Mikailalsys Journal of
Mathematics and Statistics

e-ISSN : 3030-816X
p-ISSN : 3030-8399
Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJMS.v3i2.5265

Renyi Entropy Derivation for a Modified Skewed Student-*t* Distribution

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Feb 26, 2025	Mar 11, 2025	Mar 23, 2025	Mar 28, 2025

Abstract

This paper derives the Renyi entropy for a modified Skew Student-*t* distribution (SStD). The skew Student-*t* distribution was modified using DUS transformation technique. The final expression of the Renyi entropy was derived using the probability density function of the transformed (SStD).

Keywords: DUS Transformation, Entropy, Renyi entropy, Skew distribution

INTRODUCTION

The work of Shannon (1948) based on papers by Nyquist (1924), Nyquist (1928), and Hartley (1928) on Entropy opened the area of research called Information theory. Shannon states that a measure of the amount of information H(p) contained in a series of events $p_1 \ldots p_N$ should satisfy.

$$H(p) = -K\sum_{i=1}^{N} p_i \ln p_i \tag{1}$$

where, K is a positive constant. This quantity has since become known as the Shannon entropy.

Renyi (1961) defines the entropy of a probability distribution not only as a measure of uncertainty but also as a measure of information. It generalizes various notions of entropy, such as the Shannon Entropy, Hartley entropy, collision entropy, and the min-entropy.

The amount of information of order c concerning a continuous random variable x obtained from observing an event E is defined as:

$$R_c = \frac{1}{1-c} log[E_c(\alpha)], \tag{2}$$

where, $E_c(\alpha) = \int_{-\infty}^{+\infty} f(x,\alpha)^c dx$ and $c > 0, c \neq 0$, and $f(x,\alpha)$ the PDF of distribution.

Each value of c gives a possible entropy measure. All are additive for independent random variables. The, max-entropy (c = 0) for all p_i being positive, the $\boldsymbol{R_c}$ is known as the max-entropy, or Hartley entropy. Shannon entropy (c = 1), when c = 1 the more familiar Shannon entropy is obtained. Collision entropy (c = 2), when the order c is not specified, it's implicit default value is 2, this case is also called collision entropy and is used in quantum information theory. Min-entropy (c = ∞), in the limit as c goes to ∞, the Rényi entropy of *the random variable* converges to the negative log of the probability of the most probable outcome. Derivation of Renyi entropy for different modified distributions have always been done and this can be seen in the researches of David *et al.* (2024a and 2024b), Mathew *et al.* (2024), Adubisi et al. (2023), Eghwerido et al. (2020), etc,.

METHODS

In this section the theoretical Renyi entropy derivation for the skew Student-*t* distribution modified through DUS transformation is presented.

Renyi Entropy Derivation for DUS Skew Student-*t* Distribution (DUS-SStD)

Let *x* be an independent and identically distributed (iid) random variable. The random variable *x* follows a DUS-SStD with a density function given as (Nkombou, 2025):

$$f(x,\alpha) = \frac{1}{e-1}\left[\frac{\alpha}{2(\alpha+x^2)^{3/2}}\right] e^{\left[\frac{1}{2}\left(1+\frac{x}{\sqrt{\alpha+x^2}}\right)\right]}, \quad -\infty < x < +\infty, \alpha > 0. \tag{3}$$

Lemma: The Renyi measure of entropy for a DUS-SStD is given as:

$$R_c = \frac{1}{1-c} log \left[\begin{cases} 0 & \text{for} \quad h\ odd \\ B_{i,h,c,\alpha} \dfrac{\Gamma(\frac{1}{2}(h-1))\Gamma(\frac{1}{2}(3c-1))}{\Gamma(\frac{1}{2}(h+3c))} & \text{for} \quad h\ even \end{cases} \right]$$

where,

$$B_{i,h,c,\alpha} = \frac{1}{(e-1)^c} \sum_{i=0}^{\infty} \sum_{h=0}^{i} \frac{c^i}{i!} (0.5)^{c+1} \binom{h}{i} \alpha^{0.5(h+1+2c)}$$

RESULTS

Proof: Equation (1) defines the Renyi entropy measure and $E_c(\alpha)$ is evaluated by substituting Equation (3) in its expression. This gives;

$$E_c(\alpha) = \int_{-\infty}^{+\infty} [\frac{1}{e-1} \frac{\alpha}{2(\alpha+x^2)^{\frac{3}{2}}} e^{\frac{1}{2}(1+\frac{x}{\sqrt{\alpha+x^2}})}]^c dx$$

$$= \frac{1}{(e-1)^c} \int_{-\infty}^{+\infty} [\frac{\alpha}{2(\alpha+x^2)^{\frac{3}{2}}} e^{\frac{1}{2}(1+\frac{x}{\sqrt{\alpha+x^2}})}]^c dx$$

$$= \frac{1}{(e-1)^c} (\alpha)^c \left(\frac{1}{2}\right)^c \int_{-\infty}^{+\infty} (\alpha+x^2)^{\frac{-3c}{2}} e^{\frac{c}{2}(1+\frac{x}{\sqrt{\alpha+x^2}})} dx$$

$$= \frac{1}{(e-1)^c} (\alpha)^c \left(\frac{1}{2}\right)^c \int_{-\infty}^{+\infty} (\alpha+x^2)^{\frac{-3c}{2}} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{c}{2}\right)^i (1+\frac{x}{\sqrt{\alpha+x^2}})^i dx$$

$$= \frac{1}{(e-1)^c} \sum_{i=0}^{\infty} \alpha^c c^i \left(\frac{1}{2}\right)^{c+i} \frac{1}{i!} \int_{-\infty}^{+\infty} (\alpha+x^2)^{\frac{-3c}{2}} (1+\frac{x}{\sqrt{\alpha+x^2}})^i dx$$

$$= \frac{1}{(e-1)^c} \sum_{i=0}^{\infty} \frac{\alpha^c c^i}{i!} \left(\frac{1}{2}\right)^{c+i} \int_{-\infty}^{+\infty} (\alpha+x^2)^{\frac{-3c}{2}} \sum_{h=0}^{i} \binom{h}{i} x^h (\alpha+x^2)^{\frac{-h}{2}} dx$$

$$= \frac{1}{(e-1)^c} \sum_{i=0}^{\infty} \sum_{h=0}^{i} \frac{\alpha^c c^i}{i!} \left(\frac{1}{2}\right)^{c+i} \binom{h}{i} \int_{-\infty}^{+\infty} (\alpha+x^2)^{-\frac{1}{2}(h+3c)} x^h dx$$

Using the concept of Taboga (2017),

$$\int_{-\infty}^{+\infty} x^n f(x,\alpha) dx = \left[1-(-1)^n\right] \int_0^{+\infty} x^n f(x,\alpha) dx, \tag{4}$$

where, $f(x,\alpha)$ is the PDF of a given probability distribution. Also, let $p=\frac{x^2}{\alpha}$ and $x=(p\alpha)^{\frac{1}{2}}$, by differentiating the expression of w.r.t. x, it yields $dx=\frac{\alpha.dp}{2(p\alpha)^{\frac{1}{2}}}$.

Using the above substitution and Equation (4) in the expression of $E_c(\alpha)$, it becomes:

$$E_c(\alpha) =[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\int_0^{+\infty}x^h(\alpha+x^2)^{-\frac{1}{2}(h+3c)}dx$$

$$=[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{-\frac{1}{2}(h+3c)}\int_0^{+\infty}x^h(1+\frac{x^2}{\alpha})^{-\frac{1}{2}(h+3c)}dx$$

$$=[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{-\frac{1}{2}(h+3c)}\int_0^{+\infty}x^h(1+\frac{x^2}{\alpha})^{-\frac{1}{2}(h+3c)}dx$$

$$=[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{-\frac{1}{2}(h+3c)}\int_0^{+\infty}(p\alpha)^{\frac{h}{2}}(1+p)^{-\frac{1}{2}(h+3c)}\frac{\alpha dp}{2(p\alpha)^{\frac{1}{2}}}$$

$$=[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{-\frac{1}{2}(h+3c)}\int_0^{+\infty}\frac{1}{2}p^{\frac{1}{2}(h-1)}(1+p)^{-\frac{1}{2}(h+3c)}dp\alpha^{\frac{1}{2}(h+1)}$$

$$=[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\frac{1}{2}\alpha^{\frac{1}{2}(h+1)}\int_0^{+\infty}p^{\frac{1}{2}(h-1)}(1+p)^{-\frac{1}{2}(h+3c)}dp$$

$$=\frac{1}{2}[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{\alpha^c c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{\frac{1}{2}(h+1)}\int_0^{+\infty}p^{\frac{1}{2}(h-1)}(1+p)^{-\frac{1}{2}(h+3c)}dp$$

$$=\frac{1}{2}[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{\frac{1}{2}(h+1+2c)}\int_0^{+\infty}p^{\frac{1}{2}(h-1)}(1-p)^{-\frac{1}{2}(h+3c)}dp$$

Following Gamma function integral, that is,

$$\int_0^{+\infty}\frac{x^a}{(1+x)^b}dx=\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)} \tag{5}$$

where, $a=\frac{1}{2}(h-1)$, $b=\frac{1}{2}(h+3c)$, then $a+1=\frac{1}{2}(h+1)$ and $b-a-1=\frac{1}{2}(3c-1)$

The integral part of the expression of $E_c(\alpha)$ using the Gamma function is evaluated as:

$$\int_0^{+\infty} p^{\frac{1}{2}(h-1)}(1-p)^{-\frac{1}{2}(h+3c)}dp = \frac{\Gamma(\frac{1}{2}(h-1))\Gamma(\frac{1}{2}(3c-1))}{\Gamma(\frac{1}{2}(h+3c))}$$

And finally,

$$= \frac{1}{2}[1-(-1)^h]\frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{\frac{1}{2}(h+1+2c)}\frac{\Gamma(\frac{1}{2}(h-1))\Gamma(\frac{1}{2}(3c-1))}{\Gamma(\frac{1}{2}(h+3c))}$$

Or, $E_c(\alpha) = \begin{cases} 0 & \text{for} \quad h\ odd \\ B_{i,h,c,\alpha}\dfrac{\Gamma(\frac{1}{2}(h-1))\Gamma(\frac{1}{2}(3c-1))}{\Gamma(\frac{1}{2}(h+3c))} & \text{for} \quad h\ even \end{cases}$

where,

$$B_{i,h,c,\alpha} = \frac{1}{(e-1)^c}\sum_{i=0}^{\infty}\sum_{h=0}^{i}\frac{c^i}{i!}\left(\frac{1}{2}\right)^{c+i}\binom{h}{i}\alpha^{\frac{1}{2}(h+1+2c)}$$

Therefore,

$$R_c = \frac{1}{1-c}log\left[\begin{cases} 0 & \text{for} \quad h\ odd \\ B_{i,h,c,\alpha}\dfrac{\Gamma(\frac{1}{2}(h-1))\Gamma(\frac{1}{2}(3c-1))}{\Gamma(\frac{1}{2}(h+3c))} & \text{for} \quad h\ even \end{cases}\right]$$

CONCLUSION

In this paper, the derivation of Renyi entropy for DUSSStD has been successfully presented. It is clear that other measures of entropies can be derived for the DUSSStD and this can be done in future research.

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