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# Inference and Asymmetric GARCH-Model with a New Distributed Innovation

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**Abstract**

A novel generalized-odd-generalized exponentiated skew-t (GOGEST) innovation density for the generalized autoregressive conditional heteroskedasticity (GARCH) models is proposed. The features of the proposed distribution were derived. The parameter estimates of the proposed distribution through simulation were carried-out with maximum likelihood estimation technique. The performance of the asymmetric GARCH-GOGEST model relative to five other asymmetric GARCH-various existing innovation densities in volatility modeling was investigated using the Bitcoin log-returns. The empirical results showed that the asymmetric GARCH-GOGEST models were superior over the other asymmetric GARCH models. However, the threshold GARCH-GOGEST model outperformed the other models in terms of volatility predictability (out-of-sample).

**Keywords**: Asymmetric-volatility models; Hybrid distribution; Simulation study; Bitcoin, Innovation distributions


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## INTRODUCTION

Excess kurtosis, skewness, heavy tail, volatility clustering, and leverage effects are well-known stylized properties of financial time series. This frequently causes overestimation or underestimating of actual volatility estimates (Adubisi et al., 2022a). Thus, skewness and leptokurtic behaviour of asset returns have been described using heavy-tailed and skewed distributions (Nelson, 1991; Altun et al., 2018). Several authors have investigated the performance of GARCH-type models under heavy-tailed distributions in modeling and predicting asset return volatility (Ural and Demireli, 2019; Samson et al., 2020a; Samson et al., 2020b; Yelamanchili, 2020; Gyamerah, 2019; Maree et al., 2017; Kosapattarapim, 2013; Nagamani and Tripathy, 2018; Ngunyi et al., 2019; Adubisi et al., 2022b; Adubisi et al., 2022c, Mubarak et al., 2024). Volatility is a major concern in portfolio allocation, risk management, investment decisions, option pricing, microeconomic analysis, and policy formulation. In this setting, volatility projections can serve as a trigger element for both the financial markets and the economy (Poon & Granger, 2003). As a result, predicting asset return volatility is critical in finance since volatility is usually regarded as the most essential aspect in financial investment (Mandelbrot, 1963).

Harvey and Sucarrat (2013) proposed the EGARCH-beta-skew Student-t density model to accurately predict daily volatility. Agboola et al. (2018, 2019) assessed daily volatility for stock index returns using a new generalization of the skew Student-t distribution, demonstrating that it outperformed the skew normal, skew Student-t, and skew generalized error densities. Harvey and Chakravarty (2008) and Harvey (2013) proposed using the beta-student-t distribution for the exponential GARCH (EGARCH) model to estimate volatility, which captures the dynamics better than existing densities. Employing both simulated and real data, Nakajima and Omori (2012) conducted a Bayesian study of the stochastic model with a generalized hyperbolic skew Student-t distribution using an effective Markov chain Monte Carlo estimation approach. Nelson (1991) concluded that volatility models that do not allow for asymmetries in conditional variance typically provide poor volatility forecasts. Liu and Hung (2010) discovered that more flexible GARCH-type models are highly acceptable in volatility prediction under all density assumptions. In terms of the innovation distribution, Feng and Shi (2017) discovered that leptokurtic distributions in GARCH-type models could provide more accurate volatility projections. However, the construction of robust distributions is critical in improving the accuracy of the financial risk system. The goal for constructing the new model was to generate a more flexible distribution with symmetric, right and left skewed, and unimodal properties.

The purpose of this study is to propose a new conditional innovation density that would yield more accurate volatility forecasts than the current routinely utilized innovation densities in GARCH volatility models. For this reason, the generalized odd generalized exponentiated skew-t (GOGEST) distribution is created, which provides unimodal, skewed, and fat-tail shapes for the probability density function (pdf). In addition, a new dynamic asymmetric GARCH-GOGEST model for predicting daily volatility is developed, based on different asymmetric GARCH-type volatility models with GOGEST innovation density, since existing innovation densities are quite incapable of accounting for excess kurtosis and skewness in financial datasets.

## METHODS

### Generalized-odd-generalized-exponentiated-skew-t-model

Alizadeh et al. (2017) created the generalized odd generalized exponential (GOGE) family of distributions with the cumulative distribution function (CDF) specified as

$$F(y;\varphi,\tau) = \left[1 - e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}}\right]^{\tau} \tag{1}$$

and the probability density function (PDF) is specified as

$$f(y;\varphi,\tau) = \frac{\varphi\tau g(y)G(y)^{\varphi-1}}{[1-G(y)^{\varphi}]^2} e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}} \left[1 - e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}}\right]^{\tau-1} \tag{2}$$

where the CDF and PDF of the parent model is represented with $G(y)$ and $g(y)$. $\varphi > 0, \tau > 0$ are two shape parameters. In this research, the parent distribution is the skew student-t (ST) model introduced by Jones and Faddy (2003). The CDF of the ST is given by

$$G_{ST}(y) = \frac{1}{2}\left(1 + \frac{y}{\sqrt{\upsilon+y^2}}\right), \quad y \in \Re, \tag{3}$$

and the corresponding PDF is

$$g_{ST}(y) = \frac{\upsilon}{2(\upsilon+y^2)^{\frac{3}{2}}}, \quad y \in \Re \tag{4}$$

where, $\upsilon$ controls the skewness.

Inserting the ST of Jones and Faddy (2003) into the GOGE of Alizadeh et al. (2017). The CDF and PDF of $GOGE_{ST}$ are gotten as follows:

$$F(y;\varphi,\tau)=\left[1-e^{\frac{-\Psi(y)^{\varphi}}{1-\Psi(y)^{\varphi}}}\right]^{\tau},\quad y\in\Re \tag{5}$$

$$f(y;\varphi,\tau)=\frac{\varphi\tau\psi(y)}{[1-\Psi(y)^{\varphi}]^{2}}\Psi(y)^{\varphi-1}e^{\frac{-\Psi(y)^{\varphi}}{1-\Psi(y)^{\varphi}}}\left[1-e^{\frac{-\Psi(y)^{\varphi}}{1-\Psi(y)^{\varphi}}}\right]^{\tau-1},\quad y\in\Re \tag{6}$$

Here, the PDF and CDF of the ST is represented with $\psi(y)$ and $\Psi(y)$. $\varphi>0,\tau>0$ are shape parameters and $\upsilon$ is the skewness parameter. Figure 1 presents PDF charts for the $GOGE_{ST}$. As illustrated in Figure 1, the $GOGE_{ST}$ provides new opportunities to describe the unimodality, skewness, leptokurtic, and heavy-tailed patterns of most datasets from many domains. Furthermore, the $GOGE_{ST}$ is seen as a viable candidate for overcoming the lack of modeling capabilities of existing distributions in terms of precise volatility prediction, as excess kurtosis and skewness are common traits in most financial series.

**Mathematical properties of the $GOGE_{ST}$ distribution**

**Quantile Function**

The quantile function is obtained by inverting Equation (5). The quantile function of the $GOGE_{ST}$ is specified as

$$Q(U)=\upsilon^{\frac{1}{2}}\frac{\left(\left\{2\left[\frac{-\ln\left(1-u^{\frac{1}{\tau}}\right)}{\left(1-\ln\left(1-u^{\frac{1}{\tau}}\right)\right)}\right]^{\frac{1}{\varphi}}\right\}-1\right)}{\sqrt{1-\left(\left\{2\left[\frac{-\ln\left(1-u^{\frac{1}{\tau}}\right)}{\left(1-\ln\left(1-u^{\frac{1}{\tau}}\right)\right)}\right]^{\frac{1}{\varphi}}\right\}-1\right)^{2}}},\quad U\in(0,1) \tag{7}$$

The quantile function is thought to be extremely effective for generating random variables from any continuous distribution. As a result, the numerical values of the median (M), skewness (S), and kurtosis (K) in Table 1 are simulated using the $GOGE_{ST}$ quantile function in Equation (7). The results reveal that the new model is capable of handling leptokurtic, negative and positive skewed data.

**Table 1.** Numerical values of some useful statistics

| υ | φ | T | M | S | K |
|---|---|---|---|---|---|
| 0.5 | 0.8 | 0.7 | -0.4351 | -0.3173 | -1.1428 |
| | 1.0 | 0.9 | -0.1700 | -0.1801 | -0.5791 |
| | 1.2 | 1.0 | -0.035 | -0.1181 | -0.3656 |
| | 1.7 | 1.3 | 0.207 | -0.0327 | -0.0999 |
| | 2.0 | 1.8 | 0.366 | 0.0019 | 0.0006 |
| 1.0 | 0.8 | 0.7 | -0.6153 | -0.3173 | -1.1428 |
| | 1.0 | 0.9 | -0.2399 | -0.1801 | -0.5792 |
| | 1.2 | 1.0 | -0.050 | -0.1181 | -0.3656 |
| | 1.7 | 1.3 | 0.293 | -0.0327 | -0.0999 |
| | 2.0 | 1.8 | 0.518 | 0.0019 | 0.0006 |
| 2.0 | 0.8 | 0.7 | -0.8701 | -0.3173 | -1.1428 |
| | 1.0 | 0.9 | -0.3393 | -0.1801 | -0.5792 |
| | 1.2 | 1.0 | -0.070 | -0.1181 | -0.3656 |
| | 1.7 | 1.3 | 0.415 | -0.0327 | -0.0999 |
| | 2.0 | 1.8 | 0.732 | 0.0019 | 0.0006 |

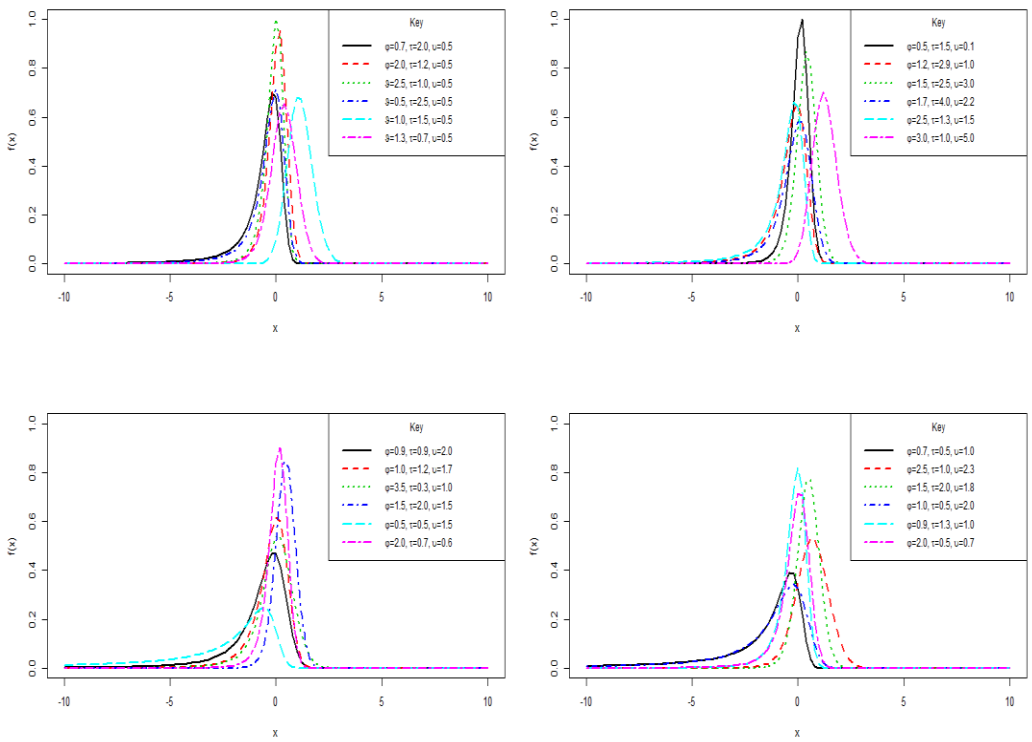


**Figure 1:** The GOGE$_{ST}$ density function plots with designated value of the parameters.

### Linear representation

In this subsection, the linear form of the density and distribution functions which are very useful in studying the mathematical features of $GOGE_{ST}$ are presented. Firstly, the expansion form of the PDF is gotten by applying the generalized binomial series expansion in Equation (5), hence the PDF is specified as

$$f(y) = \frac{\varphi\tau\upsilon[\Psi(y)]^{\varphi-1}}{2(\upsilon+y^2)^{\frac{3}{2}}[1-[\Psi(y)]^{\varphi}]^2}\sum_{i=0}^{\infty}(-1)^i\binom{\tau-1}{i}e^{\frac{-(i+1)[\Psi(y)]^{\varphi}}{1-[\Psi(y)]^{\varphi}}} \quad (8)$$

But,

$$e^{\frac{-(i+1)[\Psi(y)]^{\varphi}}{1-[\Psi(y)]^{\varphi}}} = \sum_{j=0}^{\infty}(-1)^j\frac{(i+1)^j}{j!}\left(\frac{[\Psi(y)]^{\varphi}}{1-[\Psi(y)]^{\varphi}}\right)^j \quad (9)$$

Hence,

$$f(y) = \frac{\varphi\tau\upsilon}{2(\upsilon+y^2)^{\frac{3}{2}}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j[\Psi(y)]^{\varphi(j+1)-1}}{j![1-[\Psi(y)]^{\varphi}]^{j+2}}\binom{\tau-1}{i} \quad (10)$$

Using the generalized binomial series, the PDF of the GOGEST may be established as an infinite linear combination of the ST written as

$$f(y) = \varphi\tau\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(i+1)^j}{j!}\binom{\tau-1}{i}\binom{-(j+2)}{k}\left(\frac{\upsilon}{2(\upsilon+y^2)^{\frac{3}{2}}}\right)[\Psi(y)]^{\varphi(j+k+1)-1} \quad (11)$$

If $\varphi$and $\tau$are integers the index $\mathrm{i}$ stops at $(\tau-1)$ and $\mathrm{k}$ stops at $-(j+2)$. When $\varphi$ is non-integer, a more general form is expressed as

$$f(y) = \sum_{i,j,k=0}^{\infty}\eta_{i,j,k}\left(\frac{1}{2(\upsilon+y^2)^{\frac{3}{2}}}\right)(1-\{1-[\Psi(y)]\})^{\varphi(j+k+1)-1} \quad (12)$$

where $\eta_{i,j,k} = \varphi\tau\upsilon\frac{(-1)^{i+j+k}(i+1)^j}{j!}\binom{\tau-1}{i}\binom{-(j+2)}{k}$

Using the generalized binomial series, the $GOGE_{ST}$ density function $f(y)$ is given as

$$f(y) = w_{i,j,k,l,h}y^h(\upsilon+y^2)^{-\left(\frac{h+3}{2}\right)} \quad (13)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty}\frac{(-1)^{l+h}}{2^{l+1}}\eta_{i,j,k}\binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

Secondly, the distribution function of the $GOGE_{ST}$ is specified as

$$F(y) = \psi_{i,j,k,l,m}y^m(\upsilon+y^2)^{-\frac{m}{2}} \quad (14)$$

where $w_{i,j,k,l,m} = \sum_{i,j,k,l,m=0}^{\infty} \frac{(-1)^{i+j+k+l+m}}{j!2^{l}} \binom{\tau}{i}\binom{j}{k}\binom{\varphi(j+k)}{l}\binom{l}{m}$

**Raw Moments**

The g^th raw moment of the $GOGE_{ST}$ is derived. Let $Y \sim \mathrm{GOGE_{ST}}(\varphi, \tau, \upsilon)$ be a random-variable (r.v), the g^th raw-moment is

$$\mu'_g = E(y^g) = \int_{-\infty}^{+\infty} y^r f(y)dy \quad (15)$$

Taboga (2017) showed that the g^th moment in Equation (15) can be specified as

$$\mu'_g = E(y^g) = (1 + (-1)^g)\int_0^{+\infty} y^g f(y)dy \quad (16)$$

In this context, $f(y)$ is the series expansion of the PDF in Equation (13). Hence, the g^th moment expression of the $GOGE_{ST}$ distribution is given as:

$$\mu'_g = (1 + (-1)^g)\int_0^{+\infty} y^g w_{i,j,k,l,h} y^h (\upsilon + y^2)^{-\left(\frac{h+3}{2}\right)} dy \quad (17)$$

So, the gth raw-moment of the $GOGE_{ST}$ is distinct for $r = 2,4,6,8,\ldots$ (even numbers). Then

$$\mu'_g = w_{i,j,k,l,h}\upsilon^{\frac{g-2}{2}} B\left(\frac{g+h+1}{2}, \frac{2-g}{2}\right) \quad (18)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

**Corollary 1:** The incomplete moment (IM) of the $GOGE_{ST}$ is investigated. This suffices that the g^th IM, $\varphi'_r(t)$ is

$$\varphi'_g(t) = E(Y^g | Y < t) = \int_0^t y^g f(y)dy$$

In this context, $f(y)$ is the series expansion of the pdf in Equation (13). Hence, the g^th IM of the $GOGE_{ST}$ is specified as

$$\varphi'_r(t) = \int_0^t y^r w_{i,j,k,l,h} y^h (\upsilon + x^2)^{-\left(\frac{h+3}{2}\right)} dy \quad (19)$$

Simplifying Equation (19), the g^th IM of the $GOGE_{ST}$ is specified as

$$\varphi'_g(t) = \frac{w_{i,j,k,l,h}}{2}\upsilon^{\frac{g-2}{2}} B\left(t, \frac{g+h+1}{2}, \frac{2-g}{2}\right) \quad (20)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

**Remark:** The first IM $\varphi_1'(t) = \int_0^t yf(y)dy$ of the GOGE$_{ST}$ is derived by introducing $g = 1$ into Equation (20).

**Corollary 2:** The characteristics function (CF) of the GOGE$_{ST}$ is investigated. This suffices that the CF, $\Phi_Y(t)$ is

$$\Phi_Y(t) = E\left(e^{itY}\right) == \int_{-\infty}^{+\infty} e^{ity} f(y)dy = \sum_{g=0}^{\infty} \frac{(it)^g}{g!} \mu_g' \quad (21)$$

In this context, $\mu_g'$ is the g$^{th}$ raw moment of the GOGE$_{ST}$ in Equation (18). The CF of the GOGE$_{ST}$ is specified as

$$\Phi_Y(t) = \sum_{g=0}^{\infty} \frac{(it)^g}{g!} w_{i,j,k,l,h} \upsilon^{\frac{g-2}{2}} B\left(\frac{g+h+1}{2}, \frac{2-g}{2}\right) \quad (22)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l} \binom{l}{h}$

**Rényi-Tsallis-Q Entropies**

The Rényi entropy which quantifies the disparity of indecision is specified as

$$I_\delta(Y) = \frac{1}{1-\delta} \log \int_{-\infty}^{+\infty} f(y)^\delta dy, \ \delta > 0 \text{ and } \delta \neq 1 \quad (23)$$

By means of Equation (5), $f(y)^\delta$is expressed in the series expansion form as:

$$f(y)^\delta = w_{i,j,k,l,h} y^h (\upsilon + y^2)^{-\left(\frac{h+3\delta}{2}\right)} \quad (24)$$

where

$$w_{i,j,k,l,h} = (\varphi\tau\upsilon)^\delta \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{i+j+k+l+h}(i+\delta)^j}{j!\,2^{l+\delta}} \binom{\delta(\tau-1)}{i} \binom{-(j+2\delta)}{k} \binom{\varphi(j+k+\delta)-\delta}{l} \binom{l}{h}$$

Therefore, the Rényi-entropy of the GOGE$_{ST}$ is specified as:

$$I_\delta(Y) = \frac{1}{1-\delta} \log \int_{-\infty}^{+\infty} w_{i,j,k,l,h} y^h (\upsilon + y^2)^{-\left(\frac{h+3\delta}{2}\right)} dy \quad (25)$$

Simplifying and solving Equation (25) based on Taboga (2017), the Rényi-entropy of the GOGE$_{ST}$ is specified as:

$$I_\delta(X) = \frac{1}{1-\delta} \log\left( w_{i,j,k,l,h} \upsilon^{\frac{1-3\delta}{2}} B\left(\frac{h+1}{2}, \frac{3\delta-1}{2}\right)\right) \quad (26)$$

Similarly, the Tsallis q-entropy is expressed as

$$H_q(Y) = \frac{1}{q-1}\left(1 - \left[\int_{-\infty}^{+\infty} f(y)^q dy\right]\right), q > 0 \text{ and } q \neq 0 \qquad (27)$$

Using the steps in Equations (24)-(26), the q-entropy of the GOGE$_{ST}$ is given as:

$$H_q(Y) = \frac{1}{q-1}\left\{1 - \left(w^*_{i,j,k,l,h}\upsilon^{\frac{1-3q}{2}} B\left(\frac{h+1}{2}, \frac{3q-1}{2}\right)\right)\right\} \qquad (28)$$

where

$$w^*_{i,j,k,l,h} = (\varphi\tau\upsilon)^q \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{i+j+k+l+h}(i+q)^j}{j!\,2^{l+q}} \binom{q(\tau-1)}{i} \binom{-(j+2q)}{k} \times \binom{\varphi(j+k+q)-q}{l}\binom{l}{h}$$

**Order Statistics**

The r$^{th}$ order statistic $Y_{r:n}$ is presented as

$$f_{r:n}(y) = \frac{g(y)}{B(r,n-r+1)}[G(y)]^{r-1}[1-G(y)]^{n-r} \qquad (29)$$

given that the random sample and order statistics are $Y_1, Y_2, \dots, Y_n$ and $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$, respectively. The function in Equation (29) is expressed as:

$$f_{r:n}(y) = \frac{1}{B(r,n-r+1)}\sum_{l=0}^{n-r}(-1)^l \binom{n-r}{l}[G(y)]^{r+l-1}g(y) \qquad (30)$$

Injecting Equations (5) and (6) into Equation (30) and expanding using the binomial series, the r$^{th}$ order statistics for the GOGE$_{ST}$ is specified as

$$f_{r:n}(y) = \frac{1}{B(r,n-r+1)}\vartheta_{l,i,j,k,g,h}\, y^h(\upsilon+y^2)^{-\left(\frac{h+3}{2}\right)} \qquad (31)$$

Where

$$\vartheta_{l,i,j,k,g,h} = \varphi\tau\upsilon\sum_{l=0}^{n-r}\sum_{i,j,k,g,h=0}^{\infty}\frac{(-1)^{i+j+kl+g+h}(i+1)^j}{j!2^{g+1}}\binom{\tau(r+l)-1}{i} \times \binom{-(j+2)}{k}\binom{n-r}{l}\binom{\varphi(j+k+1)-1}{g}\binom{g}{h}$$

**Remark:** The minimum and maximum order statistics is obtained by injecting $r = 1$ and $r = n$ into Equation (31).

**Maximum Likelihood Estimation for the GOGE$_{ST}$**

The observed random-values from the $\mathrm{GOGE_{ST}}(y;\varphi,\tau,\upsilon)$ are denoted as $y_1, y_2, \dots, y_n$. The GOGE$_{ST}$ log-likelihood-function (LLF) is specified as

$$l(\varphi,\tau,\upsilon) = n\log\varphi + n\log\tau + n\log\upsilon - n\log 2 - \frac{3}{2}\sum_{i=1}^{n}\log(\upsilon + y_i^2)$$

$$+ (\varphi - 1)\sum_{i=1}^{n}\log w - \sum_{i=1}^{n}\left\{\frac{w^{\varphi}}{1 - w^{\varphi}}\right\}$$

$$+(\tau - 1)\sum_{i=1}^{n}\log\left\{1 - e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\} - 2\sum_{i=1}^{n}\log(1 - w^{\varphi}) \qquad (32)$$

where $w = \left(\frac{1}{2}\left(1 + \frac{y_i}{\sqrt{\upsilon + y_i^2}}\right)\right)$. Differentiating Equation (32) with respect to $\varphi, \tau$ and $\upsilon$ gives the follows:

$$\frac{\partial l}{\partial\varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n}\log w - \sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\{1 - w^{\varphi}\}^2} - (\tau - 1)\sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\{1 - w^{\varphi}\}^2\left\{1 - e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\}}e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}$$

$$+ 2\sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\{1 - w^{\varphi}\}}$$

$$\frac{\partial l}{\partial\tau} = \frac{n}{\tau} + \sum_{i=1}^{n}\log\left\{1 - e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\}$$

$$\frac{\partial l}{\partial\upsilon} = \frac{n}{\upsilon} - \frac{3}{2}\sum_{i=1}^{n}\frac{1}{(\upsilon + y_i^2)} + (\varphi - 1)\sum_{i=1}^{n}\frac{y_i}{4(\upsilon + y_i^2)^{\frac{3}{2}}w} - \varphi\sum_{i=1}^{n}\frac{y_i w^{\varphi-1}}{4(\upsilon + y_i^2)^{\frac{3}{2}}(1 - w^{\varphi})^2}$$

$$-\varphi(\tau - 1)\sum_{i=1}^{n}\frac{y_i(\upsilon + y_i^2)^{\varphi-1}}{4(\upsilon + y_i^2)^{\frac{3}{2}}(1 - w^{\varphi})^2\left(1 - e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right)}$$

$$-\varphi\sum_{i=1}^{n}\frac{y_i w^{\varphi-1}}{2(\upsilon + y_i^2)^{\frac{3}{2}}(1 - w^{\varphi})}$$

The ML estimates $\langle \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\upsilon}} \rangle$ of the parameters $\langle \boldsymbol{\varphi}, \boldsymbol{\tau}, \boldsymbol{\upsilon} \rangle$, are found by solving the following $\frac{\partial l}{\partial \varphi} = 0$, $\frac{\partial l}{\partial \tau} = 0$ and $\frac{\partial l}{\partial \upsilon} = 0$. Also, R (optim function) can be used in obtaining the estimates of the parameters.

**Asymmetric GARCH-type Models**

For consultants and economic experts alike, financial time series volatility modeling is an essential field. To model time-varying volatility, the autoregressive conditional heteroscedasticity (ARCH) model was created (Engle, 1982; Bollerslev, 1986). To account for leverage effects in the conditional variance, asymmetric GARCH-type models were developed, including the exponential GARCH (EGARCH) model (Nelson, 1991), the nonlinear asymmetric GARCH (AGARCH) model (Engle and Ng, 1993), the threshold GARCH (TGARCH) model by (Zakoian, 1994), the asymmetric power ARCH (APARCH) model (Ding et al., 1993), the quadratic GARCH (QGARCH) model (Sentana, 1995), and the Glosten-Jagannathan-Runkle GARCH (GJRGARCH) model Glosten et al. (1993). The log-return is indicated as $r_t$, and the GARCH-(1,1)-model is specified as

$$r_t = \mu + \psi_t$$

$$\psi_t = z_t\sqrt{h_t^2}, \quad z_t \sim \text{i.i.d.}$$

$$h_t^2 = \omega + \eta_1\psi_{t-1}^2 + \eta_2 h_{t-1}^2, \tag{33}$$

where $\omega > 0$, $\eta_1 > 0$, $\eta_2 > 0$, $z_t$ is the innovation density with $E(z_t) = 0$ and $\text{var}(z_t) = 1$, $\mu_t$, $h_t^2$ and $\psi_t$ denote the conditional-mean and-variance, and error. However, the EGARCH, TGARCH and APARCH models are considered in this study. The conditional variance specifications of EGARCH (1,1), TGARCH (1,1) and APARCH (1,1) models are specified as:

$$\ln h_t^2 = \omega + \eta_1 \frac{\psi_{t-1}}{h_{t-1}} + \eta_3 \left|\frac{\psi_{t-1}}{h_{t-1}}\right| + \eta_2 \ln h_{t-1}^2 \tag{34}$$

$$h_t = \omega + \eta_1\psi_{t-1}^2 + \eta_3 H_{t-1}\psi_{t-1}^2 + \eta_2 h_{t-1}, \text{ with } H_{t-1} = \begin{cases} 1, & \text{if} \quad \psi_{t-1} < 0, \\ 0, & \text{if} \quad \psi_{t-1} \geq 0, \end{cases} \tag{35}$$

$$h_t^\delta = \omega + \eta_1(|\psi_{t-1}| - \eta_3\psi_{t-1})^\delta + \eta_2 h_{t-1}^\delta \tag{36}$$

where $\eta_3$ is the leverage effect parameter, $\text{In}(h_t^2)$, $\text{In}(h_{t-1}^2)$ are the conditional log variance for present and preceding days, respectively, $\psi_{t-1}$ is the preceding day error and $\delta$ is the Taylor effect parameter. The existing innovation densities commonly used in the asymmetric GARCH framework are the Skew-Normal-Density (SND), Skew-Student-t-Density (SSD), Skew-Generalized-Error-Density (SGED), Generalized-Hyperbolic-Density (GHD) and Johnson-Reparametrized (SU) Density (JSUD).

**Generalized-Odd-Generalized-Exponentiated-Skew-t-Innovation Density ($GOGE_{ST}$)**

The standardized density function of the $GOGE_{ST}$ is specified as

$$f(\varphi,\tau,\upsilon,\sqrt{h_t^2}) = \left(\frac{1}{\{h_t^2\}^{\frac{1}{2}}}\right)\frac{\varphi\tau\upsilon[\vartheta_t]^{\varphi-1}e^{\frac{-(\vartheta_t)^{\varphi}}{1-(\vartheta_t)^{\varphi}}}}{2\left(\upsilon+\left(\frac{\vartheta_t}{\sqrt{h_t^2}}\right)^2\right)^{\frac{3}{2}}[1-(\vartheta_t)^{\varphi}]^2}\left\{1-e^{\frac{-(\vartheta_t)^{\varphi}}{1-(\vartheta_t)^{\varphi}}}\right\}^{\tau-1} \tag{37}$$

The log likelihood function (LLF) is specified as

$$\begin{aligned}\mathrm{LLF}(\varphi,\tau,\upsilon,h_t^2) &= n\log\varphi + n\log\tau + n\log\upsilon - n\log 2 - \frac{3}{2}\sum_{t=1}^{n}\log\left(\upsilon+\left(\frac{\vartheta_t}{\sqrt{h_t^2}}\right)^2\right)\\ &+(\varphi-1)\sum_{t=1}^{n}\log(\vartheta_t) - 2\sum_{t=1}^{n}\log\{1-(\vartheta_t)^{\varphi}\} - \sum_{t=1}^{n}\left\{\frac{(\vartheta_t)^{\alpha}}{1-(\vartheta_t)^{\alpha}}\right\}\\ &+(\tau-1)\sum_{t=1}^{n}\log\left\{1-e^{\frac{-(\vartheta_t)^{\varphi}}{1-(\vartheta_t)^{\varphi}}}\right\} - \frac{1}{2}n\log h_t^2\end{aligned} \tag{38}$$

where $\vartheta_t = \frac{1}{2}\left(1+\frac{\frac{\psi_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\psi_t}{\sqrt{h_t^2}}\right)^2}}\right)$ and $h_t^2$ is the asymmetric GARCH-type models per unknown vector parameters.

**Selection Criteria for the Models**

The models presented in Section 3 are chosen based on the information criterion. The models' correctness is evaluated using the updated Brooks and Burke (2003) information criteria. The adjusted AIC and BIC are as follows:

$$\mathrm{AIC} = \frac{2c}{T} - \frac{2\mathrm{LLF}}{T}$$

$$\mathrm{BIC} = \frac{c\log_e(T)}{T} - \frac{2\mathrm{LLF}}{T} \tag{39}$$

where T is the total number of estimated parameters, c is the number of observations, and LLF stands for the estimated log-likelihood value. When it comes to the designated innovation density, the model that performs the best is the one with the lowest AIC and BIC values.

**Forecasts Evaluation**

The root mean square root (RMSE), mean absolute error (MAE), and mean square error (MSE) are used to assess the prediction performance of the asymmetric GARCH-type models. For the volatility forecasts, the MSE, RMSE, and MAE are provided as

$$\mathrm{MSE} = \frac{1}{\mathrm{T}}\sum_{\mathrm{t}=1}^{\mathrm{T}}(\Phi_{\mathrm{t}})^2$$

$$\mathrm{RMSE} = \sqrt{\frac{1}{\mathrm{T}}\sum_{\mathrm{t}=1}^{\mathrm{T}}(\Phi_{\mathrm{t}})^2} \qquad (40)$$

$$\mathrm{MAE} = \frac{1}{\mathrm{T}}\sum_{\mathrm{t}=1}^{\mathrm{T}}|\Phi_{\mathrm{t}}|$$

where $\Phi_{\mathrm{t}} = \hat{\mathrm{h}}_{\mathrm{t}} - \mathrm{h}_{\mathrm{t}}$, $\mathrm{T}$ is the sample size, $\hat{\mathrm{h}}_{\mathrm{t}}$ and $\mathrm{h}_{\mathrm{t}}$ denote volatility forecast and realized volatility. For predicting the daily return volatility, the model with the lowest MSE, RMSE, and MAE values is regarded as the most appropriate.

## RESULTS AND DISCUSSION

**Simulation-Study**

The simulation process for the MLE is presented as follows: we generated N = 10,000 samples from the $\mathrm{GOGE}_{\mathrm{ST}}$ with sizes n = 50, 300 and 1000 using R-programming. The accuracy of the ML estimates is appraised via the 'average estimates (AEs)', 'absolute bias (Abs)', 'mean square errors (MSEs)' and 'root mean square roots (RMSEs)'.

**Table 2:** Numerical values of AEs, Abs, MSEs and RMSEs

| | | $\varphi = 1.0$ $\tau = 1.5$ $\upsilon = 1.3$ | | | | | | $\varphi = 1.2$ $\tau = 1.7$ $\upsilon = 1.5$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n | Par. | AE | Abs | MSE | RMSE | n | Par. | AE | Abs | MSE | RMSE |
| 50 | $\varphi$ | 0.987 | 0.013 | 0.060 | 0.245 | 50 | $\varphi$ | 1.163 | 0.037 | 0.049 | 0.221 |
| | $\tau$ | 1.772 | 0.272 | 0.581 | 0.762 | | $\tau$ | 1.937 | 0.237 | 0.343 | 0.586 |
| | $\upsilon$ | 1.400 | 0.100 | 0.264 | 0.514 | | $\upsilon$ | 1.582 | 0.082 | 0.180 | 0.424 |
| 300 | $\varphi$ | 0.995 | 0.005 | 0.023 | 0.153 | 300 | $\varphi$ | 1.176 | 0.024 | 0.022 | 0.148 |
| | $\tau$ | 1.602 | 0.102 | 0.184 | 0.428 | | $\tau$ | 1.816 | 0.116 | 0.142 | 0.377 |
| | $\upsilon$ | 1.348 | 1.348 | 0.080 | 0.282 | | $\upsilon$ | 1.558 | 0.058 | 0.062 | 0.249 |
| 1000 | $\varphi$ | 0.997 | 0.003 | 0.010 | 0.100 | 1000 | $\varphi$ | 1.181 | 0.239 | 0.012 | 0.109 |

| n | Par. | AE | Abs | MSE | RMSE | n | Par. | AE | Abs | MSE | RMSE |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | τ | 1.545 | 0.045 | 0.069 | 0.262 | | τ | 1.773 | 0.111 | 0.071 | 0.266 |
| | υ | 1.324 | 0.024 | 0.031 | 0.176 | | υ | 1.541 | 0.131 | 0.029 | 0.172 |

| | | $\varphi = 2.2 \quad \tau = 2.0 \quad \upsilon = 2.5$ | | | | | | $\varphi = 3.5 \quad \tau = 3.2 \quad \upsilon = 3.0$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n | Par. | AE | Abs | MSE | RMSE | n | Par. | AE | Abs | MSE | RMSE |
| 50 | φ | 2.370 | 0.170 | 0.459 | 0.677 | 50 | φ | 3.832 | 0.3316 | 1.087 | 1.042 |
| | τ | 2.087 | 0.087 | 0.390 | 0.625 | | τ | 3.383 | 0.183 | 0.828 | 0.910 |
| | υ | 2.426 | 0.074 | 0.409 | 0.640 | | υ | 2.875 | 0.125 | 0.707 | 0.841 |
| 300 | φ | 2.216 | 0.016 | 0.138 | 0.372 | 300 | φ | 3.521 | 0.021 | 0.383 | 0.619 |
| | τ | 2.070 | 0.070 | 0.163 | 0.403 | | τ | 3.315 | 0.115 | 0.316 | 0.562 |
| | υ | 2.536 | 0.036 | 0.162 | 0.402 | | υ | 3.056 | 0.056 | 0.287 | 0.536 |
| 1000 | φ | 2.192 | 0.008 | 0.075 | 0.273 | 1000 | φ | 3.475 | 0.025 | 0.211 | 0.459 |
| | τ | 2.055 | 0.055 | 0.092 | 0.304 | | τ | 3.283 | 0.083 | 0.166 | 0.407 |
| | υ | 2.542 | 0.042 | 0.090 | 0.300 | | υ | 3.065 | 0.065 | 0.159 | 0.398 |

Based on the simulation process, the MLE is a suitable method for estimating the GOGEST parameters. Table 2's results show that the parameter estimates are quite stable and very close to the true parameter values for the various sample sizes. The AE' tend to be closer to the true parameter values, and the Absbias, MSEs, and RMSEs decrease as the sample size increases.

## Practical Finding

### Dataset description

The Bitcoin (BTC) cryptocurrency price index is used to evaluate how well the asymmetric GARCH-type models forecast daily volatility. 3402 daily log-returns from 02/02/2012 to 31/05/2021 make up the used data. The 151 daily log-returns from the year 2021 are utilized to evaluate the out-of-sample forecasting performance, while the 3251 daily log-returns for nine years (2012–2020) are used for the estimate procedure. Figure 2 shows the graphical plots, while Table 3 summarizes the log-return statistics for Bitcoin (BTC). The daily log-returns are clearly not normally distributed, as evidenced by the huge Jarque-Bera (JB) test statistic value ($p < 0.001$), which is a result of the obvious negative skewness and excess kurtosis. Furthermore, the conditional variance's incidence of ARCH effects is specified by the Lagrange-multiplier (LM) test.

**Table 3:** Descriptive statistics for BTC.

| | Values |
|---|---|
| Number of observations | 3251 |
| Mean | 0.2604 |
| Median | 0.1534 |
| Minimum | -48.090 |
| Maximum | 30.830 |
| Standard Deviation | 4.4438 |
| Skewness | -0.8566 |
| Kurtosis | 13.3007 |
| Jarque-Bera | 24398 ($p < 0.001$) |
| ARCH (2) | 314.71 ($p < 0.001$) |

**Estimation**

The SND, SSD, SGED, JSUD, GHD, and $GOGE_{ST}$ innovation densities are used to estimate the asymmetric models. The rugarch package in R-program is used to get the parameter estimates of the models under SND, SSD, SGED, JSUD and GHD densities. The models' parameters under the $GOGE_{ST}$ innovation density are estimated using the Optim function in R program. The parameter estimates for six innovation densities for the EGARCH (1,1), TGARCH (1,1), and APGARCH (1,1) models are shown in Table 4. The daily log-returns have a leverage effect, as evidenced by the highly statistically significant parameter estimates of the conditional variance and the parameter's significance at the standard level. Because of this, the shocks' effects are asymmetrical, meaning that negative shocks have a greater effect on volatility than positive shocks of the same magnitude.

Table 5 provides crucial statistics for the selection of models. It indicates that, in comparison to other models, the EGARCH-$GOGE_{ST}$ model has the highest log-likelihood and the lowest AIC and BIC values. The best asymmetric model for the conditional variance of the BTC log-returns is thus determined to be the EGARCH-$GOGE_{ST}$ model.

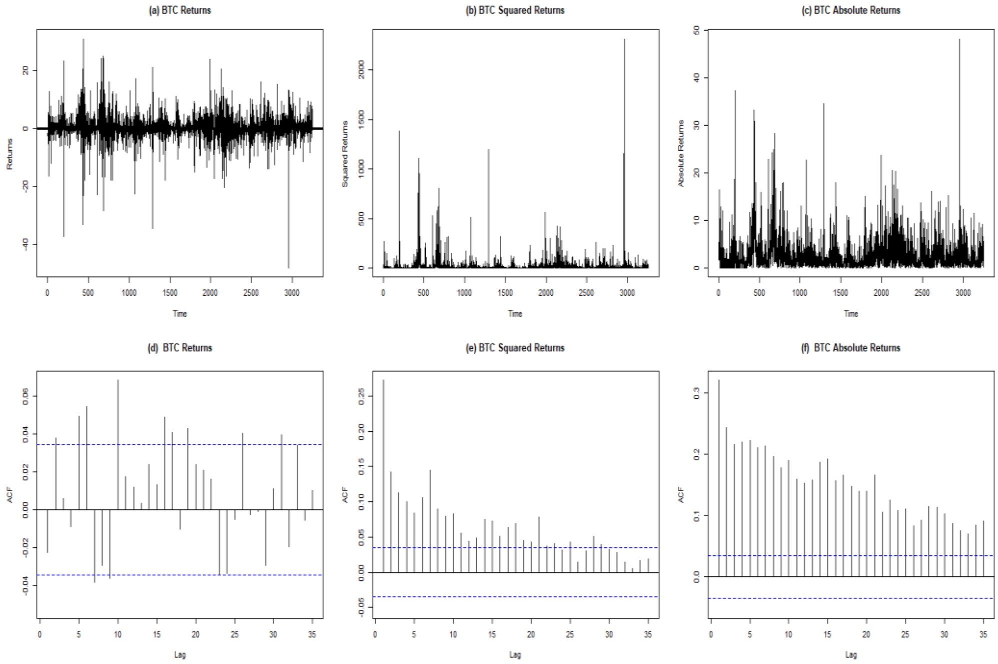


**Figure 2:** BTC daily log-returns, squared and absolute log-returns and sample autocorrelations.

The BTC daily log-returns don't exhibit any discernible patterns. As seen in Figure 2(b, c), there is some persistence in the squared and absolute log-returns. Plots depicting low volatility values followed by low volatility values and high volatility values followed by high volatility values, respectively, are specific examples of volatility clustering. Furthermore, as seen in Figures 2(e, f), the squared and absolute log-returns exhibit positive autocorrelation, but the daily log-returns do not clearly demonstrate serial correlation.

**Table 4:** Estimates of the asymmetric models

| | | | **EGARCH (1,1)** | | | |
|---|---|---|---|---|---|---|
| **Par.** | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
| **μ** | 0.0000 | 0.1671‘***’ | 0.0000 | 0.1025‘’ | 0.1398‘*’ | 4.402E-04 |
| **ω** | 0.2447‘***’ | 0.0684‘***’ | 0.0850‘***’ | 0.0674‘***’ | 0.0660‘***’ | -1.713E-04 |
| **$\eta_1$** | -0.0232‘.’ | 0.0419‘*’ | 0.0153‘’ | 0.0362‘.’ | 0.0359‘.’ | 0.0332‘.’ |
| **$\eta_2$** | 0.9248‘***’ | 0.9772‘***’ | 0.9670‘***’ | 0.9752‘***’ | 0.9753‘***’ | 0.8119‘***’ |
| **$\eta_3$** | 0.2933‘***’ | 0.3236‘***’ | 0.2852‘***’ | 0.2928‘***’ | 0.2905‘***’ | -0.0350 |
| **α** | 0.8873‘***’ | - | - | 0.2500‘.’ | - | - |
| **ξ** | - | 0.9856‘***’ | - | - | - | - |
| **ν** | - | 2.6900‘***’ | 0.8226‘***’ | - | - | - |
| **η** | - | - | 1.0000‘***’ | - | - | - |
| **β** | - | - | - | -0.1765 | - | - |
| **λ** | - | - | - | -1.0394‘***’ | - | - |
| **υ** | - | - | - | - | -0.0433 | - |

| | | | | | | |
|---|---|---|---|---|---|---|
| **ϑ** | - | - | - | - | 1.0531‘***’ | 3.0281‘***’ |
| **γ** | - | - | - | - | - | 7.395E+01 ‘***’ |
| **τ** | - | - | - | - | - | 0.2964 |

**TGARCH (1,1)**

| Par. | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
|---|---|---|---|---|---|---|
| **μ** | 0.0075 | 0.1490‘*’ | 0.0000 | 0.0908‘ ’ | 0.1272‘*’ | 3.95E-03 |
| **ω** | 0.2854‘***’ | 0.0902‘*’ | 0.1168‘***’ | 0.0895‘*’ | 0.0886‘***’ | 0.2853‘**’ |
| **$\eta_1$** | 0.1681‘***’ | 0.2050‘***’ | 0.1801‘***’ | 0.1857‘***’ | 0.1843‘***’ | 1.2513‘***’ |
| **$\eta_2$** | 0.8184‘***’ | 0.8652‘***’ | 0.8521‘***’ | 0.8621‘***’ | 0.8626‘***’ | 0.3644‘***’ |
| **$\eta_3$** | 0.1036‘.’ | -0.0941‘.’ | -0.0929 | -0.0890 | -0.0883 | 0.0549 |
| **α** | 0.8881‘***’ | - | - | 0.2500‘.’ | - | - |
| **ξ** | - | 0.9803‘***’ | - | - | - | - |
| **ν** | - | 2.6880‘***’ | 0.8220‘***’ | - | - | - |
| **η** | - | - | 1.0000‘***’ | - | - | - |
| **β** | - | - | - | -0.1848‘ ’ | - | - |
| **λ** | - | - | - | -1.0349‘***’ | - | - |
| **υ** | - | - | - | - | -0.0491 | - |
| **ϑ** | - | - | - | - | 1.0519‘***’ | 1.5285‘***’ |
| **γ** | - | - | - | - | - | 1.716E+02 ‘***’ |
| **τ** | - | - | - | - | - | 1.227E+01 |

**APARCH (1,1)**

| Par. | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
|---|---|---|---|---|---|---|
| **μ** | 0.0631 | 0.1467‘***’ | 0.0000 | 0.0908‘ ’ | 0.1275‘.’ | 3.31E-04 |
| **ω** | 0.4302‘***’ | 0.0793‘***’ | 0.1023‘***’ | 0.0802‘***’ | 0.0788‘.’ | 0.0307‘*’ |
| **$\eta_1$** | 0.1732‘***’ | 0.1957‘***’ | 0.1736‘***’ | 0.1795‘***’ | 0.1781‘***’ | 0.0959‘***’ |
| **$\eta_2$** | 0.8121‘***’ | 0.8673‘***” | 0.8545‘***’ | 0.8641‘***’ | 0.8647‘***’ | 0.7386‘***’ |
| **$\eta_3$** | 0.0778‘.’ | -0.0994‘ ’ | -0.0085 | -0.0937 | -0.0931 | 8.314E-03 |
| **δ** | 1.3160‘***’ | 0.8933‘***’ | 0.8730‘***’ | 0.8995‘***’ | 0.8956‘***’ | 1.0677‘***’ |
| **α** | 0.8921‘***’ | - | - | 0.2500‘.’ | - | - |
| **ξ** | - | 0.9797‘***’ | - | - | - | - |
| **ν** | - | 2.6859‘***’ | 0.8207‘***’ | - | - | - |
| **η** | - | - | 1.0000‘***’ | - | - | - |
| **β** | - | - | - | -0.1838‘ ’ | - | - |
| **λ** | - | - | - | -1.0331‘***’ | - | - |
| **υ** | - | - | - | - | -0.0488 | - |
| **ϑ** | - | - | - | - | 1.0512‘***’ | 1.5395‘***’ |
| **γ** | - | - | - | - | - | 1.382E+02 ‘***’ |
| **τ** | - | - | - | - | - | 3.879E+01 |

Significance levels: 0 ‘***’, 0.001 ‘**’, 0.01 ‘*’, 0.05 ‘.’, 0.1 ‘ ’, 1

The conditional variance parameter estimates for the asymmetric models under the $GOGE_{ST}$ density are highly statistically significant, and the parameter $\eta_3$ is significant at the standard level, indicating that the daily log-returns have a leverage effect. Because of this, the shocks' effects are asymmetrical, meaning that negative shocks have a greater effect on volatility than positive shocks of the same magnitude. Table 5 provides crucial statistics for the selection of models. It indicates that, in comparison to other models, the asymmetric models under $GOGE_{ST}$ innovation density have the lowest AIC and BIC values and the maximum log-likelihood. But because it has the lowest AIC and BIC values, the EGARCH-GOGEST model is chosen as the optimal volatility model for the conditional variance of the BTC log-returns.

**Table 5:** Comparison of the Models for Selection

| Model | LL | AIC | BIC |
|---|---|---|---|
| **EGARCH- SND** | -8930.696 | 5.4978 | 5.5090 |
| **EGARCH- SSD** | -8409.936 | 5.1781 | 5.1912 |
| **EGARCH-SGED** | -8418.751 | 5.1835 | 5.1966 |
| **EGARCH-GHD** | -8405.539 | 5.1760 | 5.1909 |
| **EGARCH-JSUD** | -8404.530 | 5.1747 | 5.1878 |
| **EGARCH- $GOGE_{ST}$** | -5929.735 | 3.6529 | 3.6678 |
| | | | |
| **TGARCH- SND** | -8931.886 | 5.4985 | 5.5098 |
| **TGARCH- SSD** | -8404.229 | 5.1745 | 5.1877 |
| **TGARCH-SGED** | -8413.927 | 5.1805 | 5.1936 |
| **TGARCH-GHD** | -8399.637 | 5.1723 | 5.1873 |
| **TGARCH-JSUD** | -8398.837 | 5.1712 | 5.1843 |
| **TGARCH- $GOGE_{ST}$** | -7778.089 | 4.7900 | 4.8049 |
| | | | |
| **APARCH- SND** | -8930.696 | 5.4978 | 5.5090 |
| **APARCH- SSD** | -8409.936 | 5.1781 | 5.1912 |
| **APARCH-SGED** | -8418.751 | 5.1835 | 5.1966 |
| **APARCH-GHD** | -8405.539 | 5.1760 | 5.1909 |
| **APARCH-JSUD** | -8404.530 | 5.1747 | 5.1878 |
| **APARCH- $GOGE_{ST}$** | -6620.921 | 4.0781 | 4.0931 |

The GARCH-type-$GOGE_{ST}$ and other models' diagnostic tests are shown in Table 6. The squared standardized residuals from the fitted models show no evidence of serial correlation, according to the Ljung-Box statistic. Similarly, the standardized residuals from the estimated models show no additional ARCH processes, according to the ARCH-LM statistic, suggesting that the conditional variance equations are given appropriately. As a result, every model performed admirably in characterizing the dynamical aspects represented by the log-returns.

**Table 6:** Diagnostic Tests for the Models

| Model | Ljung-Box Statistic | p-value | ARCH-LM Statistic | p-value |
|---|---|---|---|---|
| **EGARCH-SND** | 2.568 | 0.9998 | 1.685 | 0.9982 |
| **EGARCH-SSD** | 3.138 | 0.9995 | 2.026 | 0.9961 |
| **EGARCH-SGED** | 3.249 | 0.9993 | 2.046 | 0.9960 |
| **EGARCH-GHD** | 3.157 | 0.9994 | 2.033 | 0.9961 |
| **EGARCH-JSUD** | 3.155 | 0.9995 | 2.036 | 0.9961 |
| **EGARCH-$GOGE_{ST}$** | 21.297 | 0.8784 | 20.062 | 0.9149 |
| **TGARCH-SND** | 3.429 | 0.9991 | 2.641 | 0.9887 |
| **TGARCH-SSD** | 4.158 | 0.9972 | 3.176 | 0.9770 |
| **TGARCH-SGED** | 3.806 | 0.9983 | 2.749 | 0.9867 |
| **TGARCH-GHD** | 4.107 | 0.9974 | 3.119 | 0.9785 |
| **TGARCH-JSUD** | 4.114 | 0.9973 | 3.129 | 0.9782 |
| **TGARCH-$GOGE_{ST}$** | 4.849 | 0.9933 | 4.274 | 0.9341 |
| **APARCH-SND** | 2.948 | 0.9996 | 2.064 | 0.9958 |
| **APARCH-SSD** | 4.902 | 0.9929 | 3.946 | 0.9497 |
| **APARCH-SGED** | 4.436 | 0.9959 | 3.395 | 0.9705 |
| **APARCH-GHD** | 4.755 | 0.9940 | 3.788 | 0.9564 |
| **APARCH-JSUD** | 4.798 | 0.9937 | 3.839 | 0.9543 |
| **APARCH-$GOGE_{ST}$** | 4.179 | 0.9971 | 3.499 | 0.9671 |

**Forecasts**

Table 7 displays the asymmetric models' out-of-sample performance, and Table 8 shows a ranking-based comparison of the innovation densities. The evaluation metrics show that the asymmetric models with the lowest MSE', RMSE', and MAE' values are those under the $GOGE_{ST}$ density. But with the lowest MSE', RMSE', and MAE' values, it shows that the TGARCH-$GOGE_{ST}$ model is statistically efficient and performs better than the rest in predicting BTC volatility. Furthermore, as determined by the MSE', RMSE', and MAE' for the EGARCH (1,1), TGARCH (1,1), and APARCH (1,1) models, the GOGEST innovation density has the lowest overall rank value.

**Table 7:** Forecasts Evaluation

| Model | MSE' | RMSE' | MAE' |
|---|---|---|---|
| **EGARCH- SND** | 23.5393 | 4.8517 | 3.5653 |
| **EGARCH- SSD** | 23.5110 | 4.8488 | 3.5709 |
| **EGARCH-SGED** | 23.5394 | 4.8517 | 3.5653 |
| **EGARCH-GHD** | 23.5153 | 4.8493 | 3.5678 |
| **EGARCH-JSUD** | 23.5118 | 4.8489 | 3.5695 |
| **EGARCH- $GOGE_{ST}$** | 22.8307 | 4.7781 | 3.4785 |
| | | | |
| **TGARCH- SND** | 23.5369 | 4.8515 | 3.5652 |
| **TGARCH- SSD** | 23.5114 | 4.8488 | 3.5699 |
| **TGARCH-SGED** | 23.5393 | 4.8517 | 3.5653 |
| **TGARCH-GHD** | 23.5170 | 4.8494 | 3.5673 |
| **TGARCH-JSUD** | 23.5127 | 4.8490 | 3.5689 |
| **TGARCH- $GOGE_{ST}$** | 20.4212 | 4.5190 | 3.1792 |
| | | | |
| **APARCH- SND** | 23.5221 | 4.8499 | 3.5664 |
| **APARCH- SSD** | 23.5115 | 4.8489 | 3.5698 |
| **APARCH-SGED** | 23.5393 | 4.8517 | 3.5653 |
| **APARCH-GHD** | 23.5170 | 4.8494 | 3.5673 |
| **APARCH-JSUD** | 23.5127 | 4.8490 | 3.5689 |
| **APARCH- $GOGE_{ST}$** | 22.7278 | 4.7674 | 3.4577 |

**Table 8:** Innovation Densities Ranking in Forecasts Appraisal

| | EGARCH (1,1) | | | | | |
|---|---|---|---|---|---|---|
| | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
| **MSE** | 5 | 2 | 6 | 4 | 3 | 1 |
| **RMSE** | 5.5 | 2 | 5.5 | 4 | 3 | 1 |
| **MAE** | 2.5 | 6 | 2.5 | 4 | 5 | 1 |
| **TOTAL** | 13 | 10 | 14 | 12 | 11 | 3 |
| | **TGARCH (1,1)** | | | | | |
| | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
| **MSE** | 5 | 2 | 6 | 4 | 3 | 1 |
| **RMSE** | 5 | 2 | 6 | 4 | 3 | 1 |
| **MAE** | 2 | 6 | 3 | 4 | 5 | 1 |
| **TOTAL** | 12 | 10 | 15 | 12 | 11 | 3 |
| | **APARCH (1,1)** | | | | | |
| | SND | SSD | SGED | GHD | JSUD | $GOGE_{ST}$ |
| **MSE** | 5 | 2 | 6 | 4 | 3 | 1 |
| **RMSE** | 5 | 2 | 6 | 4 | 3 | 1 |
| **MAE** | 3 | 6 | 2 | 4 | 5 | 1 |
| **TOTAL** | 13 | 10 | 14 | 12 | 11 | 3 |

The use of the asymmetric GARCH model with $GOGE_{ST}$ density for estimating and forecasting the volatility of the BTC log-returns is generally supported by the comparison of the volatility models in this study. Furthermore, for the asymmetric GARCH (1,1) models taken into consideration in this study, the suggested $GOGE_{ST}$ innovation density appears to be typically the best.

## CONCLUSION

The GOGEST distribution is derived in this study and is highly effective in representing skewness, leptokurtic behaviour, and unimodal forms. Three asymmetric GARCH models with GOGEST innovation density in relation to the SND, SSD, SGED, SHD, and JUSD are used to estimate the volatility of the BTC cryptocurrency. Three forecast performance measures are used to assess the accuracy model. According to the selection criteria, the asymmetric GARCH-GOGEST models are optimally the best models. More so, the EGARCH-GOGEST model, which mirrors the fundamental process in relation to leptokurtic innovation, serial correlation, and asymmetric volatility clustering, is the most capable in describing the dynamic features of the BTC returns. The study's conclusions support the idea that asymmetric GARCH models enhance prediction accuracy. Moreover, the outcomes confirm that the TGARCH-GOGEST model outperforms other models in out-of-sample forecasting when it comes to modeling BTC volatility. According to this study, the majority of financial returns exhibit non-normal characteristics such skewness, asymmetric volatility clustering, heavy tails, and excess kurtosis.

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