewedness of 0.0104 shows the log stock price is approximately normal. The kurtosis reveals a distribution with flat tails which indicates a small outlier in the distribution. The normality plot in Figure 2 and test give more clarity picture.

The box-plot in Figure 1 is presented to detect if data falls within the normality limit or to show the presence of outlier(s) in stock price data.

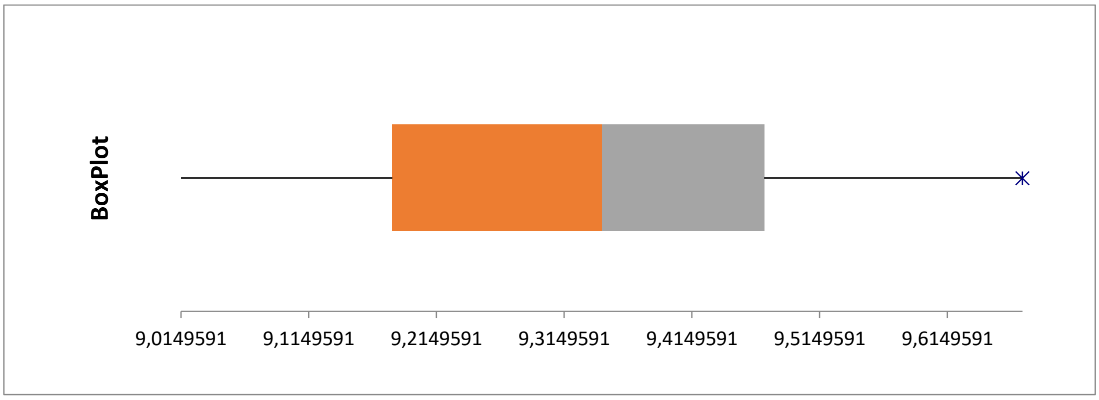

Figure 1: Box Plot of Stock price

From Figure1, we observed the presence of an outlier in the data, which is obviously the maximum value from the summary statistics, that is, 9.6723. To determine if the data is normally distributed, the Histogram/density plot is given in Figure 2 followed by the One Sample Kolmogorov-Smirnov test of normality.

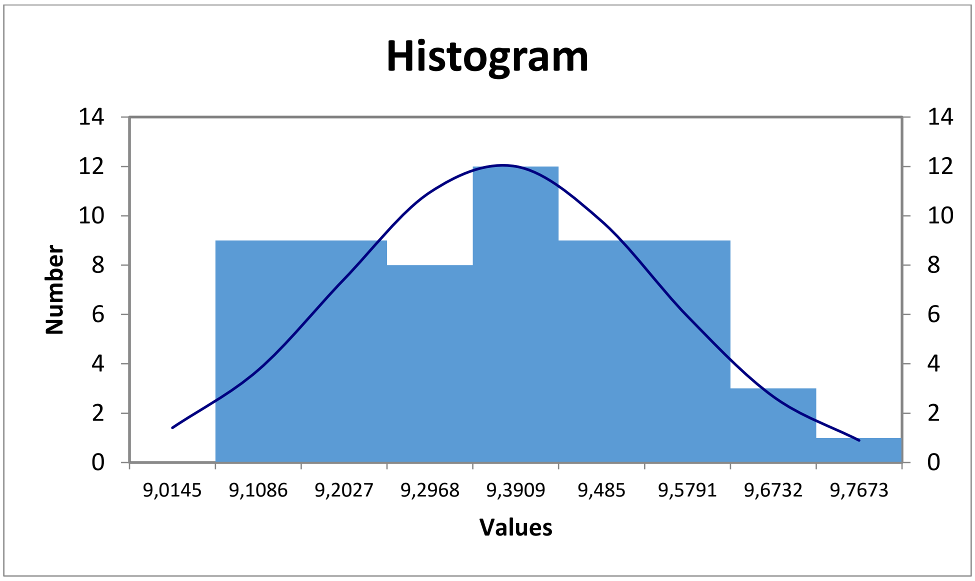

Figure 2: Histogram and Probability Plot of Log Stock Price

One-sample Kolmogorov-Smirnov test

D = 0.095, p-value = 0.617

The plot shown in Figure 2 displays a normally distributed data backed-up by the K-S test which gives a p-value of $0.617 > 0.05$ level of significance. This is an indication that the log stock price data is normally distributed, or rather, since the D-statistics is lesser than the p-value, it is reasonable to conclude that the log of the stock price data follows a normal distribution which satisfies the assumption of a Geometric Brownian Motion model that the log of stock prices follows a normal distribution.

Expected Stock Price using GBM Model

The stock price from the Nigerian Stock Exchange expected values were estimated considering the lognormal drift and volatility. The outcome of the result from the geometric Brownian motion model is shown in Table 3. Measure of accuracy using the mean absolute percentage error was recorded to determine the accuracy and error in the estimation from the model.

Table 3: Expected Stock Price using Geometric Brownian Motion Model

Jan'15 - Dec'19 data Based (60 months)			
Months	**Actual Price**	**Expected Price**	**Absolute Percentage Error**
1	9,846.63	9,892.63	0.47
2	10,044.55	9,892.56	1.51
3	10,717.53	9,892.48	7.70
4	11,786.95	9,892.42	16.07
5	11,658.81	9,892.35	15.15
6	11,421.02	9,892.28	13.39
7	10,344.42	9,892.20	4.37
8	10,336.86	9,892.16	4.30
9	10,728.90	9,892.07	7.80
10	10,027.78	9,892.00	1.35
11	9,495.50	9,891.93	4.17
12	9,850.61	9,891.85	0.42
13	8,225.21	8,262.91	0.46
14	8,452.46	8,262.86	2.24
15	8,704.87	8,262.80	5.08
16	8,621.01	8,262.72	4.16
17	9,500.90	8,262.68	13.03
18	10,165.34	8,262.63	18.72
19	9,619.99	8,262.55	14.11
20	9,478.87	8,262.52	12.83
21	9,733.37	8,262.46	15.11
22	9,349.56	8,262.39	11.63
23	8,720.80	8,262.31	5.26
24	9,246.92	8,262.27	10.65
25	8,972.90	9,013.27	0.45
26	8,765.92	9,013.21	2.82
27	8,828.96	9,013.13	2.09
28	8,912.90	9,013.05	1.12
29	10,197.73	9,013.02	11.62
30	11,452.12	9,012.95	21.30
31	12,705.45	9,012.89	29.06
32	12,237.48	9,012.81	26.35
33	12,216.93	9,012.76	26.23
34	12,694.94	9,012.70	29.01
35	13,214.58	9,012.60	31.80
36	13,609.47	9,012.57	33.78
37	15,895.88	15,966.04	0.44
38	15,549.79	15,965.93	2.68
39	14,992.96	15,965.82	6.49

40	14,948.51	15,965.73	6.80
41	13,802.61	15,965.60	15.67
42	13,866.42	15,965.46	15.14
43	13,409.71	15,965.36	19.06
44	12,722.38	15,965.21	25.49
45	11,962.26	15,965.10	33.46
46	11,852.70	15,964.99	34.70
47	11,271.48	15,964.88	41.64
48	11,720.72	15,964.78	36.21
49	11,394.93	11,444.24	0.43
50	11,829.53	11,444.15	3.26
51	11,672.10	11,444.06	1.95
52	10,958.72	11,443.99	4.43
53	13,684.62	11,443.95	16.37
54	13,205.54	11,443.80	13.34
55	13,154.61	11,443.73	13.01
56	13,391.01	11,443.67	14.54
57	13,450.44	11,443.56	14.92
58	12,829.67	11,443.48	10.80
59	13,032.66	11,443.42	12.19
60	12,968.59	11,443.30	11.76
Mean Absolute Percentage Error			**12.67%**

The model accuracy depends on the percentage of the error. The smaller the MAPE value the more accurate the estimate. Table 2 displays a mean percentage absolute error of 12.67% in the estimation of the expected stock price, this present the model as good for this analysis.

The Expected Stock Price as Function of Time

The graph in Figure 3 shows uncertainty and randomness in its path. It also shows that the expected stock price is continuous in time and value depicting roughness and exponential growth. Hence, the Geometric Brownian Motion modelling of theexpected value of stock prices with drift showedit is an increasing function of time.

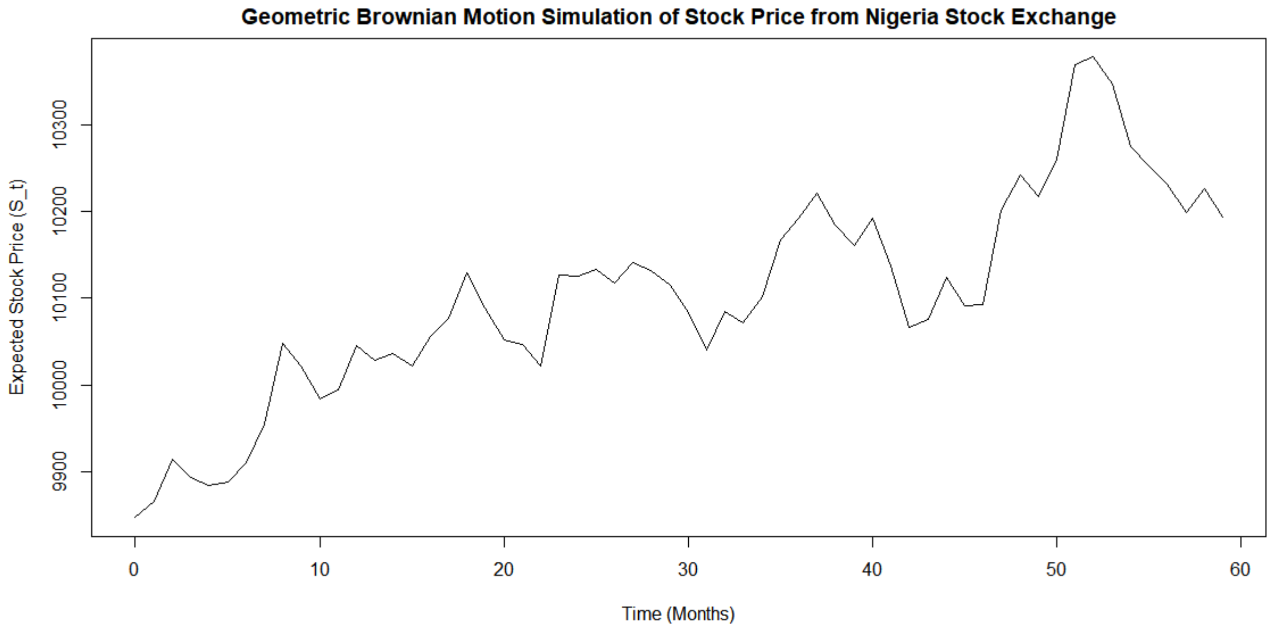

Figure 3: Geometric Brownian Motion of Expected Stock Price

The Expected Volatility

Figure 4 shows a rise, fall and fluctuation on the trend of the graph and there is no certainty to how future market price is expected to perform as every month's price was characterized by randomness, which agrees with the work. The volatility has been the major reason why fundamental and technical analysis has not been able to beat market prices of stocks. The presence ofvolatility in the expected stock price is visible and clear enough, with an increasing trend, displaying roughness and unpredictable properties in its path.

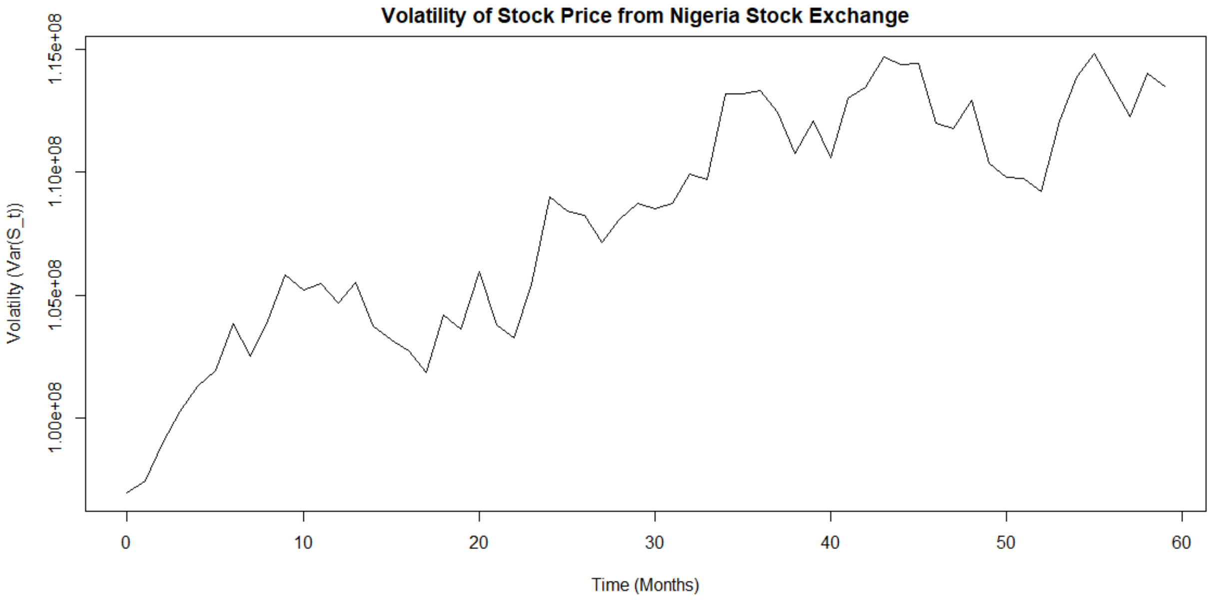

Figure 4: Expected Volatility using Geometric Brownian Motion

CONCLUSION

From the results of this study, the expected stock price and the expected volatility of stock prices are increasing functions of time. Stock price follows the random walk theory and therefore contains martingale property. The results of this research work are consistent with the study by Abraham et al., (2002) who examined the random walk properties and weak form efficiency of three growing up gulf stock markets and Borges (2011) using Lisbon stock market to test for randomness in stocks and concluded that stock market follows a random walk. The results are also consistent with the report of Rossano (2006) that the variance(volatility) in stock prices an increasing function of time.

Recommendations

i. Financial managers and investors should learn to diversify their portfolios to be able to curtail or minimize risk in investment.

ii. Investors need no waste funds in hiring fund managers as most forecast and predictions don't come true as the result of market fluctuations from the historical prices. We need not waste money and hire fund managers to manage our resources or funds.

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