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Lebesgue Measure and Integration on Subsets of $\mathbb{R}^d$

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Abstract

Henri Lebesgue, a French mathematician, discovered centuries ago that the Riemann Integral does not work well on unbounded functions. It prompts him to consider another way to integration known as Lebesgue Integral. This paper discusses the Riemann integral's shortcomings and introduce a more thorough concept of integration, the Lebesgue integral, repeated integration. There is also some debate about the Lebesgue measure, which determines the Lebesgue integral. Some examples are given, such as $F_\sigma - set$, G_δ -set and Cantor function. In this article, we first look into a unified theory for d-dimensional volume based on the concept of a measure, and then we will use that theory to build a stronger and more flexible theory for integration.

Keywords: Measurable function, Lebesgue measure, Lebesgue integral, Supremum, Infimum, Subsets of $\mathbb{R}^d$

Introduction

In the mathematical field of measure theory, subsets of higher-level Euclidean n-spaces are assigned a measure using the Lebesgue measure, which bears the name of the French mathematician Henri Lebesgue. It tells to the usual length, area, or volume measurement for smaller dimensions.

$n = 1, 2$, or 3.

N-dimensional volume is another name for it [1]. The notion of Lebesgue integration is one instance of its frequent usage in practical mathematics. A Lebesgue measure may be assigned to a set, which is known as a Lebesgue-measurable set. The measure of set A is represented by $\lambda(A)$ in this instance. Lebesgue integral and this measure were defined by Henri Lebesgue in 1901 and 1901, respectively. Part of his dissertation, both were published in 1902 [2].

A non-negative function of a single variable's integral is the region between its graph and the X-axis. Named for the French mathematician Henri Lebesgue, the Lebesgue integral is a weighted sum limit. There exist functions that satisfy both Lebesgue and Riemann integrability, in addition to those that satisfy both [3].

Mathematicians found that an integral of polygon approximation techniques may be used to describe the area under a curve, but as irregular functions became more complex, more challenging approximation techniques were needed.

It supplies the abstractions needed for this. It is useful in many different fields of mathematics, such as real analysis and probability theory. The integral (Lebesgue 1904) was invented by Henri Lebesgue (1875–1941), in whose honor it is called [4]. Lebesgue integration is a part of axiomatic probability theory, describing it as a generic measure.

It is a mathematical concept that spreads the classical notion of integration to a larger class of functions. This integration is named after French mathematician Henri Léon Lebesgue, who introduced the integral in the early 20th century as a way to overcome some limitations of the Riemann integral [4].

The Lebesgue integral is particularly useful when dealing with functions that are not necessarily continuous or even defined on a set of points. It provides a more flexible and powerful framework for integration, allowing the integration of a broader class of functions, including those with discontinuities.

Here are some key concepts related to the Lebesgue integral:

Measurable Sets

In the Lebesgue integral, integration is defined on measurable sets. A set is measurable if its "size" can be assigned in a consistent way. The collection of measurable sets forms a σ-algebra, which is a mathematical structure that satisfies certain properties.

Measurable Functions

The Lebesgue integral is defined for measurable functions if the pre-image of measurable set under the function is a measurable set. This ensures that the integration process is well-defined.

Lebesgue Measure

The Lebesgue measure is a way to assign a "size" or "length" to measurable sets. It extends the notion of length to more general sets, including those with irregular shapes or fractal-like properties.

Integration Process

Lebesgue integral is defined as the supremum (upper bound) of step functions that are less than or equal to the given function. For general functions, the Lebesgue integral can be decomposed into the difference of the integrals of its positive and negative parts. The Lebesgue integral has several advantages over the Riemann integral, such as its ability to handle a wider class of functions and its convergence properties. It plays a crucial role in modern real analysis, probability theory, and various branches of mathematics [4].

We will first look into a unified theory for d-dimensional volume based on the concept of a measure, and then we will use that theory to build a stronger and more flexible theory for integration. Let this a strategy reversal: rather than using integration to determine volumes. We will create an integration theory based on a more important understanding of volume.

Volume in $\mathbb{R}^d$ will be covered for all d$\in \mathbb{N}$ including d = 1 and d = 2. To avoid confusion, we will refer to the length of a set in $\mathbb{R}$ and the area of a set in $\mathbb{R}^2$ as one- and two-dimensional volumes, respectively. To get a sense of what we're striving for, suppose we wish to measure the volume of subsets A$\subset \mathbb{R}^3$ and designate the volume of A by $\mu(A)$. Details and proofs can be found in real analysis texts like [5], [6], [7], [8], [9].

[10],[11],[12],[13].While some proofs are straightforward, others are very tough, and there are other perplexing complications with Lebesgue measure and integration.

Outer Measure in $\mathbb{R}^d$

A subsets A of $\mathbb{R}^d$ is called an open box if there are numbers,

$$A = \left(a_1{}^{\{1\}},\ a_2{}^{\{1\}}\right) \quad \times \quad \left(a_1{}^{\{2\}},\ a_2{}^{\{2\}}\right) \quad \cdots\cdots \times \quad \left(a_1{}^{\{d\}},\ a_2{}^{\{d\}}\right) \tag{1}$$

We define the volume $|A|$ of A to be 0 if A is the empty set, and otherwise,

$|A| = \left(a_2{}^{\{1\}} - a_1{}^{1}\right) \ \left(a_2{}^{\{2\}} - a_1{}^{\{2\}}\right) \cdots\cdots \left(a_2{}^{\{d\}} - a_1{}^{\{d\}}\right)$, where d = 1, 2, and 3.

$|A|$ is the length, area, volume of A.

If $\mathcal{A} = \{A_1, A_2 \cdots\cdots A_n \cdots\}$ is a countable collection of open boxes, then it's size $|\mathcal{A}|$

$|\mathcal{A}| = \sum 1$Ak $|Ak|$ where $|A| = \infty$

A covering of a set B$\subset 1R^d$ is a countable collection A = $\{A_1, A_2 ,,,, A_n,,,,\}$

of open boxes B $\subset \bigcup_{n=1}^{\infty} A_n$. The outer measure of a set B $\in 1R^d$ is defined by μ^*(B) = inf $\{|\mathcal{A}| : \mathcal{A}$ is cover of B by open boxes $\}$.

Lebesgue Measure

Definition 3.1

A set E$\subseteq \mathbb{R}^d$ is Lebesgue measurable, if

$$\forall\, \varepsilon > o\ , \exists \ \text{open} \ \cup \supseteq \exists\ \forall, \ |U \backslash E|_e \ \leq \in \tag{2}$$

It is exterior Lebesgue measure and represented by $|E| = |E|_{e.}$

Theorem 3.2.

A measurable subset of $\mathbb{R}^d$ is called $\sigma -$ algebra if there exists

(i) $\emptyset$ and $\mathbb{R}^d$ are set of measurable function.

(ii) If $E_1,\ E_2, \cdots\cdots$ are set then it's sum $\cup\, E_k$ is also measurable set.

(iii) $\mathbb{R}^d \setminus$ E is also measurable set if E is a measurable set.

(iv) All of $\mathbb{R}^d$ is open and closed subsets are measurable set.

(v) Every E ⊂ $\mathbb{R}^d$ with $|E|_e = 0$, then it is measurable set.

Because complements and countable union preserve measurability. It is also retained in the case of countable intersection. We'll provide an equal concept of measurability [5].

Definition 3.3

(a) The set H ⊆ $\mathbb{R}^d$ is a G_δ- set, then many open sets are defined by U_k ∀ H = ∩ U_k.

(b) The set H⊆ $\mathbb{R}^d$ is an F_σ- set, then many closed sets are defined by F_k ∀, H = ∪F_k.

Theorem 3.4

If E ⊆ $\mathbb{R}^d$ then, we get

(i)E is a measurable.

(ii)For $\varepsilon > 0$, have a closed set as F⊆ E, ∀ $|E \setminus F|_e \leq \varepsilon$.

(iii) For E= H\Z, H is G_δ- set and $|Z|$ =0.

(iv) For E = H ∪Z, H is an F_δ- set and $|Z|$ = 0.

Theorem 3.5

Suppose E and E_k are measurable set for k∈ ℕ and there exists subsets of $\mathbb{R}^d$.

(i) **Associativity**: Let E_1, E_2 are measurable set in $\mathbb{R}^d$ defined as

$$|\cup_{k=1}^{\infty} F_k| = \sum_{k=1}^{\infty} |E_k|.$$

(ii) For $E_1 \subseteq E_2$ and $|E_1| < \infty$, then $|E_1 \setminus E_2| = |E_2| - |E_1|$.

(iii) **Continuity beneath**: For, $E_1 \subseteq E_2 \subseteq \cdots\cdots$, results $|\cup E_k| = \lim_{k \to \infty} |E_k|$.

(iv)**Continuity upstairs**: For, $E_1 \supseteq E_2 \supseteq \cdots\cdots\cdots\cdots$, results $|E_1| < \infty$.

Results $|\cap E_k| = \lim_{k \to \infty} |E_k|$.

(v) **Constance of transformation**: Let h ∈ $\mathbb{R}^d$, then it results $|E + H| = |E|$, E+H $= \{x + h: x \in E\}$.

(vi) **Variance of variables**: Let T: $\mathbb{R}^d \rightarrow \mathbb{R}^d$ defined

T(E) is also measurable, $|T(E)| = |det(T)|\ |E|$

(vii) **Cartesian product**: Let E$\subseteq \mathbb{R}^m$ and F$\subseteq \mathbb{R}^n$ then

E$\times F \subseteq \mathbb{R}^{m+n}$, is also measurable and we get $|E \times F| = |E||F|$ [5].

Definition 3.6.

Almost everywhere (a. e)

In mathematics, the word "almost everywhere" (abbreviated as "a. e.") is normally used, particularly in measure theory and analysis. It designates a feature that is true for every point in a set, maybe with the exclusion of a subset of points with measure zero. The phrase "almost everywhere" makes it possible to make mathematical claims that are more general and cover the common situations while also allowing a small number of brilliant but insignificant situations. It refers to holds except may be on a set of measure zero.

Let C be the traditional Cantor middle-third set, for example, then $|C| = 0$. As a result, except for those t in C, the characteristics function $\mathcal{X}_c$ satisfies $\mathcal{X}_c(t) = 0$, except, for t $\in C$.

So, $\mathcal{X}_c(t)$ for almost every t.

Definition 3.7. The supremum of a function f: E→ $\mathbb{R}$ is

sup f(t) = inf $\{M: F \leq M\ a.e.\}$. Then f is bounded if $SUP_{t \in E}|f(t)| < \infty$.

Measurable Functions

In Lebesgue theory of integration, the class of measurable functions plays a critical role. It will occupy a position equivalent to that of the class of bounded and continuous functions almost everywhere in the Riemann theory of integration, as well as functions of bounded variation in the case of Stieltjes integrals. A function is integrable if its behavior is not too irregular and its values are not too huge too frequently. The second criteria are identical to the existence of the upper and lower integrals being equal. We now introduce the concept of measurability, which provides the exact requirements required for integrability, provided that the function is not too huge. In many circumstances, it is easier to explore a function's measurability rather than its upper and lower integrals directly.

The class of measurable functions on subsets of $\mathbb{R}^d$ includes real-valued $\pm\infty$ and complex-valued functions, with extended teal-valued functions often considered, as detailed in [5].

Definition 4.1 (Measurable Functions with Real Values)

f is a Lebesgue measurable functions for F : E⟶ $\mathbb{R}$ be given for E∈ $\mathbb{R}^d$

if $f^{-1}(\sigma,\infty)$= $\{t \in E: f(t) > \sigma\}$ is a measurable subset of $\mathbb{R}^d$ for each $\sigma \in \mathbb{R}$.

Every open set is measurable, as is any continuous function f: $\mathbb{R}^d \longrightarrow$.

An open set's inverse image under a continuous function is also measurable. A measurable function, though, does not always need to be continuous. For the most portion, measurable operations like addition, multiplication, and limits reserve measurable data. While conformations need to be used wisely, measurability is guaranteed, when a continuous function and measurable function are shared in the accurate sequence.

Theorem 4.2. Suppose E⊆ $\mathbb{R}^d$ is measurable.

(i)Let f: E⟶ $\mathbb{R}$ is measurable for g = $f_{a,e}$ then g is again measurable.

(ii)Let f, g: E⟶ $\mathbb{R}$ is measurable for f+ g also measurable.

(iii) The measurability of f: E⟶ $\mathbb{R}$ and continuity of $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ guarantees the measurability of $\varphi 0 f$.

$|f|, f^2, f^+, f^-$, and $|f|^p$ are all measurable for p> 0.

(iv)Let f, g: f → $\mathbb{R}$ are measurable, then f g exists.

(v)Let functions f_n :E⟶ $\mathbb{R}$ is measurable then, there occurs sup f_n, inf f_n, lim sup f_n , and lim inf f_n.

(vi)Let functions f_n :E⟶ $\mathbb{R}$ is measurable for f(t) = $lim_{n\to\infty} f_n$(t) occurs for a. e. then f is also continuous [5].

Definition 4.3 Complex Function

Complex numbers are represented by a+ib, where i is the imaginary number and a and b are real numbers (i^2) = -1. Complex functions are mathematical functions that work on complex numbers. The mapping of complex numbers to other complex numbers is done via complex functions. In many areas of mathematics and science, these functions are vital.

Let a measurable E $\subseteq \mathbb{R}^d$ and f: E $\rightarrow \mathbb{C}$, where f = f_r + if_i in real and imaginary parts. If both f_r and f_i are measurable, then f is also measurable.

If a set has finite measure and pointwise convergence is equals to the uniform convergence (Egoroff' s Theorem).

Theorem 4.4 Egoroff 's Theorem

The convergence of almost everywhere in sequences of measurable functions is the subject of the Egoroff's theorem. It gives an approach to virtually uniform convergence, strengthening pointwise convergence. This tool is crucial for researching the interactions between sequences of functions that exhibit uniform and pointwise convergence.

Suppose E $\subseteq \mathbb{R}^d$ and |E| < ∞. If f : E ⟶ $\mathbb{C}$ is defined by

f_n(t) ⟶ f(t) for a. e. t ∈ E, for every ε > 0, then set A ⊆ E

∀ from |A| < ε and f_n converges consistently to f on E\A,

$\lim_{n\to\infty} \left(\sup_{t \notin A} |f(t) - f_n(t)| \right) = 0.$

Lebesgue integral

We begin with simple functions and work our way up to non-negative functions.

Definition 5.1

Suppose E $\subseteq \mathbb{R}^d$ is a measurable function.

(i) E is simple measurable function, φ: E ⟶ $\mathbb{C}$, distinct values give iff

$$\emptyset = \sum_{k=1}^{N} a_k \mathcal{X}_{E_k} \tag{3}$$

where N > 0, $a_k \in \mathbb{C}$ and $E_k \subset$ E.

(ii)Let a_1,... ,$a_N \in \mathbb{C}$ are the distinct values of φ and if E_k = {t ∈ E: ∅(t) = a_k}, then equation (3) is evaluated and sets $E_1, E_2 \cdots\cdots E_N$ are partition and is the standard representation.

(iii)Let ∅ is a non-negative function defined on E with typical illustration

$\emptyset = \sum_{k=1}^{N} a_k \mathcal{X}_{E_k}$ then the Lebesgue integral is given by

$$\int_E f = \int_E f(t)dt = \sum_{k=1}^{N} a_k |E_k| \tag{4}$$

(iv)Let f : E ⟶ [0,∞] is a measurable, then it is given by

$$\int \emptyset_E = \int \emptyset(t)_E \, dt = \sup\left\{\int \emptyset_E \ : 0 \le \ \emptyset \ \le f, \emptyset \ simple\right\}.$$

A ⊂ E, gives $\int_A f = \int_E f \mathcal{X}_A$.

The fundamental characteristics of integrals of non-negative functions are as follows [5, 6].

When dealing with integrals of non-negative functions, there are several fundamental characteristics and properties that are vital to comprehend. These properties are often applied in calculus and analysis.

Theorem 5.2. Let E is a measurable subset of $\mathbb{R}^d$ and f, g : E ⟶ [0,∞) are also measurable.

(i) Let ∅ is defined on E, then integrals ∅ given in (iii) and (iv) of definition 5.1 overlap.

(ii) Let f ≤ g then there exists $\int_E f \le \int_E g$.

(iii) **Tchebyshev Inequality**: Let α > 0, then there exists | {t ∈ E : f(t) > α} | ≤ $\frac{1}{\sigma} \int_E f$

.

(iv) $\int_E f = 0$, iff f = 0 almost everywhere.

The definition of $\int_E f$ given in definition 5.1 is repeatedly ungainly to implement.

If f_1(t) ≤ f_2(t) ≤ · · · for every t, then a series of real-valued functions $\{f_n\}$n∈ N is monotonic growing. $f_n \nearrow f$ is $\{f_n\}_{n\in\mathbb{N}}$ is monotonic growing as f_n(t) ⟶ f(t) is pointwise [5].

Theorem 5.3 Theorem of Monotone Convergence

Let E ⊆ $\mathbb{R}^d$ and $\{f_n\}_{n\in\mathbb{N}}$ are measurable on E such that $f_n \nearrow$ f.

We have

$$n \to \infty \int_E f_n = \int_E f .$$

Theorem 5.4. Let E ⊆ $\mathbb{R}^d$ and f : E → [0,∞) are both measurable for functions

$\emptyset_n$ ∀ $\emptyset_n \nearrow$ f, and so $\int_E \emptyset_n \nearrow \int_E f$.

Corollary 5.5 Let$\{ f_n\}_{n\epsilon\mathbb{N}}$ is a measurable sequence of non-negative functions set E $\subseteq$ $\mathbb{R}^d$.

$$\therefore \quad \int_E(\sum_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} f_n .$$

Definition 5.6 Let E$\subseteq$ $\mathbb{R}^d$ is measurable.

(i)A measurable function f : E $\rightarrow$ $\mathbb{R}$ defined by

f^+(t) = max {f(t), 0}, f^-(t) = max{ −f(t), 0 }

So, there exists, $f^+, f^- \geq 0$, f = $f^+ - f^-$ and |f| = $f^+ + f^-$

(5) Lebesgue integral of f on E is defined by

$$\int_E f = \int_E f^- + \int_E f^- \tag{6}$$

(ii)Let f : E $\rightarrow$ $\mathbb{C}$ is a measurable function and the complex analytic f = fr + if_i .

Let $\int_E f_r$ and $\int_E f_i$ are f on E is defined by

$\int_E f = \int_E fr + i \int_E fi$ and otherwise, vague.

Theorem 5.7 Let f be a measurable function in E $\subseteq$ $\mathbb{R}^d$ and $\int_E f$ is a finite scalar when $\int_E |f| < \infty$ and $| \int_E f | \leq \int_E | f |$.

Theorem 5.8. Lebesgue Differentiation

Let f $\in$ L^1 [a, b] and for almost every x $\in$ (a, b), there exists

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)dy = \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy = f(x) \tag{7}$$

Thus, integral of f

F(x) = $\int_a^x f(y)\, dy$ and F ' = f a.e.

Repeated integration

Suppose E $\subseteq$ R^m and F $\subseteq$ R^n are measurable and integrals of f over E × F if it is a measurable on E × F. First integral of f over the set E × F $\subseteq$ R^{m+n} is the double integral.

$$\iint_{E\times F} f = \iint_{E\times F} f(x, y)\, (dx\, dy).$$

$$\int_F (\int_E f(x,y)\,)\, \mathrm{dx}$$

Even if all integrals exist, they need not be comparable. Fubini and Tonelli's theorems provide sufficient reverse the sequence of integration [6].

Theorem 7. Tonelli Theorem

The Tonelli theorem is used to analyze certain types of functionals in the calculus of variations, particularly in the context of finding critical points for functionals defined on function spaces [6].

Let E and F are measurable subset of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let f : E × F → [0,∞) is measurable, then following statements are drawn.

(i)For each x ∈ E, f_x(y) = f (x, y)

(ii) For each y in F, f^y(x) = f (x, y)

(iii) g(x) = $\int_F f_x(y)\, dy$ is a measurable.

(iv) h(y) = $\int_E f^y(x)$ dx is a measurable.

(v) $\iint_{E\times F} f(x,y)(dxdy) = \int_F \left(\int_E f(x,y)dx\right)\mathrm{dy} = \int_E \left(\int_F f(x,y)dy\right)\mathrm{dx}$.

in the sense that all are infinite [5].

Theorem 7.1 Fubini Theorem

The Fubini's theorem is a important result in real analysis and measure theory, named after the Italian mathematician Guido Fubini. There are two versions of Fubini's theorem: one for Lebesgue integrals and another for Riemann integrals. We provide a brief overview of the Lebesgue version, which is more commonly encountered. Let E and F are measurable subset of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Let f ∈ L^1 (E × F), then [5].

(a) For x in E, f_x(y) = f (x, y)

(b) For y in F, f^y (x) = f (x, y)

(c) $\int_F f_x(y)dy$ = g(x) is a measurable and also integrable.

(d) $\int_E f^y(x)$ dx = h(y) is integrable.

(e) Now, we have

$$\iint_{E\times F} f(x,y)(dxdy) = \int_F\left(\int_E f(x,y)dx\right)\mathrm{dy} = \int_E\left(\int_F f(x,y)dy\right)\mathrm{dx}.$$

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[1] The term volume is also used, more strictly, as a synonym of 3-dimensional volume

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