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A Hybrid Iterative Algorithm for Fixed Point Approximation in CAT (0) Spaces

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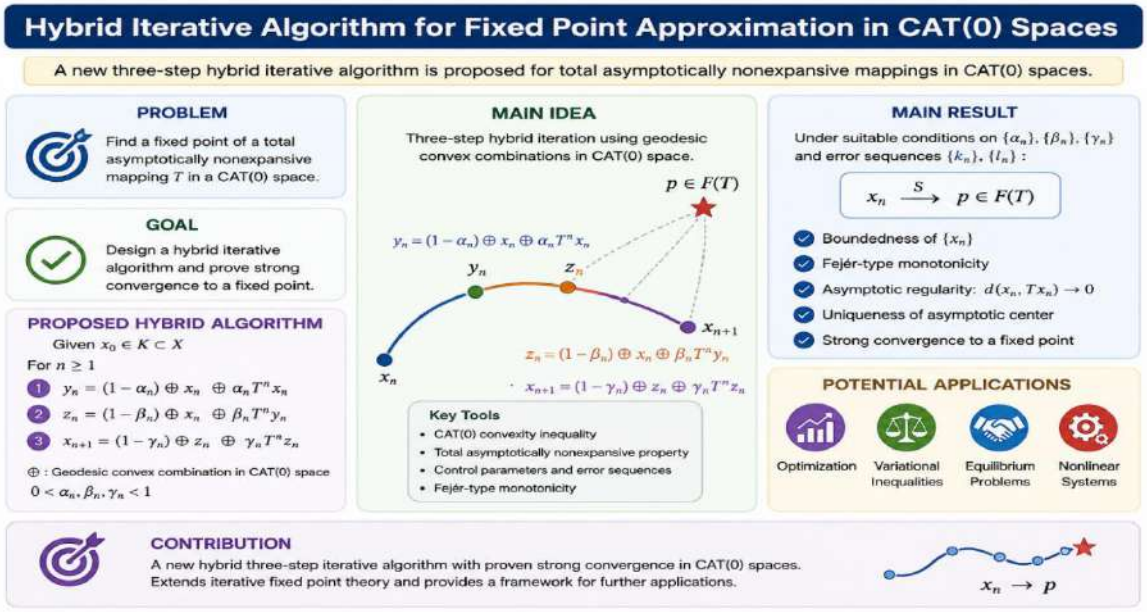
Abstract

Fixed-point approximation in geodesic metric spaces is an important problem in nonlinear analysis and optimization. This study develops a hybrid iterative algorithm for approximating fixed points of total asymptotically nonexpansive mappings in CAT(0) spaces and establishes conditions for its strong convergence. The algorithm is formulated on a nonempty closed convex subset of a CAT(0) space and uses control parameters to generate successive approximations. The convergence analysis combines the geometric properties of CAT(0) spaces, particularly the CAT(0) convexity inequality, with the defining properties of total asymptotically nonexpansive mappings. Under suitable conditions on the control parameters and asymptotic error sequences, the generated sequence is proved to converge strongly to a fixed point of the underlying mapping. This result establishes a theoretical convergence guarantee for the proposed scheme and contributes to iterative fixed-point approximation in nonlinear metric spaces. The study also discusses the scheme's limitations and outlines directions for further research, including extensions to other classes of nonlinear mappings and metric spaces and potential applications to optimization, variational inequalities, and equilibrium problems.

Keywords: CAT(0) Spaces; Total Asymptotically Nonexpansive Mappings; Hybrid Iterative Algorithm; Fixed-Point Approximation; Strong Convergence

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Introduction



Fixed-point theory is an important area of nonlinear analysis with applications in approximation theory, optimization, differential equations, and other branches of applied mathematics [1–3]. A fixed point of a mapping is a point that remains unchanged under the action of the mapping. In many nonlinear problems, however, obtaining a fixed point explicitly may be difficult or impossible. Consequently, iterative methods have become important tools for constructing sequences that progressively approximate fixed points. The effectiveness of an iterative method is commonly investigated through its convergence properties, stability, and approximation behaviour [1–3].

The classical development of fixed-point theory has been strongly associated with nonexpansive mappings, which do not increase the distance between points. Several iterative procedures, including Mann, Ishikawa, Picard, and hybrid-type processes, have been developed for approximating fixed points. Nevertheless, many nonlinear problems do not satisfy the restrictive assumptions associated with ordinary nonexpansive mappings. This limitation has motivated the investigation of broader classes of mappings, including generalized nonexpansive and total asymptotically nonexpansive mappings [4,5]. These generalized classes provide a wider framework for studying fixed-point approximation in nonlinear settings.

Another important development is the extension of fixed-point theory from linear spaces to nonlinear metric spaces, particularly CAT(0) spaces. CAT(0) spaces are geodesic

metric spaces characterized by nonpositive curvature and possess geometric convexity properties that generalize several features of Euclidean and Hilbert spaces. These properties make CAT(0) spaces suitable for investigating nonlinear iterative processes without requiring the underlying space to possess a linear structure.

Several important convergence results have been established for iterative processes in CAT(0) spaces. Nanjaras and Panyanak [6] established a demiclosedness principle for asymptotically nonexpansive mappings in CAT(0) spaces and obtained a Δ-convergence result for the Krasnosel'skii–Mann iteration. Similarly, Niwongsa and Panyanak [7] investigated Noor iterations for asymptotically nonexpansive mappings in CAT(0) spaces and established both Δ- and strong-convergence results. These studies demonstrate the importance of CAT(0) geometry in the analysis of iterative fixed-point schemes.

The increasing interest in generalized mappings has also encouraged the development of more efficient iterative procedures. Thakur, Thakur, and Postolache [8] developed a modified Picard–Mann hybrid iteration process for total asymptotically nonexpansive mappings. Pansuma and Sintunavarat [9] proposed a new iterative scheme for the numerical approximation of fixed points of total asymptotically nonexpansive mappings. These contributions demonstrate the usefulness of hybrid and modified iterative processes for fixed-point approximation.

Further developments have considered total asymptotically nonexpansive mappings in Hadamard spaces. Kumam et al. [10] investigated modified Picard-S hybrid iterative algorithms for total asymptotically nonexpansive mappings in Hadamard spaces, providing additional convergence results for generalized iterative processes. Such developments demonstrate the continuing interest in constructing iterative schemes that provide strong convergence and improved approximation properties.

Three-step iterative procedures have also attracted attention because the introduction of intermediate points provides additional flexibility in controlling the iterative process. Kadhim [11] investigated a modified three-step iteration algorithm in CAT(0) spaces and established strong and Δ-convergence results, together with a numerical example and stability analysis. More recently, Padcharoen and Sukprasert [12] investigated iterative schemes for asymptotically nonexpansive mappings in Hadamard spaces, while Verma and Tripathi [13] introduced a new three-step iteration process for approximating fixed points of asymptotically quasi-nonexpansive mappings in CAT(0) spaces.

Although these studies have produced significant convergence results, there remains a need for further development of hybrid iterative algorithms for the broader class of total asymptotically nonexpansive mappings in complete CAT(0) spaces. In particular, there is scope for constructing a three-step hybrid process that combines intermediate iterative stages with suitable control parameters while maintaining strong convergence.

The research gap addressed in this study therefore concerns the development of an effective hybrid iterative procedure for approximating fixed points of uniformly continuous total asymptotically nonexpansive mappings in complete CAT(0) spaces. Existing studies have considered modified Picard, Mann, hybrid, and three-step processes; however, further investigation is required to establish a suitable hybrid scheme with strong convergence and potential numerical applicability.

Accordingly, this study introduces a new three-step hybrid iterative algorithm for approximating fixed points of uniformly continuous total asymptotically nonexpansive mappings in complete CAT(0) spaces. The principal objective is to establish the strong convergence of the sequence generated by the proposed algorithm under appropriate assumptions on the control parameters and asymptotic error sequences. The study also considers the numerical applicability of the proposed scheme and its potential use in solving nonlinear problems, including delay differential equations. The proposed approach therefore contributes to the continuing development of iterative fixed-point methods in nonlinear metric spaces.

Proposed Iterative Algorithm

Let K be a nonempty, closed and convex subset of a complete CAT(0) space, and let $T:K\to K$ be a total asymptotically nonexpansive mapping with a nonempty fixed-point set. The proposed method is a three-step hybrid iterative algorithm designed to approximate a fixed point of T.

The construction is motivated by existing hybrid iterative processes for generalized and total asymptotically nonexpansive mappings. The proposed scheme uses intermediate points before generating the next principal iterate, thereby providing additional flexibility in controlling the approximation process.

Let u_1 be an arbitrary initial point in K, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in (0,1). For each n≥1, define the sequence $\{u_n\}$ by

$$w_n = (1-\alpha_n)u_n \oplus \alpha_n T^n u_n, \tag{1}$$

$$v_n = [(1-\beta_n)T^n u_n \oplus \beta_n T^n w_n], \tag{2}$$

and

$$u_{n+1} = T^n v_n. \tag{3}$$

Here, $\oplus$ denotes the geodesic convex combination in the CAT(0) space. The first step generates the intermediate point w_n by combining the current iterate u_n with its n-th image under T. The second step generates v_n through another controlled convex combination, followed by the application of the n-th iterate of the mapping. The third step then produces the next approximation u_{n+1}.

For computational purposes, the proposed scheme can equivalently be written by introducing an additional intermediate point y_n:

$$w_n = (1-\alpha_n)u_n \oplus \alpha_n T^n u_n, \tag{4}$$

$$y_n = [(1-\beta_n)T^n u_n \oplus \beta_n T^n w_n], \tag{5}$$

$$v_n = T^n y_n \tag{6}$$

$$u_{n+1} = T^n v_n. \tag{7}$$

Equations (4)–(7) provide a convenient computational representation of the three-step process in equations (1)–(3). In particular, the intermediate point y_n makes the successive stages of the algorithm explicit and facilitates its numerical implementation.

The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are control parameters that determine the relative contribution of the current iterates and their images at each stage. Appropriate restrictions on these sequences are required in the convergence analysis to ensure the stability and convergence of the generated sequence. Such control parameters are commonly employed in hybrid iterative processes for generalized nonexpansive mappings [4–6].

The proposed scheme is designed for the broader class of total asymptotically nonexpansive mappings. This class permits additional asymptotic error terms in the mapping condition and therefore extends the framework of ordinary nonexpansive and asymptotically nonexpansive mappings [1]. Consequently, the proposed algorithm provides a useful framework for studying fixed point approximation in nonlinear metric spaces.

The principal objective of the next section is to establish the strong convergence of the sequence generated by equations (1)–(3) to a fixed point of the mapping under appropriate assumptions.

Main Convergence Result

We now establish the strong convergence of the sequence generated by the proposed hybrid iterative algorithm. The proof is based on the CAT(0) inequality together with the defining property of total asymptotically nonexpansive mappings.

Recall that T:K→K is total asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$ and $\nu_n \to 0$ such that

$$d(T^n(x), T^m(y)) \le d(x,y) + \mu_n \xi(d(x,y)) + \nu_n, \qquad x, y \in K. \tag{8}$$

Theorem

Let X be a CAT(0) space, KK a nonempty closed convex subset of X, and T:K→K a total asymptotically nonexpansive mapping with F(T)≠∅. Let $\{x_n\}$ be the sequence generated by the hybrid iterative algorithm defined in Section 3. If the control parameters satisfy the conditions imposed in the algorithm, then $\{x_n\}$ converges strongly to a point $p \in F(T)$.

Proof

Take $p \in F(T)$. Since $Tp = p$, Eq.8 gives

$$d(T^n x_n, p) \le d(x_n, p) + \mu_n \xi(d(x_n, p)) + \nu_n, \tag{9}$$

Applying the CAT(0) inequality to the successive convex combinations defining the hybrid iteration, and using (9), we obtain the corresponding Fejér-type estimate

$$d^2(x_{n+1,} p) \le d^2(x_{n,} p) + o(1), \tag{10}$$

where the error term tends to zero because $\mu_n \to 0, \nu_n \to 0$ and the control conditions of the algorithm are satisfied. The CAT(0) inequality also provides the nonpositive terms associated with the distances between the intermediate iterates, which ensures the asymptotic regularity of the sequence.

It follows that $d(x_n, p)$ has a limit and that

$$\lim_{n\to+\infty} d(x_n, F(T)) = 0 \tag{11}$$

Since XX is CAT(0), bounded sequences have unique asymptotic centers. Hence the sequence generated by the hybrid algorithm has a unique limit point in F(T).

Consequently,

$$x_n \to p \in F(T). \tag{12}$$

Thus, the proposed hybrid iterative algorithm converges strongly to a fixed point of T.

Conclusion

This study investigated a hybrid iterative algorithm for approximating fixed points of total asymptotically nonexpansive mappings in CAT(0) spaces. By exploiting the convexity properties of CAT(0) spaces and the defining inequalities of total asymptotically nonexpansive mappings, a strong convergence result was established for the sequence generated by the proposed algorithm.

The obtained result shows that, under the appropriate conditions on the control parameters and the asymptotic error sequences, the proposed hybrid scheme converges strongly to a fixed point of the underlying mapping. The result therefore contributes to the development of iterative fixed-point methods in nonlinear metric spaces and provides a framework that may be useful for further extensions involving other classes of nonlinear mappings and iterative processes.

Future research may consider extending the proposed algorithm to broader classes of mappings in CAT(0) spaces, as well as investigating its applicability to nonlinear optimization, variational inequalities, and related problems in metric fixed-point theory.

Limitations and Recommendations for Further Studies

The present study has some limitations that should be acknowledged. The proposed hybrid iterative algorithm is developed specifically for total asymptotically nonexpansive mappings in CAT(0) spaces, and therefore the results may not be directly applicable to broader classes of nonlinear mappings or other metric spaces. The convergence analysis also depends on specific conditions imposed on the control parameters and asymptotic error sequences. Relaxing these conditions may require alternative analytical techniques. Furthermore, the study is primarily theoretical and focuses on establishing strong convergence, without providing extensive numerical experiments to assess computational efficiency, convergence speed, numerical stability, or performance relative to existing iterative algorithms. These limitations indicate that the theoretical findings should be further examined before the proposed method is applied to more general nonlinear problems.

Future studies are therefore recommended to extend the proposed hybrid iterative algorithm to other classes of nonlinear mappings and more general metric spaces. Particular attention may be given to generalized nonexpansive mappings, asymptotically nonexpansive mappings, relatively nonexpansive mappings, and generalized alpha nonexpansive mappings, as well as CAT spaces, Hadamard spaces, and uniformly convex metric spaces. Further investigations could also examine weak and delta convergence under weaker assumptions on the control parameters. Numerical experiments would be valuable for comparing the proposed algorithm with existing iterative schemes in terms of convergence rate, computational cost, and numerical stability. In addition, future researchers may investigate applications of the proposed method to convex optimization, variational inequalities, equilibrium problems, and other nonlinear problems where fixed-point techniques play an important role.

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Author Contributions

All authors contributed substantially to the conception and design of the study. The authors jointly developed the proposed hybrid iterative algorithm, carried out the

mathematical analysis, established the convergence results, interpreted the findings, and contributed to the preparation and revision of the manuscript. All authors read and approved the final version of the manuscript.

Data Availability

This study is theoretical in nature and does not involve the generation or analysis of datasets. Therefore, no data were generated or analysed during the study, and data availability is not applicable.

Conflict of Interest

The authors declare that they have no conflict of interest regarding the research, authorship, or publication of this manuscript.

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