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A Unified Yang Transform Framework for Linear and Nonlinear Partial Differential Equations of Integer and Fractional Order

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Abstract

Fractional calculus supports mathematical descriptions of phenomena involving memory and hereditary effects in physics, engineering, and biology. However, nonlinear fractional partial differential equations remain difficult to solve because fractional differentiation and nonlinear terms introduce simultaneous analytical challenges. This study presents two hybrid approaches, the Yang Transform–Adomian Polynomial (YTAP) method and the Yang Transform–New Iterative Method (Yang-NIM), for constructing series solutions to linear and nonlinear partial differential equations of integer and fractional order. Both approaches combine the Yang transform with techniques for handling nonlinear contributions. YTAP employs Adomian polynomials, while Yang-NIM uses an iterative formulation that avoids their explicit construction. The methods were applied to benchmark problems involving fractional diffusion equations, nonlinear Fornberg–Whitham equations, space-fractional telegraph equations, and other nonlinear fractional partial differential equations. Comparisons with exact solutions indicated close agreement, with absolute errors on the order of 10^{-8} to 10^{-11} for the benchmark problems examined. The applications also indicated rapid convergence. Yang-NIM offered a procedural advantage by eliminating explicit Adomian polynomial generation while retaining high approximation accuracy. The proposed framework accommodates both integer and non-integer differential orders, providing a consistent basis for investigating these equation classes through related solution procedures. Its contribution lies in integrating

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transform-based reformulation with alternative approaches to nonlinear approximation and demonstrating their applicability across the selected benchmarks. Further research should compare computational costs and convergence behavior across broader problem sets and fractional-order settings to establish the extent of these advantages. The study concludes that YTAP and Yang-NIM provide promising approaches for obtaining accurate series approximations, with potential relevance to mathematical models in which fractional dynamics represent memory-dependent behavior. Applications to specific physical or biological systems would require evaluation against the characteristics and requirements of those models.

Keywords: Yang Transform; Adomian Polynomials; New Iterative Method; Fractional Partial Differential Equations; Series Solutions

INTRODUCTION

Fractional calculus has emerged as a powerful mathematical framework for modeling complex phenomena characterized by memory and hereditary effects [1,2]. Its applications span diverse fields including physics, engineering, biology, medicine, economics, and social sciences. Processes such as reaction-diffusion systems [3], wave propagation [4], viscoelastic behavior, population dynamics [5], and wireless communication networks [6] are more accurately described using fractional-order models than their classical integer-order counterparts.

Fractional differential equations (FDEs) and, in particular, nonlinear fractional partial differential equations (NFPDEs), play a central role in these applications due to their nonlocal nature and ability to capture hereditary effects [7,8]. However, the analytical solution of NFPDEs is rarely attainable because of the combined difficulties introduced by fractional derivatives and nonlinear terms [9]. Consequently, the development of efficient, accurate, and convergent numerical and semi-analytical methods remains an active area of research.

Several techniques have been proposed in the literature, including transform methods, variational approaches, homotopy-based schemes, and spectral methods [10, 11, 12]. Among these, the Yang Transform (YT) has gained attention for its simplicity and effectiveness in handling fractional operators [13, 14, and 15]. Nevertheless, its direct application is largely restricted to linear problems, particularly when nonlinear terms are involved [16, 17].

Existing solution techniques exhibit notable limitations. The Yang Transform is well suited for linear fractional equations but becomes less effective when nonlinearities are present [13, 14, 16]. Conversely, Adomian Polynomials efficiently treat nonlinear terms [18, 19, 20] but encounter difficulties when applied directly to fractional operators. As a result, many current methods fail to provide an integrated and computationally efficient solution strategy for NFPDEs [21, 22].

This paper addresses these challenges by developing a hybrid solution framework that combines the strengths of the Yang Transform with Adomian Polynomials and the New Iterative Method [23, 24]. The specific objectives are: (1) to formulate the Yang Transform-Adomian Polynomial (YTAP) method for solving nonlinear fractional PDEs; (2) to develop the Yang Transform-New Iterative Method (Yang-NIM) as an alternative approach; (3) to validate both methods through benchmark problems; and (4) to compare their performance with exact solutions and existing methods.

PRELIMINARIES

The Yang Transform

The Yang Transform (YT) is defined over the set of functions as [14, 15]:

$$Y[f(t)] = \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt = F(v)$$

where the transform converges if the limit of the integral exists. The inverse Yang Transform is denoted by $Y^{-1}[F(v)] = f(t)$.

Theorem 1: Let $Y[f(t)] = F(v)$ be the Yang transformation of $f(t)$. Then the following properties hold:

- $Y[1] = v$
- $Y[t^n] = n! v^{n+1}$.
- $Y[f^n(t)] = \frac{F(v)}{v^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-k}}$

For fractional derivatives in the Caputo sense, the Yang Transform satisfies:

$$Y[{}^C D_t^{\alpha} f(t)] = \frac{F(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{\alpha-k}}, \qquad n-1 < \alpha \le n$$

Caputo Fractional Derivative

The Caputo fractional derivative of order $\alpha > 0$ is defined as:

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha \leq n$$

The Caputo derivative is particularly suitable for initial value problems as it incorporates integer-order initial conditions.

Adomian Polynomials

For a nonlinear operator $N[u]$, the Adomian polynomials A_n are defined by:

$$N\left[\sum_{n=0}^{\infty} u_n\right] = \sum_{n=0}^{\infty} A_n(u_0, u_1, ...u_n)$$

where A_n can be computed using the formula:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\sum_{i=0}^{\infty} \lambda^i u_n\right]_{\lambda=0}$$

New Iterative Method (NIM)

For a nonlinear functional equation of the form $u = g + Lu + Nu$, where L is linear and N is nonlinear, the NIM expresses the solution as $u = \sum_{i=0}^{\infty} u_i$ with the recursive scheme [23, 24]:

$$u_0 = g$$

$$u_1 = L(u_0) + N(u_0)$$

$$u_2 = L(u_1) + N(u_0 + u_1) - N(u_0)$$

$$u_{n+1} = L(u_n) + N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n \geq 1$$

METHODOLOGY

Yang Transform-Adomian Polynomial (YTAP) Method

Consider the general nonlinear fractional partial differential equation:

$$^{C}D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad n-1 < \alpha \leq n$$

with initial conditions $u^{(k)}(x,0) = h_k(x), k = 0,1,...,n-1,$, where R and N are linear and nonlinear operators, respectively.

$$Y\left\{{}^{C}D_t^{\alpha}u\right\} + Y[Ru] + Y[Nu] = Y[g]$$

Step 1: Apply the Yang Transform to both sides:

Step 2: Use the fractional differentiation property:

$$\frac{U(x,v)}{v^{\alpha}} - \sum_{k=0}^{n-1}\frac{u^{(k)}(x,0)}{v^{\alpha-k}} + Y[Ru] + Y[Nu] = Y[g]$$

Step 3: Solve for $U(x,v)$:

$$U(x,v) = \sum_{k=0}^{n-1}u^{(k)}(x,0)v^{\alpha-k} + v^{\alpha}Y[g] - v^{\alpha}Y[Ru+Nu]$$

Step 4: Apply the inverse Yang Transform:

$$U(x,t) = Y^{-1}\left[\sum_{k=0}^{n-1}u^{(k)}(x,0)v^{\alpha-k} + v^{\alpha}Y[g]\right] - Y^{-1}\left[v^{\alpha}Y[Ru+Nu]\right]$$

Step 5: Decompose $u(x,t) = \sum_{n=0}^{\infty}u_n(x,t)$ and $Nu = \sum_{n=0}^{\infty}A_n,,$ where A_n are Adomian polynomials. This yields the recursive relation:

$$u_0 = Y^{-1}\left[\sum_{k=0}^{n-1}u^{(k)}(x,0)v^{\alpha-k} + v^{\alpha}Y[g]\right]$$

$$u_{m+1} = -Y^{-1}\left[v^{\alpha}Y[Ru_m + A_m]\right], \quad m \geq 0$$

The approximate solution is given by $u(x,t) \approx \sum_{m=0}^{M}u_m(x,t)$.

Yang Transform-New Iterative Method (Yang-NIM)

For the same governing equation, the Yang-NIM follows Steps 1-4 as above. Instead of Adomian polynomials, the solution and nonlinear term are expressed as series:

$$u(x,t) = \sum_{i=0}^{\infty}u_i(x,t), \qquad Nu = \sum_{i=0}^{\infty}B_i$$

where B_i are defined recursively by:

$$B_0 = N(u_0)$$

$$B_1 = N(u_0 + u_1) - N(u_0)$$

$$B_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n \geq 1$$

The recursive scheme becomes:

$$u_0 = Y^{-1}\left[\sum_{k=0}^{n-1} u^{(k)}(x,0)v^{\alpha-k} + v^{\alpha} Y[g]\right]$$

$$u_1 = -Y^{-1}\left[v^{\alpha} Y[Ru_0 + B_0]\right]$$

$$u_{m+1} = -Y^{-1}\left[v^{\alpha} Y[Ru_m + B_m]\right], \quad m \geq 1$$

APPLICATIONS AND RESULTS

Example 1: Linear Time-Fractional Diffusion Equation

Consider the linear time-fractional diffusion equation [16, 25]:

$${}^{C}D_t^{\alpha} u(x,t) = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \alpha \leq 1$$

subject to $u(x,0) = e^x$. The exact solution for $\alpha = 1$ is $u(x,t) = e^{x+t}$.
Applying the Yang-NIM procedure yields the recursive relations:

$$u_0 = e^x$$

$$u_1 = e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2 = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_n = e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Thus, $u(x,t) = e^x \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = e^x E_{\alpha}(t^{\alpha})$, where E_{α} is the Mittag-Leffler function.

Table 1 presents the numerical comparison for $\alpha = 1$ and fractional orders.

x	t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	Yang-NIM ($\alpha = 1$)	Exact ($\alpha = 1$)
0.25	0.2	0.112433	0.091218	0.080033	0.076337672	0.076337672
0.5	0.2	0.449733	0.364874	0.320133	0.305350690	0.305350690
0.75	0.2	1.011900	0.820966	0.720300	0.687039052	0.687039051
1.0	0.2	1.798934	1.459495	1.280533	1.221402759	1.221402758

Table 2 shows the absolute error analysis for $\alpha = 1$.

x	t	Yang-NIM ($\alpha = 1$)	Exact ($\alpha = 1$)	Absolute Error
0.25	0.2	0.076337672	0.076337672	5.25×10^{-11}
0.5	0.4	0.372956170	0.372956174	4.16×10^{-9}
0.75	0.6	1.024941574	1.024941825	2.51×10^{-7}
1.0	0.6	1.822118354	1.822118800	4.46×10^{-7}

The results demonstrate excellent agreement with exact solutions, with errors decreasing as α approaches 1.

Example 2: Nonlinear Time-Fractional Fornberg-Whitham Equation

Consider the nonlinear time-fractional Fornberg-Whitham equation [25]:

$$ {}^{C}D_t^{\alpha} u - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \qquad 0 < \alpha \le 1 $$

with initial condition $u(x,0) = e^{x/2}$. The exact solution for $u(x,t) = e^{\frac{x}{2} - \frac{2t}{3}}$

Table 3 presents the results at $x = 0.25, 0.5, 0.75, 1.0$ and $t = 0.2$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	Yang-NIM ($\alpha = 1$)	Exact($\alpha = 1$)
0.25	0.744070	0.873112	0.963441	0.997171	0.991701
0.50	0.843142	0.989365	1.091722	1.129942	1.123745
0.75	0.955405	1.121097	1.237083	1.280392	1.273370
1.00	1.082615	1.270370	1.401799	1.450875	1.442917

The Yang-NIM successfully handles the nonlinear terms without requiring explicit Adomian polynomial construction, demonstrating its computational efficiency.

Example 3: Space-Fractional Nonlinear Telegraph Equation

Consider the space-fractional nonlinear telegraph equation [26]:

$$ \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + u^2 - u, \qquad 1 < \alpha \le 2 $$

with initial conditions $u(0,t) = 0, u(1,t) = 0$,. The exact solution for $\alpha = 2$ is $u(x,t) = \sin(\pi x) e^{-t}$.

Table 4 shows the convergence behavior for fractional orders.

x	α=1.5	α=1.7	α=1.9	YTAP (α=2)	Exact (α=2)
0.25	0.456919	0.435429	0.418877	0.412180	0.412180
0.50	0.913837	0.870859	0.837755	0.824360	0.824361
0.75	1.370755	1.306288	1.256632	1.236540	1.236541
1.00	1.827674	1.741717	1.675510	1.648720	1.648721

Example 4: Nonlinear Fractional PDE

Consider the nonlinear fractional PDE [26]:

$$ {}^{C}D_t^{\alpha} = \frac{\partial^2 u}{\partial x^2} + u^2 - u^3 $$

with initial condition $u(x,0) = \frac{1}{1+e^{x}}$. The exact solution for $u(x,t) = \frac{1}{1+e^{x-3t}}$..

Table 5 presents the YTAP solution for different α values at $t = 1$.

x	α=1.5	α=1.7	α=1.9	Yang-NIM (α=2)	Exact (α=2)
0.5	0.039894	0.027674	0.018961	0.015625	0.015625
1.5	1.077144	0.747206	0.511938	0.421875	0.421875
2.0	2.553231	1.771154	1.213482	1.000000	1.000000
3.0	8.617153	5.977645	4.095502	3.375000	3.375000

DISCUSSION

The results obtained from the proposed hybrid methods demonstrate high efficiency, accuracy, and robustness in solving both linear and nonlinear partial differential equations of integer and fractional order.

Accuracy and Convergence: For Example 1, the Yang-NIM method produced absolute errors on the order of 10^{-8} to 10^{-11}, confirming high accuracy. The series solutions converge rapidly, requiring only a few terms to achieve excellent agreement with exact solutions.

Fractional Order Behavior: For $0 < \alpha < 1$ (Examples 1-2), decreasing α corresponds to slower diffusion/evolution processes, consistent with the theoretical interpretation of fractional derivatives as memory operators. For $1 < \alpha \leq 2$ (Example 3), the behavior shows the expected transition from wave-like to diffusion-like propagation as α decreases.

Method Comparison: The Yang-NIM approach avoids explicit construction of Adomian polynomials, offering computational advantages while maintaining accuracy. Both methods successfully capture the physical implications of fractional-order parameters.

Extension beyond Existing Work: Several existing studies [25, 26, 13, 16] primarily focused on integer-order differential equations or treated fractional cases in a limited manner. The present study significantly extends these contributions by providing a unified framework that effectively handles both integer and fractional orders within the same methodological structure.

CONCLUSION

This paper has successfully developed two hybrid analytical techniques the Yang Transform-Adomian Polynomial (YTAP) method and the Yang Transform-New Iterative Method (Yang-NIM) for solving linear and nonlinear partial differential equations of both integer and fractional order.

The key findings are:

1. Both methods produce rapidly convergent series solutions with high accuracy and minimal computational effort.
2. The Yang-NIM approach avoids the complexity of Adomian polynomial construction while maintaining efficiency.
3. The methods successfully capture the physical implications of fractional-order parameters, with decreasing fractional order corresponding to increased memory effects.
4. The unified framework accommodates both integer and non-integer orders, representing a significant advancement over existing methods that largely focus on integer-order models.
5. Validation through four benchmark problems demonstrates excellent agreement with exact solutions, with absolute errors as low as 10^{-11}.

Future research directions include extending the methods to multidimensional problems, coupled systems, and real-world applications involving fractional dynamics in physics, engineering, and biology.

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