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POISSON-NEW QUADRATIC-EXPONENTIAL DISTRIBUTION

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Abstract

This proposed distribution is a discrete compound probability distribution with only one parameter. To get this distribution, Poisson distribution has been mixed with the New Quadratic-Exponential distribution of Sah (2022). Hence, it is named as "Poisson-New Quadratic-Exponential Exponential Distribution (PNLED)". The important statistical characteristics needed to check the validity of this distribution have been derived and clearly explained. To check the validity of the theoretical works of this distribution, while using goodness of fit on some over-dispersed count data, what we have been found that this distribution seems a better alternative of Poisson-Lindley distribution (PLD) of Sankaran (1970), Poisson Mishra distribution (PMD) of Sah (2017) and Poisson-Modified Mishra distribution (PMMD) of Sah and Sahani (2023).

Keywords: Probability distribution, Poisson-Lindley distribution, Modified Mishra distribution, New Quadratic-Exponential distribution, Moments, Estimation, Mixing

INTRODUCTION

In the statistical literature, we can find many types of countable and continuous mixtures of Poisson distribution which have been obtained by using two, three and even four parameters to show a better alternative of PLD [See,6] for modeling of over-dispersed count data. What our experience suggest is that the importance of all distributions varies according to its applied fields, but if we want to compare any two or more than two distributions, it would be very appropriate if the mathematical structure and the number of parameters of theses distributions are similar and the same respectively.

The main purpose of proposing this distribution is to give a better alternative of PLD [see, 7], PMD [see,10] and PMMD [see,13] for statistical modeling of over-dispersed count data based on a single parameter. The probability mass function of PLD, PMD and PMMD have been given as in the expression (1), (2) and (3) respectively.

$$P_1(Y;\alpha)=\frac{\alpha^2(y+\alpha+2)}{(\alpha+1)^{y+3}}\quad ;y=0,1,2,\ldots\quad ;\alpha>0 \tag{1}$$

$$P_2(Y;\alpha)=\frac{\alpha^3[(1+\alpha)(y+\alpha+2)+(1+y)(2+y)]}{(\alpha^2+\alpha+1)(1+\alpha)^{y+3}}\quad ;y=0,1,2,\ldots\quad ;\alpha>0 \tag{2}$$

$$P_3(Y;\alpha)=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\frac{(1+\alpha)(1+\pi+\pi\alpha+y)+(1+y)(2+y)]}{(1+\alpha)^{y+3}}\right]\quad ;y=0,1,2,\ldots\quad ;\alpha>0 \tag{3}$$

Sankaran obtained PLD (1) by mixing Poisson distribution with Lindley distribution of Lindley to model count data [See, 7]. Lindley obtained a continuous probability distribution having a single parameter, which was named as Lindley distribution (LD) [See, 6] given by its probability density function (pdf) as

$$f_1(y;\alpha)=\frac{\alpha^2}{(1+\alpha)}(1+y)e^{-\alpha y}\ ;y>0,\alpha>0 \tag{4}$$

Ghitany et. al state that it is especially useful for modeling in mortality studies [See, 4]. Sah showed that PMD is a better alternative of PLD for modeling of over-dispersed count data [See, 10]. PMD (2) has been obtained by mixing Poisson distribution with Mishra distribution (MD) [See, 9] given by its pdf

$$f_2(y;\alpha)=\frac{\alpha^3}{(2+\alpha+\alpha^2)}(1+y+y^2)e^{-\alpha y}\ ;y>0;\alpha>0 \tag{5}$$

PMMD (3) has been obtained by mixing Poisson distribution with MMD [see,12] given by its density function

$$f_3(y;\alpha) = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}(\pi + y + y^2)e^{-\alpha y} \; ; y > 0; \alpha > 0 \tag{6}$$

Research process never end and hence there is always possibility to find a better concept with greater acceptability of a new idea. In this way, Sah and Sahani obtained New Quadratic-Exponential distribution (NQED), [See, 15],

a better alternative of LD, MD, Quadratic-Exponential distribution (QED) [See, 14] and MMD, for modeling of over-dispersed survival lifetime data. Probability density function of NQED has been given by

$$f_4(y;\alpha) = \frac{\alpha^3}{(2+\pi\alpha^2)}(\pi + y^2)e^{-\alpha y} \; ; y > 0; \alpha > 0 \tag{7}$$

An idea has developed on the basis of above-mentioned concept that if NQED (7) is a better alternative of LD (4), MD (5), and MMD (6) than it is also possible that PNQED, the proposed distribution, would be a better alternative of PLD (1), PMD (2), and PMMD (3). Sah explained the applications of Generalised Poisson-Mishra distribution (GPMD) in accident proneness in better way [see, 11]

To give a better and systematic shape of this paper, it has been classified under following headings and sub-headings. In the first section, the introduction needed for this paper has been presented. In the second section, material and methods has been presented. In the third section, the important part of the paper which is known to be results has been presented. To make the work of this section simple and systematic, it is divided into different sub-headings

- Poisson-New Quadratic-Exponential Distribution (PNQED) and Its Important Characteristics
- Statistical Moments and Essential Descriptive Measures of Statistics required for PNQED
- Methods of Estimation, and
- Applications and Goodness of Fit of PNQED.

In the fourth and the last section, the conclusion of this paper has been presented through scientific and statistical methods.

METHODS

The work of this paper is theoretical which is based on the concept of continuous mixture of Poisson distribution. Goodness of fit has been used in some over-dispersed secondary count data to check the validity of the theoretical work of this paper.

RESULTS

This section can be considered as an important part of the paper which has been derived and explained under the following different sub-headings

- Poisson-New Quadratic-Exponential Distribution (PNQED) and Important Characteristics
- Statistical Moments and Essential Descriptive Measures of Statistics required for PNQED
- Methods of Estimation, and
- Applications and Goodness of Fit of PNQED.

- *Poisson-New Quadratic-exponential Distribution (PNQED) and Its Important Characteristics:*

In this sub-heading, Probability mass function (pmf), Probability generating function (pgf) and Moment generating function (mgf) have been derived and explained. The proposed distribution has been obtained by mixing Poisson distribution with NQED (7), where NQED is a mixing distribution having a single parameter (α) and Poisson distribution is an original distribution with parameter (λ). In mixing process, the parameter λ of Poisson distribution follows NQED. The pmf of PNQED can be obtained as

$$P(Y;\alpha)=\int_0^\infty\left[\left(\frac{e^{-\lambda}\lambda^y}{y!}\right)\left(\frac{\alpha^3}{(2+\pi\alpha^2)}\right)(\pi+\lambda^2)e^{-\alpha\lambda}\right]d\lambda \;;y=0,1,2,\ldots;\lambda>0;\alpha>0$$

$$=\left(\frac{\alpha^3}{(2+\pi\alpha^2)}\frac{1}{y!}\right)\left[\pi\int_0^\infty\lambda^y e^{-(1+\alpha)\lambda}d\lambda+\int_0^\infty\lambda^{y+2}e^{-(1+\alpha)\lambda}d\lambda\right]$$

$$=\left(\frac{\alpha^3}{(2+\pi\alpha^2)}\right)\left[\frac{\pi(1+\alpha)^2+(1+y)(2+y)}{(1+\alpha)^{y+3}}\right];y=0,1,2,\ldots;\alpha>0 \tag{8}$$

The probability mass function of PNQED (8) at varying values of parameter (α) is presented in graph as follows

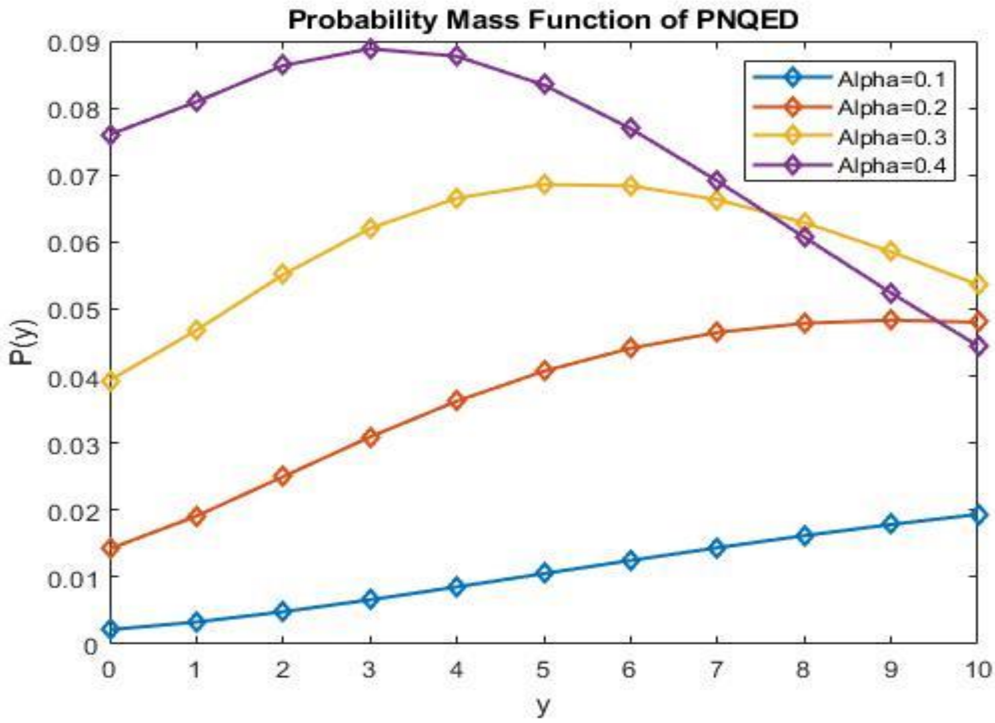

Fig.1 Graph of pmf of PNQED at $\alpha = 0.1, 0.2, 0.3, 0.4$

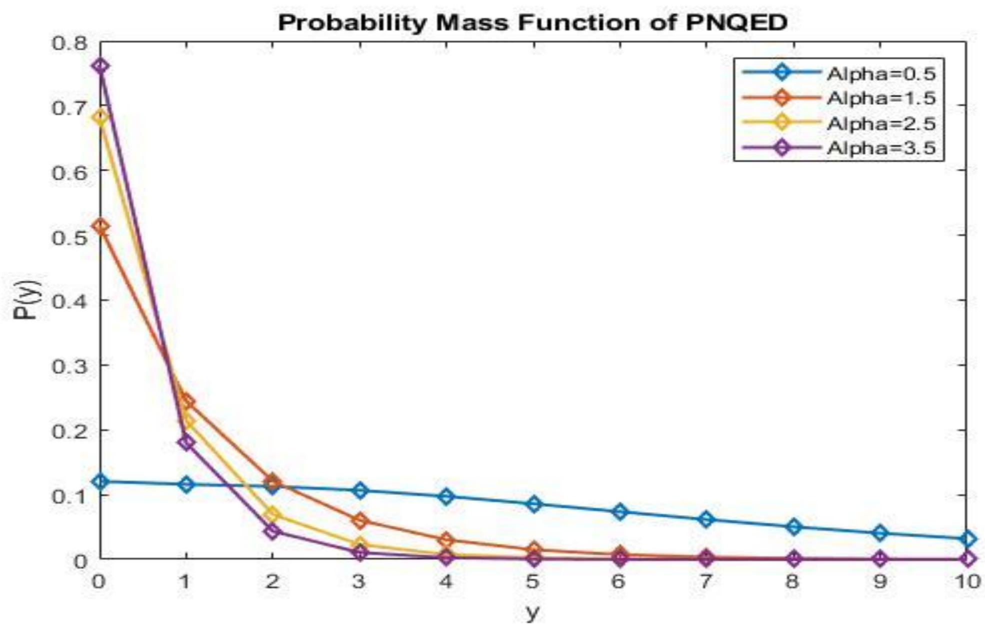

Fig.2 Graph of pmf of PNQED at $\alpha = 0.5, 1.5, 2.5, 3.5$

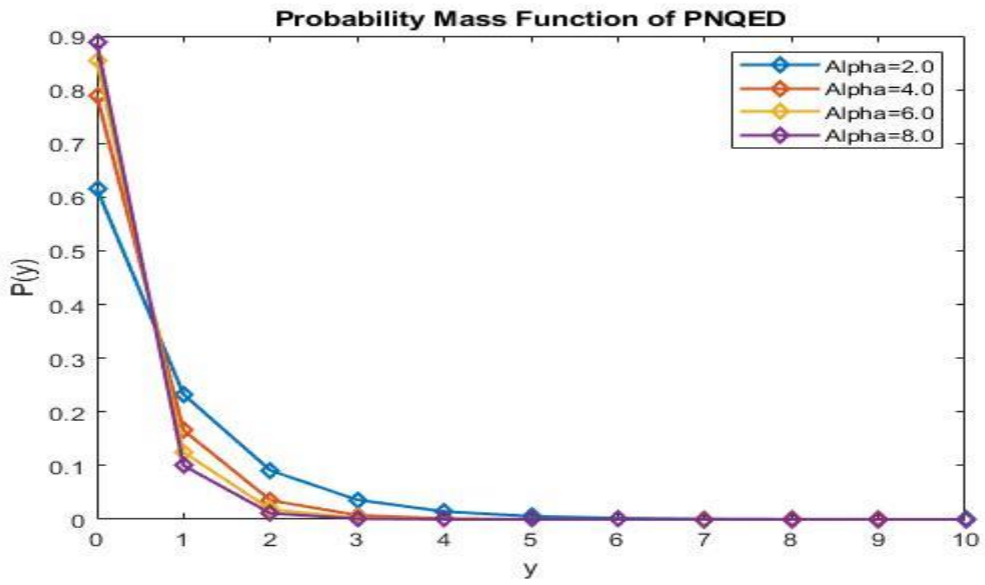

Fig.3 Graph of p.m.f. of PNQED a $\alpha = 2.0, 4.0, 6.0, 8.0$

From the above figures, it can be observed that at $y = 0$,as the value of α increases, value of $f(y)$ also increases. It can also be observed that the $f(y)$ curve touches X-axis first having the greater value of α .

Probability Generating Function(pgf.) of PNQED: It can be obtained as

$$P_Y^{(t)} = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty e^{-(1-t)\lambda}(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty(\pi+\lambda^2)e^{-(1+\alpha-t)\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\pi\int_0^\infty e^{-\lambda(1+\alpha-t)}d\lambda + \int_0^\infty \lambda^2 e^{-\lambda(1+\alpha-t)}d\lambda\right] = \left(\frac{\alpha^3}{(2+\pi\alpha^2)}\right)\left[\frac{\pi(1+\alpha-t)^2+2}{(1+\alpha-t)^3}\right];\alpha>0 \qquad (9)$$

Moment Generating Function (M.G.F.) of PMMD: It plays an important role in statistical analysis by generating the statistical moments and can be obtained as

$$M_Y^{(t)} = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty(\pi+\lambda^2)e^{-\lambda(1+\alpha-e^t)}d\lambda = \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\frac{\pi\Gamma1}{(1+\alpha-e^t)^1} + \frac{\Gamma3}{(1+\alpha-e^t)^3}\right]$$

$$= \left(\frac{\alpha^3}{(2+\pi\alpha^2}\right)\left[\frac{\pi(1+\alpha-e^t)^2+2}{(1+\alpha-e^t)^3}\right] \qquad (10)$$

The expression (9) and (10) are the pgf and mgf of PQNED (8) respectively.

- *Statistical Moments and Essential Descriptive Measures of Statistics required for PNQED:*

Statistical moments play an important role to study about shape, size and variability of any statistical distribution. That is why, it is necessary to obtain the first four moments about origin as well as about the mean of this distribution. The r[th] moment about origin of PNQED can be obtained as

$$\mu_r' = E\left[E(Y^r/\lambda\right] = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^r e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda \qquad (11)$$

Substituting $r = 1,2,3,4$ in the equation (11), the first four moments about origin of PNQED can be obtained as follows. The mean of PNQED has been obtained as

$$\mu_1' = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^1 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty(\lambda)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\int_0^\infty \pi\lambda e^{-\alpha\lambda}d\lambda + \int_0^\infty \lambda^3 e^{-\alpha\lambda}d\lambda\right] = \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\frac{\pi}{\alpha^2}+\frac{6}{\alpha^4}\right] = \frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)} \qquad (12)$$

Graphical presentation of the mean for different values of alpha is given below.

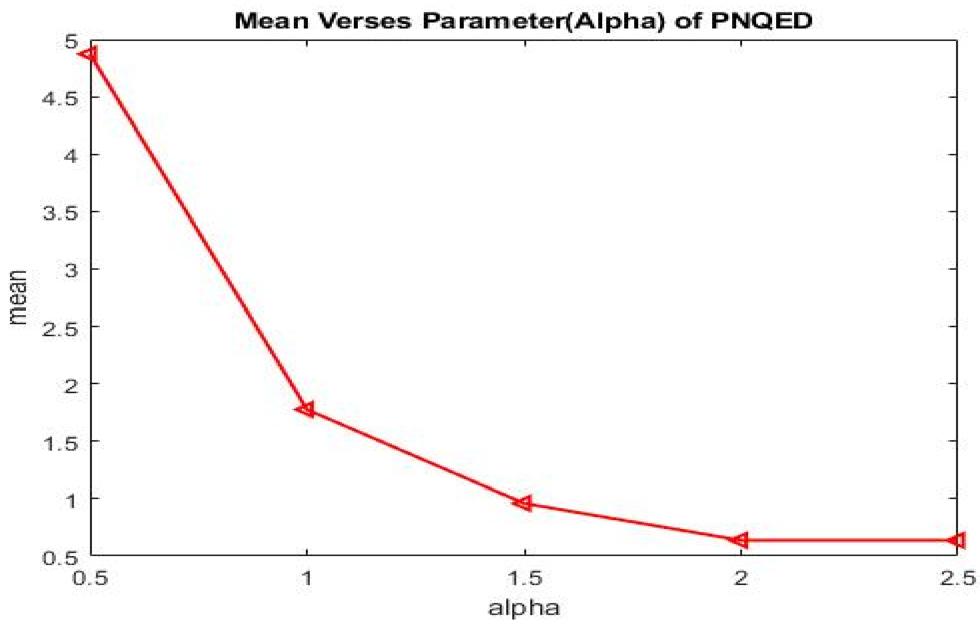

Fig.4 The mean of PNQED for $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$

It can be observed, from the figure (4), that the mean of PNQED decreases as the value of the parameter increases. Substituting $r=2$ in the equation (11), μ_2' of PNQED has been obtained as follows

$$\mu_2' = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^2 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\lambda+\lambda^2\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}++\frac{6}{\alpha^4}\right)+\left(\frac{2\pi}{\alpha^3}++\frac{24}{\alpha^5}\right)\right] = \frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)}+\frac{(2\pi\alpha^2+24)}{\alpha^2(2+\pi\alpha^2)} \tag{13}$$

Putting $r=3$ in the equation (11), μ_3' can be obtained as

$$\mu_3' = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^3 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\lambda^3+3\lambda^2+\lambda\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\pi\lambda+3\pi\lambda^2+\pi\lambda^3+\lambda^3+3\lambda^4+\lambda^5\right)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}+\frac{6}{\alpha^4}\right)+\left(\frac{6\pi}{\alpha^3}+\frac{72}{\alpha^5}\right)+\left(\frac{6\pi}{\alpha^4}+\frac{120}{\alpha^6}\right)\right]$$

$$= \frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)}+\frac{6(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)}+\frac{6(\pi\alpha^2+20)}{\alpha^3(2+\pi\alpha^2)} \tag{14}$$

Putting $r=4$ in the equation (11), the fourth moment about origin has been obtained as

$$\mu_4' = \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^4 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\lambda^4+6\lambda^3+7\lambda^2+\lambda\right)(\pi+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\int_0^\infty\left(\pi\lambda+7\pi\lambda^2+6\pi\lambda^3+\pi\lambda^4+\lambda^3+7\lambda^4+6\lambda^5+\lambda^6\right)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}+\frac{6}{\alpha^4}\right)+\left(\frac{14\pi}{\alpha^3}+\frac{168}{\alpha^5}\right)+\left(\frac{36\pi}{\alpha^4}+\frac{720}{\alpha^6}\right)+\left(\frac{24\pi}{\alpha^5}+\frac{720}{\alpha^7}\right)\right]$$

$$= \frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)}+\frac{14(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)}+\frac{36(\pi\alpha^2+20)}{\alpha^3(2+\pi\alpha^2)}+\frac{24(\pi\alpha^2+30)}{\alpha^4(2+\pi\alpha^2)} \tag{15}$$

Central Moments of PNQED:

It is essential to study about nature of variability, shape and size of PNQED. So, the first four central moments of PNQED have been be obtained as

$\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{2(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)} - \left(\frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}\right)^2$$

$$= \frac{[\pi^2\alpha^5+\pi^2\alpha^4+8\pi\alpha^3+16\pi\alpha^2+12\alpha+12]}{[\alpha(\pi\alpha^2+2)]^2} \quad (16)$$

Theorem (1): *PNQED is an over-dispersed distribution.*

Proof: *PNQED* said to be over-dispersed if $\mu_2 > \mu_1'$

Or, $\frac{[\pi^2\alpha^5+\pi^2\alpha^4+8\pi\alpha^3+16\pi\alpha^2+12\alpha+12]}{[\alpha(\pi\alpha^2+2)]^2} > \frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}$

Or, $[\pi^2\alpha^5+\pi^2\alpha^4+8\pi\alpha^3+16\pi\alpha^2+12\alpha+12] - \alpha(6+\pi\alpha^2)(2+\pi\alpha^2) > 0$

Or, $[\pi^2\alpha^4+16\pi\alpha^2+12] > 0$ (17)

The value of expression (17) is greater than zero. Hence, PNQED is an over-dispersed. Graphical presentation of μ_2 of PNQED with varying values of α is given below.

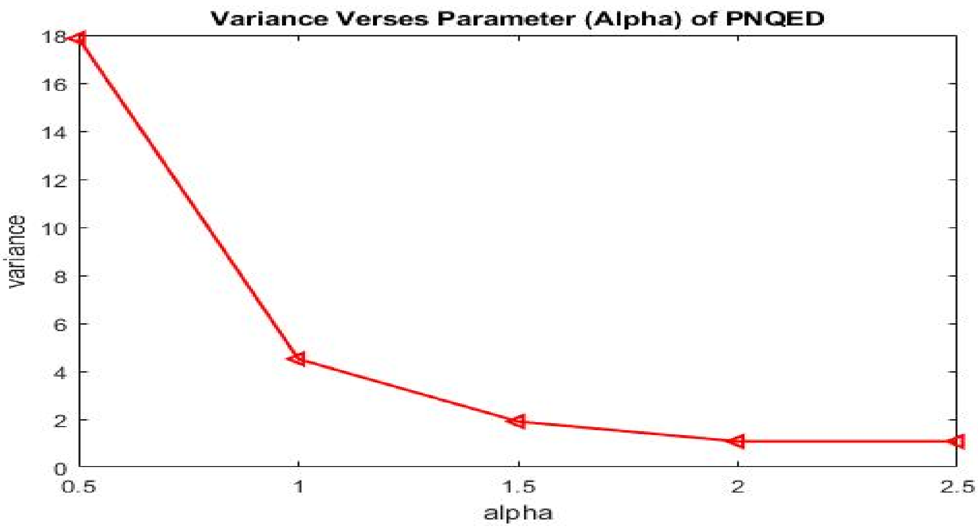

Fig.5 The Variance of PNQED for $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$

From the figure (5), it is observed that the variance of PNQED decreases as the value of α increases.

The μ_3 of PNQED can be obtained as follows

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{6(\pi\alpha^2+72)]}{\alpha^2(2+\pi\alpha^2)} + \frac{6(\pi\alpha^2+20)}{\alpha^3(2+\pi\alpha^2)}\right] - 3\left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{2(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)}\right]\left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)}\right]$$
$$+2\left[\frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}\right]^3$$

$$= \frac{\begin{array}{l}[(\pi^3\alpha^8 + 3\pi^3\alpha^7 + 2\pi^3\alpha^6 + 10\pi^2\alpha^6 + 54\pi^2\alpha^5 + 60\pi^2\alpha^4 + 28\pi\alpha^4 \\ 132\pi\alpha^3 + 72\pi\alpha^2 + 24\alpha^2 + 72\alpha + 48)]\end{array}}{[\alpha(2+\pi\alpha^2)]^3} \qquad (18)$$

The fourth central moment (μ_4) of PNQED has been obtained as

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 + 3(\mu_1')^4$$

$$= \left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{14(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)} + \frac{36(\pi\alpha^2+20)}{\alpha^3(2+\pi\alpha^2)} + \frac{24(\pi\alpha^2+30)}{\alpha^4(2+\pi\alpha^2)}\right]$$
$$-4\left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{6(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)} + \frac{6(\pi\alpha^2+20)}{\alpha^3(2+\pi\alpha^2)}\right]\left[\frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}\right]$$
$$+6\left[\frac{(\pi\alpha^2+6)}{\alpha(2+\pi\alpha^2)} + \frac{2(\pi\alpha^2+12)}{\alpha^2(2+\pi\alpha^2)}\right]\left[\frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}\right]^2 - 3\left[\frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}\right]^4$$

$$= \frac{\begin{array}{l}[\pi^4\alpha^8(\alpha^3+10\alpha^2+18\alpha+9) + \pi^3\alpha^6(12\alpha^3+188\alpha^2+5288\alpha+354) \\ +\pi^2\alpha^4(48\alpha^3+824\alpha^2+2064\alpha+1224) + \pi\alpha^2(80\alpha^3+1360\alpha^2+2880\alpha+1728) \\ +(48\alpha^3+768\alpha^2+1444\alpha+720)\end{array}}{[\alpha(2+\pi\alpha^2)]^4} \qquad (19)$$

To study about shape and size of this distribution, the co-efficient of skewness and kurtosis based on moments have been obtained as follows.

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\begin{array}{l}[(\pi^3\alpha^8 + 3\pi^3\alpha^7 + 2\pi^3\alpha^6 + 10\pi^2\alpha^6 + 54\pi^2\alpha^5 + 60\pi^2\alpha^4 + 28\pi\alpha^4 \\ 132\pi\alpha^3 + 72\pi\alpha^2 + 24\alpha^2 + 72\alpha + 48)]\end{array}}{[\pi^2\alpha^5 + \pi^2\alpha^4 + 8\pi\alpha^3 + 16\pi\alpha^2 + 12\alpha + 12]^{3/2}} \qquad (20)$$

The expression (20) represents co-efficient of skewness of PNQED based on moments. From this expression, it can be obtained that $(2/\sqrt{3}) < \gamma_1 < \infty$.

The Graphical presentation of γ_1 with different values of the parameter α is given below.

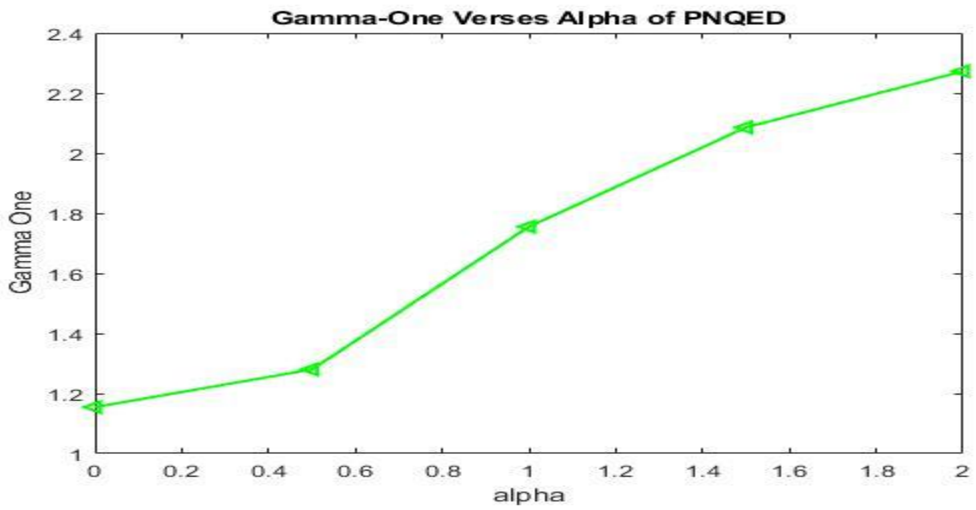

Fig.6: Skewness of PNQED for $\alpha = 0.0, 0.5, 1.0, 1.5, 2.0$

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

$$= \frac{\begin{array}{l}[\pi^4\alpha^8(\alpha^3+10\alpha^2+18\alpha+9)+\pi^3\alpha^6(12\alpha^3+188\alpha^2+5288\alpha+354) \\ +\pi^2\alpha^4(48\alpha^3+824\alpha^2+2064\alpha+1224)+\pi\alpha^2(80\alpha^3+1360\alpha^2+2880\alpha+1728) \\ +(48\alpha^3+768\alpha^2+1444\alpha+720]\end{array}}{[\pi^2\alpha^5+\pi^2\alpha^4+8\pi\alpha^3+16\pi\alpha^2+12\alpha+12]^2} \tag{21}$$

The expression (21) represents the co-efficient of kurtosis based on moments and it can be noted that $5 < \beta_2 < \infty$.Hence, it is leptokurtic by size. The graphical presentation of β_2 for different values of α is given below

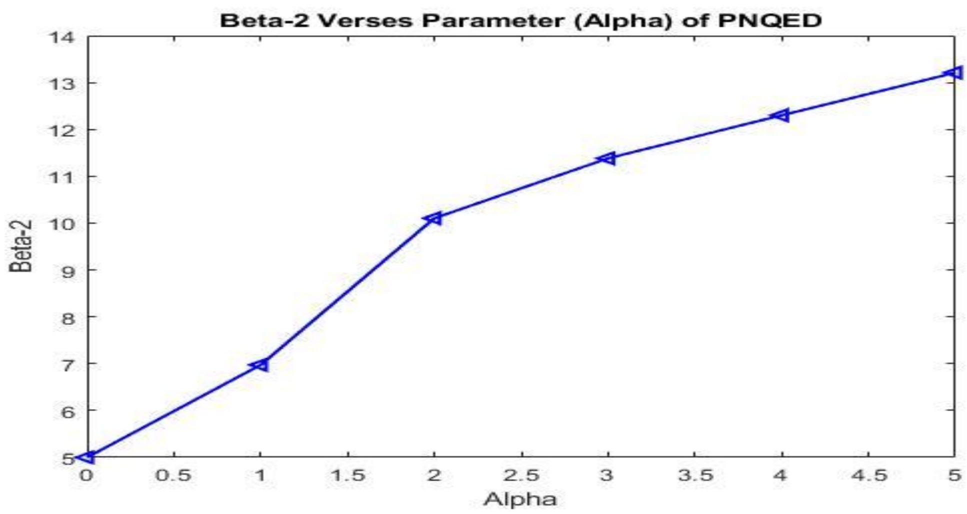

Fig.7: Kurtosis of PNQED for $\alpha = 0.0, 1.0, 1.5, 2.0, 2.5$

From figure (6), we can conclude that the values of γ_1 are directly proportional to the values of α such that $(2/\sqrt{3})<\gamma_1<\infty$. From figure (7), we can observe that the values of β_2 are directly proportional to the values of α, while the values of mean and variance are inversely proportional to the value of α.

- *Remarks:*
  - PNQE is always over-dispersed.
  - It is always positively skewed by shape, and
  - It is always leptokurtic by size.
- *Methods of Estimation of the Parameter of PNQED:*

These two methods of estimation, namely (a) Method of moments and (b) Maximum likelihood method, have been used to derive the point estimator of the parameter of this distribution.

(a) *Method of moments*:

Since this distribution depends on only one parameter, we can use the expression (12) of the mean of this distribution to get a sufficient estimator of the parameter of this distribution as follows

$$\mu_1' = \frac{(6+\pi\alpha^2)}{\alpha(2+\pi\alpha^2)}$$

Or, $\mu_1'(2\alpha+\pi\alpha^3)-(6+\pi\alpha^2)=0$ (22)

To get an estimator of the parameter of this distribution, population mean is replaced by the sample mean and solve the expression (22), the polynomial equation of α in third degree, by Regula- Falsi or Newton-Rapson method which gives a point estimator of α.

(b) *Method of maximum likelihood:*

A sample of size n is taken from the PNQED population to get the estimator of the parameter (α) from this method as follows

$$\begin{array}{ll} y_i & : y_1 \; y_2 \; y_3 \cdots y_k \\ f_i & : f_1 \; f_2 \; f_3 \cdots f_k \end{array}$$

The maximum likelihood equation has been obtained as

$$L = \left(\frac{\alpha^3}{2+\pi\alpha^2} \right)^n (1+\alpha)^{-\sum_{i=1}^{k}(y_i+3) f_i} \prod_{i=1}^{k} [\pi(1+\alpha)^2 + (1+y_i)(2+y_i)]^{f_i} \tag{23}$$

Or, $log(L) = 3n \log a - n \log(2+\pi\alpha^2) - \left(\sum_{i=1}^{k} (y_i+3) f_i \right) (\log(1+\alpha)) + \sum_{i=1}^{k} f_i \log\left(\pi(1+\alpha)^2 + (1+y_i)(2+y_i) \right)$

Or, $\frac{\partial(log(L))}{\partial\alpha} = \frac{3n}{\alpha} - \frac{2\pi n\alpha}{(2+\pi\alpha^2)} - \frac{(\bar{y}n+3n)}{(1+\alpha)} + \sum_{i=1}^{k} \frac{f_i[2\pi(1+\alpha)]}{[\pi(1+\alpha)^2 + (1+y_i)(2+y_i)]} = 0$

(24)

An estimate of α can also be obtained by solving the expression (24).

- *Goodness of Fit and Applications of PNQED:*

This distribution can be applied to the over-dispersed count data related to the fields like, Biology, ecology, accident proneness, number of errors theory and many more, for statistical modeling. In order to test the statistical validity of the theoretical work of this distribution, goodness of fit has been applied to some over-dispersed count secondary data mentioned below.

Example (1): Distribution of mistakes in copying groups of random digits, Kemp and Kemp (1965), [see.5].

Number of errors per group	0	1	2	3	4^+
Observed Frequency	35	11	8	4	2

Example (2): Distribution of Pyrausta nablilalis in 1937, Beall (1940), [see, 1].

Number of insects per leaf	0	1	2	3	4	5
Observed Frequency	33	12	6	3	1	1

Example (3): Distribution of mammalian cytogenic dosimetry lesions in rabbit lymphoblast included by Streptonigrin [NSC-45383], Exposure-70($\mu g / kg$), [see,2].

Class / Exposure ($\mu g / kg$)	0	1	2	3	4	5	6
Observed Frequency	200	57	30	7	4	0	2

Example (4): Distribution of number of red mites on apple leaves, reported by Garman (1923) [see,3].

Number of red mites per leaf	0	1	2	3	4	5	6	7
Observed Frequency	38	17	10	9	3	2	1	0

Table 1: Observed Verses Expected Frequency of Example (1)

Number of errors per group	Observed frequency	Expected frequency			
		PLD	PMD	PMMD	PNQED
0	35	33.0	32.9	33.6	34.0
1	11	15.3	15.3	14.7	14.4
2	8	6.8	6.8	6.5	6.4
3	4	2.9	3.6	2.9	2.9
4	2	2.0	1.4	2.3	2.3
Total	60	60.0	60.0	60.0	60.0
μ'_1	0.78333333				
μ'_2	1.8500				
$\hat{\alpha}$		1.7434	2.1654758	1.804566256	1.726097365
d.f.		2	2	2	2
χ^2		1.78	1.72	1.44	1.328
P-value		0.61	0.625	0.677	0.698

Table 2: Observed Verses Expected Frequency of Example (2)

Number of insects per leaf	Observed frequency	Expected frequency			
		PLD	PMD	PMMD	PNQED
0	33	31.5	31.4	32.0	32.3
1	12	14.2	14.2	13.7	13.4
2	6	6.1	6.2	5.9	5.8
3	3	2.5	2.6	2.6	2.5
4	1	1.0	1.0	1.1	1.1
5	1	0.7	0.6	0.7	0.9
Total	56	56.0	56.0	56.0	56.0
μ'_1	0.75				
μ'_2	1.8571				
$\hat{\alpha}$		1.8081	2.234	1.862795711	1.779643506
d.f.		2	2	2	2
χ^2		0.53	0.47	0.29	0.21
P-value		0.83	0.85	0.89	0.90

Table 3. Observed Verses Expected Frequency of Example (3)

Class per Exposure ($\mu g / kg$)	Observed frequency	Expected frequency			
		PLD	PMD	PMMD	PNQED
0	200	191.8	191.6	193.4	194.4
1	57	70.3	70.2	68.4	67.4
2	30	24.9	25.1	24.5	24.2
3	7	8.6	8.7	8.8	8.9
4	4	2.9	2.9	3.2	3.3
5	0	1.0	1.0	1.1	1.2
6	2	0.5	0.5	0.6	0.5
Total	300	300.0	300.0	300.0	300.0

μ_1'	0.553333333				
μ_2'	1.253333333				
$\hat{\alpha}$		2.353339	2.784976722	2.339317198	2.220534523
d.f.		3	3	3	3
χ^2		3.91	3.81	3.40	3.227
P-value		0.43	0.45	0.50	0.52

Table 4. Observed Verses Expected Frequency of Example (4)

Number of red mites per leaf	Observed frequency	Expected frequency		
		PLD	PMMD	PNQED
0	38	35.8	36.8	37.5
1	17	20.7	20.0	19.5
2	10	11.4	10.9	10.5
3	9	6.0	5.9	5.8
4	3	3.1	3.1	3.1
5	2	1.6	1.6	1.7
6	1	0.8	0.8	0.9
7^+	0	0.6	0.9	1.0
Total	80	80.0	80.0	
μ_1'	1.15			
μ_2'	3.4			
$\hat{\alpha}$		1.255891	1.3724416	1.329883264
d.f.		3	3	3
χ^2		2.47	1.15	0.85
P-value		0.61	0.82	0.87

In order to test the goodness of fit of this distribution, we have used the above mentioned four secondary data. The first example was given by Kemp and Kemp related to the number of errors per page [see, 5]. The second example was given by Beall which is based on the number of insects per leaf [see, 1]. The third example is due to Catcheside at al which is about class per exposer [See, 2] and the example (4), is related to number of red mites per leaf, was given by Garman [See, 3]. In the above-mentioned tables, the expected frequency due to PLD, PMD, PMMD and PNQED has been calculated which makes it easy for us to compare these distributions. The first three examples have already been mentioned in the doctoral thesis [see, 8].

CONCLUSION

In table-V, $d.f.$ $\chi^2_{d.f.}$ and $P-Value$ of PLD, PMD, PMMD and PNQED have been included to make comparison easy and simple.

Table 5. PLD. PMD and PMMD Verses PNQED

Table	PLD			PMD		PMMD		PNQED	
	$d.f.$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$	$\chi^2_{d.f.}$	$P-Value$
I	2	1.78	0.61	1.72	0.625	1.44	0.677	1.328	0.698
II	2	0.53	0.83	0.47	0.85	0.29	0.89	0.21	0.90
III	3	3.91	0.43	3.81	0.45	3.40	0.50	3.227	0.52
IV	3	2.47	0.61	-	-	1.15	0.82	0.85	0.87

We have drawn these various conclusions about PNQED using table- V and descriptive measures of PNQED which have been derived earlier.

- PNQED is found to be a better alternative of PLD [see, 7] and PMD [see, 10] PMMD [see, 13] for statistical modeling because P-value obtained by using PNQED is greater than the P-value obtained by using PLD, PMD and PMMD.
- Since $\mu_2 > 0$, it is always over-dispersed.
- Since $(2/\sqrt{3}) < \gamma_1 < \infty$,it is always positively skewed by shape and
- Since $5 < \beta_2 < \infty$, it is Leptokurtic by size.

Conflict of Interest

The authors of this paper have written this paper selflessly with the aim of contributing only to continuous mixtures of Poisson distribution. The authors do not intend to prejudice or offend anyone while writing this paper.

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REFERENCES

[1] Beall, G. (1940). The fit and significance of contagious distribution when applied to observations on larval insects. *Ecology*, 21, 460-474.

[2] Catcheside DG, Lea DE, Thoday JM (1946 b). The Production of chromosome structural changes in Tradescantia microspores in relation to dosage, intensity and temperature. *J Genet,* 47: 137-149.

[3] Garman, P (1923). The European red mites in Connecticut apple orchards. *Connecticut Agri. Exper. Station Bull,* 252, 103-125.
https://doi.org/10.5962/bhl.title.51017

[4] Ghitany, M.E. and Al-Mutairi, D.K. (2009): Estimation Methods for Discrete Poisson-Lindley distribution. J. Stat.Compt. Simul, 79 (1), 1-9

[5] Kemp, C.D. and Kemp, A.W. (1965). Some properties of the Hermite distribution. *Biometrika*, 52, 381-394.

[6] Lindley, D.V. (1958). Fiducial distributions and Bayes theorem, *Journal of Royal Statistical Society*. B, 20 ,102-107.

[7] Sankaran, M. (1970). The discrete Poisson-Lindley distribution. *Biometrics*, 26,145-149.

[8] Sah, B.K. (2013). Generalisations of Some Countable and Continuous Mixtures of Poisson Distribution and Their Applications. *Doctoral thesis, Patna University, Patna, India.*

[9] Sah, B.K. (2015). Mishra Distribution. *International Journal of Mathematics and Statistics Invention (IJMSI),* Volume 3(8), PP- 14-17.

[10] Sah, B.K. (2017). Poisson-Mishra Distribution. *International Journal of Mathematics and Statistics Invention (IJMSI)*, Volume 5(3), PP- 25-30.

[11] Sah, B.K. (2018). A Generalised Poisson-Mishra Distribution. *Nepalese Journal of Statistics*, Volume 2, PP- 27-36. DOI: https://doi.org/10.3126/njs.v2i0.21153

[12] Sah, B.K. (2022). Modified Mishra Distribution. *The mathematics Education*, Volume-56(2), PP- 1-17. DOI: https://doi.org/10.5281/zenodo.7024719

[13] Sah, B.K. and Sahani, S. K. (2022). Poisson-Modified Mishra Distribution. *Jilin Daxue Xuebao (Gongxueban) Journal of Jilin University (Engineering and Technology Edition),* Volume-42(1), DOI: 10.17605/OSF.IO/K5A7M

[14] Sah, B.K. (2022). Quadratic-Exponential Distribution. *The mathematics Education,* Volume-56(1), PP-1-17. DOI: https://doi.org/10.5281/zenodo.6381529

[15] Sah, B.K. and Sahani, S. K. (2022). New Quadratic-Exponential Distribution. *Journal of Pharmaceutical Negative Results,* Volume-13(4), PP-2338-2351. DOI: https://doi.org/10.47750/pnr.2022.13.S04.289