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POINCARE'S THEOREM OF ASYMPTOTIC SERIES AND ITS APPLICATION

Nand kishor Kumar[1], Dipendsra Prasad Yadav[2], Suresh Kumar Sahani[3]
[1]Trichandra Campus, Tribhuvan University, Nepal; [2]Thakur Ram Campus, Birgunj, Nepal;
[3]MIT Campus, T.U. Janakpurdham, Nepal
nandkishorkumar2025@gmail.com; dipendra.yadav2032@gmail.com

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Abstract

This article explains an important asymptotic series theorem. Poincare also demonstrates how to solve linear differentials with polynomial coefficients using asymptotic series. The significance of asymptotic series has also been discussed

Keywords: Poincare theorem, Asymptotic Series

Introduction and Historical Remarks

Poincare explored and defined the equation

$$f(\lambda z) = R(f(z)),\ z \in \mathbb{C} \qquad \text{(i)}$$

in his major articles published in 1890[1], where R(z) is a rational function and C is a constant. He demonstrated that if R(0) = 0, R' (0) = λ and $|\lambda| > 1$, a meromorphic or full solution of equation (i) occurs. After Poincare's, (i) is known as the Poincare's equation, and its solutions are known as the Poincare's functions. Later, G. Valiron [2], [3] took a significant step by discussing the equation (i) as R(z) = p(z) is a polynomial.

$$f(\lambda z) = p(f(z)),\ z \in C, \quad \text{(ii)}$$

and discovered the existence of a full solution f(z). He arrived at the asymptotic formula for

$$M(r) = \max_{|z| \le r} |f(z)|$$

$$\log M(r) \sim r\, \varrho F \ \log r \log |\lambda| \ ,\ r \to \infty \quad \text{(iii)}$$

F(z) is a one-periodic function that is restricted by two positive constants, = log d log | | and d = deg p(z). Borel developed his well-known sum ability theory of oscillating series around 1896. Borel's theory is more precise and accurate than Poincare's. Mittag-Leffler constructed and defined asymptotic series with greater precision. The studies [4], [5], [6], [7] investigated various elements of Poincare's functions.

In 1912, Poincare published a paper titled "Sur un Theoreme de Geometrie" [8], in which he presented a large amount of geometric proof in numerous special circumstances. However, he was unable to prove the theorem. Following Poincare's death in 1913, another mathematician, George David Birkhoff, published the first complete proof [9]. Unfortunately, Birkhoff's argument for the existence of the second fixed point is based on a faulty application of Poincare's theorem [10], also known as the Poincare's-Hoff index theorem in its more comprehensive form.

The indices of the fixed points of f must add to zero, according to Poincare's theorem. As a result, if f has at least one fixed point with a non-zero index, there must be at least one more fixed point. This, however, ignores the possibility that the fixed point has index zero. Birkhoff ultimately proved the general case of the theorem in 1926 in a paper titled "An extension of Poincare's last geometric theorem" using an analytic approach different from Poincare's [11].

Preliminaries

Let a function J(x) defined in inverse powers of x

$$J(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots\cdots.$$

Generally, the partial sums do not necessarily have to converse; but we consider that partial sum yields "asymptotic" formula for f.

$$\lim_{x\to\infty} x^n(J - S_n) = 0$$

(iv)

Where J(x) > $\frac{K_n}{x^{n+1}}$ and K_n depends on n, not on x. S_n is partial sum up to (n+1) terms.

Definition

Poincare Statement

If for all n, the series is asymptotic to the function. This relation (iv) can be represent as

$$J(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots\cdots$$

(v)

This series is ordered as $\frac{1}{x^{n+1}}$. For a given value of n, the first (n+1) terms of the series can be chosen to be as close to the function J(x) possible by extending x. There is an inaccuracy of order $\frac{1}{x^{n+1}}$ for each value of x and n. Because the series is now divergent, there is an optimal number of terms in the series to describe J(x) for a given value of x. This is an unavoidable mistake. As x increases, so does the optimal number of terms increase and the inaccuracy. It is noteworthy that the asymptotic series differs significantly from the traditional power law, as

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots$. For all finite values of x, sinx converges perfectively.

Some important theorem [9]

Theorem 1: Convergent asymptotic series can be added and subtracted.

Proof: From the definition of asymptotic expansion, it is obvious.

Theorem 2: Convergent asymptotic series can be multiplied.

Proof: consider two asymptotic series:

$$J(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ \ldots, \quad K(x) \sim b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ \ldots,$$

Then the formal product is

$$\Pi\,(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_2}{x^3} + \ \dots\ . \text{ Where } c_n = a_0 b_n + a_1 b_{n-1} + ..a_0 b_0.$$

We will show that the product J(x)· k(x) id represented asymptotically by Π(x).

$s_n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} \dots + \frac{a_n}{x^n}$

$T_n = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} \dots + \frac{a_n}{x^n}$

$\sum_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} \dots + \frac{c_n}{x^n}$

Denote the sums of the first (n +1) terms in these three series. Then we have:

J(x) = $s_n + \frac{P}{x^n}$, $\quad K(x) = T_n + \frac{\sigma}{x^n}$

Where ρ, σ are function of x which tend to zero as $x = \infty$. Now, by definition $\sum_n$ coincides with the product $S_n T_n$ including the terms $\frac{1}{x^n}$.

$S_n T_n = \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} \dots + \frac{a_n}{x^n}\right) \left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} \dots + \frac{a_n}{x^n}\right)$, on simplification and got the form $S_n T_n$ - $\sum_n$ consists the terms from $\frac{1}{x^{n+1}}$ to $\frac{1}{x^{2n}}$.

As x→ ∞ , J(x)→ a_o , K(x)→ b_0 and $\rho \to 0$, $\sigma \to 0$

$\lim_{x\to\infty} (x^n J(x) K(x) - \sum_n\ ) = \lim_{x\to\infty} \frac{p^n}{x^n} = 0$. So, the product $J(x)K(x)$ is represented by asymptotically by $\prod(x)$.

Theorem 3: An asymptotic series is integrated term by term to get another asymptotic series of the integral of original function.

Proof: Suppose the integration of an asymptotic series in which $a_0 = 0$, $a_1 = 0$.

If J (x) $\sim \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \cdots$

$\lim_{x\to\infty} x^n$ (J - $S_n) = 0$ so, for any $\varepsilon > 0$ there is n_0 , such that

$|x^n (J - S_n)| < \varepsilon \ or |J - S_n| < \frac{\varepsilon}{x^n}$ for any n≥ n_0.

$|J - S_n| < \frac{\varepsilon}{x^n}$ if x ≥ x_0

$\left|\int_x^\infty (J - S_n)\right| \le \int_x^\infty |J - S_n| < \int_x^\infty \frac{\varepsilon}{x^n}$

$$\left|\int_x^\infty Jdx - \int_x^\infty S_n dx\right| < \frac{\varepsilon}{(n-1)x^{n-1}}, if\ x > x_0$$

So that $\int_x^\infty Jdx$ is represented asymptotically by

$\int_x^\infty \left(\frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \cdots\right)$ dx = - $a_2 x^{-1} \int_x^\infty - \frac{a_3}{2} x^{-2} \int_x^\infty - \ \ldots . = \frac{a_2}{x} + \frac{a_3}{2x^2} + \frac{a_4}{3x^3} + \cdots$

Theorem 4: Let the derivative of an asymptotic expansion exists, then the expansion of J'(x) is the term –by term differentiation of the J(x).

Proof:

If $\phi(x)$ has a definite finite limit as x tends to ∞, then $\phi'(x)$ either oscillates or tends to zero as a limit.

If ϕ (x) tends to a definite 1imit, we can find x_0 So that $|\phi(x) - \phi(x_0)| < \varepsilon$ if $x > x_0$.

Thus, since $\phi'(\xi) = \frac{\phi(x) - \phi(X_0)}{X - X_0}$, where x $> \xi > x_0$, we find

$|\phi'(\xi)| < \frac{\varepsilon}{x - x_0}$. So, $\phi'(x)$ cannot approach any definite 1imit other than zero; but the last inequality does not exclude oscillation, since ξ may not take all values greater than x_0 as tends to ∞. $\phi'(x)$, if it has a definite limit, it must be zero.

$$\mathrm{J}\ (\mathrm{x}) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots$$

Then we have

$$\lim_{x\to\infty} x^{n+1} \{J(x) - S_{n+1}(x)\}$$

and

$$\lim_{x\to\infty} x^{n+1} \left\{J(x) - S_n(x) - \frac{a_{n+1}}{x^{n+1}}\right\} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{n+1} \{J(x) - S_n(x)\} = a_{n+1}.$$

Thus, the differential coefficient

$$x^{n+1}\{J'(x) - S'_n(x)\} + (n+1)x^n\{J(x) - S_n(x)\}$$

If it has a definite limit, must tend to 0.

But $x^n\{J(x) - S_n(x)\} \to 0$ so that, $\lim x^{n+1}\{J'(x) - S'_n(x)\} = 0.$ That is, if J' (x) has an asymptotic series, it is

$$-\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \frac{2a_3}{x^4} - \cdots .$$

Applications

The theory of Poincare is extensively used in the solution of differential equations. As the independent variable goes to infinity along a fixed path, asymptotic series can be used to solve any linear differential equation with polynomial coefficients. Poincare did not specify the areas of validity. Horn later filled the holes in a number of particular circumstances.

Barnes and Hardy used the theory of contour integration to apply Poincare's theory to the asymptotic representation of functions generated by power -series.

Conclusion

This article goes through the Poincare theorem in depth. Poincare failed to establish his geometric theorem in 1992. The geometric proof is totally supported by the Poincare-Hoff index theorem. This article also shows one of its applications.

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