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Combined Analysis of Several Experiments of Hyper Blocks Graeco Latin Sudoku Square Designs with Same Treatments under Mixed Effect Models

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Abstract

Experimental designs based on Hyper Blocks Graeco-Latin Sudoku Squares have predominantly been analyzed within single environments using fixed-effects models, limiting their applicability to experiments conducted across multiple locations or seasons. This study introduced a combined analytical framework for several experiments conducted under different environments with identical treatments. To address the limitations of conventional approaches, environmental effects and treatment-by-environment interactions were incorporated into the Hyper Blocks Graeco-Latin Sudoku Square Design (HBGLSSD) framework using mixed-effects models. Hypothetical and simulated datasets were generated and analyzed using R software. The efficiency of the multi-environment HBGLSSD was evaluated against that of the conventional HBGLSSD based on mean square error, p-values, and the statistical power of the F test. The results indicate that incorporating environmental effects and treatment-by-environment interactions improves analytical precision and efficiency, as evidenced by lower error mean squares and greater test power. The proposed framework therefore provides a more robust basis for analyzing HBGLSSD experiments conducted under heterogeneous environmental conditions. This study extends Sudoku-based experimental design methodology by accommodating cross-environment variation and offers a practical analytical approach for multi-location or multi-season experiments involving common treatments.

Keywords: Combined Analysis; Hyper Blocks Graeco-Latin Sudoku Square Design; Mixed-Effects Models; Statistical Power; Treatment-by-Environment Interaction

Introduction

Experiment may be conducted with the aid of randomized block designs (RBD), Latin square designs (LSD) or balanced incomplete designs (BID) over different environments. The reason may be due to lack of space that would accommodate all the experimental plots or with the underlying condition that the experiments must be carried out at different locations (environment) or different seasons (Danbaba *et al.* 2018). Analysis of data from these experiments is usually carried out through joint analysis of all the experiments, instead of individual experiment (Albassan & Ali, 2014) and one of the advantages of multi-environment analysis is that it increases the accuracy of evaluation (hence the accuracy of selection).

The primary goal of researchers in the experimental sector is that they are very interested in minimizing experimental errors and in obtaining as much information as possible with the least amount of resources (Kirk, 1982). Although a good design has the least amount of experimental error, a perfect design is still unattainable because of various circumstances and environmental factors (Lakic, 2002, Sahib *et al.* 2025).

The joint analysis of several experiments conducted via orthogonal Sudoku design of odd order experiment as an environmental factor that is only successful if the variance between row and column environments is substantially greater than the variance within row and column blocks; otherwise, not suitable because the loss of degrees of freedom will raise the error mean square and render the results inconsequential. However, Hyper Block Graeco-Latin Sudoku square design might be another option, but has not given attention or considered effects of different environments, i.e., when analysis may be carried out due to lack of space that could accommodate all the experimental plots or with the underlying condition that the experiment must be carried out at different locations (environment) or different seasons, also the study does not include an interaction situation.

This study have introduced multi-environment experiments from different environments (locations), on hyper block graeco-latin sudoku square design by incorporating environmental factors and their interactions with treatments. By addressing the aforementioned gap, the study aims to develop and improved models that further reduce error variance and provide more efficient estimation under both random and mixed effect models.

Although several research works has been proposed on Sudoku-based experimental designs – including Sudoku square, Graeco Sudoku square, Hyper Graeco-Latin Sudoku square, Hyper Block Graeco-Latin Sudoku square (Subramani 2013; Danbaba & Dauran, 2016; Hussain *et al.* 2017; Danbaba *et al.* 2018; Hussain *et al.* 2019; Dauran *et al.* 2020; Sahib *et al.* 2025).

Sahib et al., (2025) study, however focused only on construction and analysis of Hyper Block Graeco-Latin Sudoku square design within a single-environment under fixed effects without environmental factor effects and their interactions on several experiments. Thus, there remains a research gap in extending Hyper Block Graeco-Latin Sudoku square designs to combined analysis of several experiments with the same treatments under mixed effect models.

Objective of the research study are:

i. To develop experimental environment factor to models on Hyper Block Graeco-Latin Sudoku square designs

ii. Derive sum of squares of the modified models in (i) using numerical examples and simulation data.

iii. Compare the efficiency of modified models with some existing ones.

Method of Combined analysis of SEHBGLSSD

Assume that we have two environments with units that are arranged in "$k = m^2$" rows, "$k = m^2$" columns, "$k = m^2$" blocks, and "$k = m^2$" squares and m (number or row-block or column-block).

These experimental units are given several treatment kinds so that each treatment must appear only once in each row, once in each column, and once in each square (Box), Row-block, column-block or block of same experiment.

We can test w sets of treatment $type(i), i = 1, 2.........w$ at the same time in a single experimentation. The factors are environments, rows, columns, square row-blocks, column-blocks, environment $\times$ treatments type (i). In this study model of HBGLSSD discussed by Sahib *et al.* (2025) were modified to obtain four new models and combined analysis from each were discussed under "g" environments.

The following example explains the method of combined analysis several experiments of hyper block GLSS design (HBGD). Let the first row of the type (1) treatments Sudoku square design as: $\{A, E, I, M, B, F, J, N, C, G, K, O, H, L, P\}$. Then the first row of its orthogonal treatments (2) type Sudoku square design is $\{\alpha, \theta, \delta, \omega, \beta, \mu, \varphi, \nu, \gamma, \pi, \varepsilon, \eta, \lambda, \phi, \tau, \upsilon\}$ and the first row of its third orthogonal treatments type (3) Sudoku square design is $\{p, l, h, d, o, k, g, c, n, j, f, b, m, i, e, a\}$.

Table 1: shows the comprehensive Sudoku square design of treatments of type (1) with the original row $\{A, E, I, M, B, F, J, N, C, G, K, O, H, L, P\}$.

Table 1: Layout of Experiment of (HBGLSS) Design $(4x4)^2$, (Environment I)

		Column Block I				**Column Block II**				**Column Block III**				**Column Block IV**			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Row Block I	1	$A\alpha p$	$E\theta l$	$I\delta h$	$M\omega d$	$B\beta o$	$F\mu k$	$J\varphi g$	Nvc	$C\gamma n$	$G\pi j$	$K\varepsilon f$	$bO\eta$	$D\lambda m$	$H\phi i$	$L\tau e$	$P\upsilon a$
	2	$B\beta o$	$F\mu k$	$J\varphi g$	$N\sigma c$	$C\gamma n$	$G\pi j$	$K\varepsilon f$	$O\eta b$	$D\lambda m$	$H\phi i$	$L\tau e$	$P\upsilon a$	$E\theta l$	$I\delta h$	$M\omega d$	$A\alpha p$
	3	$C\gamma n$	$G\pi j$	$K\varepsilon f$	$O\eta b$	$D\lambda m$	$H\phi i$	$L\tau e$	$P\upsilon a$	$E\theta l$	$I\delta h$	$M\omega d$	$A\alpha p$	$F\mu k$	$J\varphi g$	Nvc	$B\beta o$
	4	$D\lambda m$	$H\phi i$	$L\tau e$	$P\upsilon a$	$E\theta l$	$I\delta h$	$M\omega d$	$A\alpha p$	$F\mu k$	$J\varphi g$	Nvc	$N\beta o$	$G\pi j$	$K\varepsilon f$	$O\eta b$	$C\gamma n$
Row Block II	5	$E\theta l$	$I\delta h$	$M\omega d$	$A\alpha p$	$F\mu k$	$J\varphi g$	Nvc	$B\beta o$	$G\pi j$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$H\phi i$	$L\tau e$	$P\upsilon a$	$D\lambda m$
	6	$F\mu k$	$J\varphi g$	$N\sigma c$	$B\beta o$	$G\pi j$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$H\phi i$	$L\tau e$	$P\upsilon a$	$D\lambda m$	$I\delta h$	$M\omega d$	$A\alpha p$	$E\theta l$
	7	$G\pi j$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$H\phi i$	$L\tau e$	$P\upsilon a$	$D\lambda m$	$I\delta h$	$M\omega d$	$A\alpha p$	$E\theta l$	$J\varphi g$	Nvc	$B\beta o$	$F\mu k$
	8	$H\phi i$	$L\tau e$	$P\upsilon a$	$D\lambda m$	$I\delta h$	$M\omega d$	$A\alpha p$	$E\theta l$	$J\varphi g$	Nvc	$B\beta o$	$F\mu k$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$G\pi j$
Row Bloc	9	$I\delta h$	$M\omega d$	$A\alpha p$	$E\theta l$	$J\varphi g$	Nvc	$B\beta o$	$F\mu k$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$G\pi j$	$L\tau e$	$P\upsilon a$	$D\lambda m$	$H\phi i$
	10	$J\varphi g$	$N\sigma c$	$B\beta o$	$F\mu k$	$K\varepsilon f$	$O\eta b$	$C\gamma n$	$G\pi j$	$L\tau e$	$P\upsilon a$	$D\lambda m$	$H\phi i$	$M\omega d$	$A\alpha p$	$E\theta l$	$I\delta h$

k III	11	Kεf	Oηb	Cγn	Gπj	Lτe	Pυa	Dλm	Hϕi	Mωd	Aαp	Eθl	Iδh	Nvc	Bβo	Fμk	Jφg
	12	Lτe	Pυa	Dλm	Hϕi	Mωd	Aαp	Eθl	Iδh	Nvc	Bβo	Fμk	Jφg	Oηb	Cγn	Gπj	Kεf
Row Block IV	13	Mωd	Aαp	Eθl	Iδh	Nvc	Bβo	Fμk	Jφg	Oηb	Cγn	Dπj	Kεf	Pυa	Dλm	Hϕi	Lτe
	14	Nσc	Bβo	Fμk	Jφg	Oηb	Cγn	Gπj	Kεf	Pυa	Dλm	Hϕi	Lτe	Aαp	Eθl	Iδh	Mωd
	15	Oηb	Cγn	Gπj	Kεf	Pυa	Dλm	Hϕi	Lτe	Aαp	Eθl	Iδh	Mωd	Bβo	Fμk	Jφg	Nvc
	16	Pυa	Dλm	Hϕi	Lτe	Aαp	Eθl	Iδh	Mωd	Bβo	Fμk	Jφg	Nvc	Cγn	Gπj	Kεf	Oηb

Table 2: Layout of Experiments of (HBGLSS) Design $(4x4)^2$, (Environment II)

		Column Block I				Column Block II				Column Block III				Column Block IV			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Row Block I	1	Cγn	Gπj	Kεf	bOη	Dλm	Hϕi	Lτe	Pυa	Aαp	Eθl	Iδh	Mωd	Bβo	Fμk	Jφg	Nvc
	2	Dλm	Hϕi	Lτe	Pυa	Eθl	Iδh	Mωd	Aαp	Bβo	Fμk	Jφg	Nσc	Cγn	Gπj	Kεf	Oηb
	3	Eθl	Iδh	Mωd	Aαp	Fμk	Jφg	Nvc	Bβo	Cγn	Gπj	Kεf	Oηb	Dλm	Hϕi	Lτe	Pυa
	4	Fμk	Jφg	Nvc	Nβo	Gπj	Kεf	Oηb	Cγn	Dλm	Hϕi	Lτe	Pυa	Eθl	Iδh	Mωd	Aαp
Row Block II	5	Gπj	Kεf	Oηb	Cγn	Hϕi	Lτe	Pυa	Dλm	Eθl	Iδh	Mωd	Aαp	Fμk	Jφg	Nvc	Bβo
	6	Hϕi	Lτe	Pυa	Dλm	Iδh	Mωd	Aαp	Eθl	Fμk	Jφg	Nσc	Bβo	Gπj	Kεf	Oηb	Cγn
	7	Iδh	Mωd	Aαp	Eθl	Jφg	Nvc	Bβo	Fμk	Gπj	Kεf	Oηb	Cγn	Hϕi	Lτe	Pυa	Dλm
	8	Jφg	Nvc	Bβo	Fμk	Kεf	Oηb	Cγn	Gπj	Hϕi	Lτe	Pυa	Dλm	Iδh	Mωd	Aαp	Eθl
Row Block III	9	Kεf	Oηb	Cγn	Gπj	Lτe	Pυa	Dλm	Hϕi	Iδh	Mωd	Aαp	Eθl	Jφg	Nvc	Bβo	Fμk
	10	Lτe	Pυa	Dλm	Hϕi	Mωd	Aαp	Eθl	Iδh	Jφg	Nσc	Bβo	Fμk	Kεf	Oηb	Cγn	Gπj
	11	Mωd	Aαp	Eθl	Iδh	Nvc	Bβo	Fμk	Jφg	Kεf	Oηb	Cγn	Gπj	Lτe	Pυa	Dλm	Hϕi
	12	Nvc	Bβo	Fμk	Jφg	Oηb	Cγn	Gπj	Kεf	Lτe	Pυa	Dλm	Hϕi	Mωd	Aαp	Eθl	Iδh
Row Block IV	13	Oηb	Cγn	Dπj	Kεf	Pυa	Dλm	Hϕi	Lτe	Mωd	Aαp	Eθl	Iδh	Nvc	Bβo	Fμk	Jφg
	14	Pυa	Dλm	Hϕi	Lτe	Aαp	Eθl	Iδh	Mωd	Nσc	Bβo	Fμk	Jφg	Oηb	Cγn	Gπj	Kεf
	15	Aαp	Eθl	Iδh	Mωd	Bβo	Fμk	Jφg	Nvc	Oηb	Cγn	Gπj	Kεf	Pυa	Dλm	Hϕi	Lτe
	16	Bβo	Fμk	Jφg	Nvc	Cγn	Gπj	Kεf	Oηb	Pυa	Dλm	Hϕi	Lτe	Aαp	Eθl	Iδh	Mωd

Model 1: *The modification* of Sahib *et al.* (2025) model into Type 1 is as follows:

In this model we assume row, column, treatments type (1, 2 & 3), and effects as in the HBGLSS designs. In addition to the assumption; row block, column block and square effects, environmental experiments and interaction between treatments and environment.

$$Y_{ij(klpqrt)x} = \mu + \theta_x + \alpha_{ix} + \beta_{jx} + \tau_k + \omega_r + \pi_t + \lambda_{lx} + \gamma_{px} + \varphi_{qx} + (\tau\theta)_{kx} + (\omega\theta)_{rx} + (\pi\theta)_{tx} + \varepsilon_{ij(klpqrt)}$$

(1) Where $i, j, k, l, p, q, r, t = 1, 2, ..., m^2;\ m = 4; k = m^2;\ x = 1, 2, ..., g.$

Where;

$\mu =$ Overall mean,
$\theta_x = x^{th}$ Environment-experiment (EE),
$\alpha_{ix} = i^{th}$ Row block effect in x^{th} environment experiment (RB),
$\beta_{jx} = j^{th}$ Column block effect in x^{th} environment experiment (CB),
$\tau_k = k^{th}$ Treatments effect of type (1),
$\omega_r =$ Effect of the r^{th} treatments of type (2),
$\pi_t =$ Effect of the t^{th} treatments type (3),
$\lambda_{lx} = l^{th}$ Row effect in x^{th} environment experiment (R),
$\gamma_{px} = p^{th}$ Column effect in x^{th} environment experiment (C),
$\varphi_{qx} = q^{th}$ Sub-Square effect in x^{th} environment experiment (SS),
$(\tau\theta)_{kx} =$ Interaction between k^{th} Treatment type (1) and x^{th} environment experiment,
$(\omega\theta)_{rx} =$ Interaction between r^{th} Treatment type (2) and x^{th} environment experiment,
$(\pi\theta)_{tx} =$ Interaction between t^{th} Treatment type (3) and x^{th} environment experiment,
$\varepsilon_{ij(klpqrt)} =$ Effect of random error (E).

$$TSS = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \frac{G^2}{gm^4} \quad , SST_k = \sum_{k}^{m^2}\frac{y^2_{..k......}}{gm^2} - \frac{G^2}{gm^4}$$

$$SST_r = \sum_{r}^{m^2}\frac{y^2_{...r.....}}{gm^2} - \frac{G^2}{gm^4}\ \ SST_t = \sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2} - \frac{G^2}{gm^4},$$

$$SSRB = \sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4} - \frac{G^2}{gm^4} \quad , SSCB = \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4} - \frac{G^2}{gm^4}$$

$$SSR = \sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{.....l...x}}{gm^4} - \frac{G^2}{gm^4}, \quad SSC = \sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{......p..x}}{gm^4} - \frac{G^2}{gm^4}, \quad SSS = \sum_{q}^{m^2}\sum_{x}^{g}\frac{\varphi^2_{........qx}}{gm^4} - \frac{G^2}{gm^4}$$

Table 3: ANOVA table for (SEHBGLSS) designs for model Type I of order 16

Source of Variations	Sum of squares	Degrees of Freedom	Mean squares
Environments	$SSEE$	$g-1$	$MSEE=\frac{SSEE}{g-1}$
Treatment type (1)	SSt_1	m^2-1	$MSt_1=\frac{SSt_1}{m^2-1}$
Treatment type (2)	SSt_2	m^2-1	$MSt_2=\frac{SSt_2}{m^2-1}$
Treatment type (3)	SSt_3	m^2-1	$MSt_3=\frac{SSt_3}{m^2-1}$
Row-block	$SSbr$	$gm-1$	$MSbr=\frac{SSbr}{gm-1}$
Column-block	$SSbc$	$gm-1$	$MSbc=\frac{SSbc}{gm-1}$
Rows	SSr	gm^2-1	$MSr=\frac{SSr}{gm^2-1}$
Columns	SSc	gm^2-1	$MSc=\frac{SSc}{gm^2-1}$
Sub-Squares	SSS	gm^2-1	$MSS=\frac{SSS}{gm^2-1}$
Environment x treatment type (1)	$SSI_{(1)}$	$(g-1)(m^2-1)$	$MSI_{(1)}=\frac{SSI_{(1)}}{(g-1)(m^2-1)}$
Environment x treatment type (2)	$SSI_{(2)}$	$(g-1)(m^2-1)$	$MSI_{(2)}=\frac{SSI_{(2)}}{(g-1)(m^2-1)}$
Environment x treatment type (3)	$SSI_{(3)}$	$(g-1)(m^2-1)$	$MSI_{(3)}=\frac{SSI_{(3)}}{(g-1)(m^2-1)}$
Error	SSE	$gm^2(m^2-6)$ $-2g(m-1)+5$	$MSE=\frac{SSE}{gm^2(m^2-6)-2g(m-1)+5}$
TOTAL	SST	gm^4-1	

$$SSEE=\sum_{x}^{g}\frac{\theta^2_{.........x}}{gm^4}-\frac{G^2}{gm^4}, \quad SSI_{(1)}=\sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2}-\sum_{k}^{m^2}\frac{y^2_{...k......}}{gm^2}-\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}+\frac{G^2}{gm^4}$$

$$SSI_{(2)}=\sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2}-\sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2}-\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}+\frac{G^2}{gm^4}$$

$$SSI_{(3)}=\sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2}-\sum_{t}^{m^2}\frac{y^2_{.....t....}}{gm^2}-\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}+\frac{G^2}{gm^4}$$

$$SSE=\sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2}Y_{ijklpqrtx}-\sum_{k}^{m^2}\frac{y^2_{..k......}}{gm^2}-\sum_{r}^{m^2}\frac{y^2_{...r.....}}{gm^2}-\sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2}-\sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4}$$
$$-\sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4}-\sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{....l...x}}{gm^4}-\sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{......p..x}}{gm^4}-\sum_{q}^{m^2}\sum_{x}^{g}\frac{\varphi^2_{.......qx}}{gm^4}-\sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2}+\sum_{k}^{m^2}\frac{y^2_{...k......}}{gm^2}$$
$$-\sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2}+\sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2}-\sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2}+\sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2}+2\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}-12\frac{G^2}{gm^4}$$
(2)

Model 2: *The modification of* Sahib et al. (2025) model to type II is as follows:

In this model it is assumed that the row effects are nested in the row block effects and the column effects are nested in the column block effects.

$$Y_{ij(klpqrt)x}=\mu+\theta_x+\alpha_{ix}+\beta_{jx}+\tau_k+\omega_r+\pi_t+\lambda(\alpha)_{l(i)x}+\gamma(\beta)_{p(j)x}+\varphi_{qx}+(\tau\theta)_{kx}+(\omega\theta)_{rx}+(\pi\theta)_{tx}+\varepsilon_{ij(klpqrt)}$$
(3)

Where $i,j,k,l,p,q,r,t=1,2,...,m^2;\ m=4; k=m^2;\ x=1,2,...,g.$

Where;

$\mu=$ Overall mean,

$\theta_x=x^{th}$ Environment-experiment,

$\alpha_{ix}=i^{th}$ Row block effect in x^{th} environmental experiment,

$\beta_{jx}=j^{th}$ Column block effect in x^{th} environmental experiment,

$\tau_k=k^{th}$ Treatments effect of type (1),

$\omega_r=$ Effect of the r^{th} treatments of type (2),

$\pi_t=$ Effect of the t^{th} treatments type (3),

$\lambda(\alpha)_{l(i)x}=l^{th}$ Row effect nested in i^{th} block (row) in x^{th} environmental experiment,

$\gamma(\beta)_{p(j)x}=p^{th}$ Column effect nested in j^{th} block (column) in x^{th} environmental experiment, $\varphi_{qx}=q^{th}$ Sub-Square effect in x^{th} environmental experiment,

$(\tau\theta)_{kx}=$ Interaction between k^{th} Treatment type (1) and x^{th} environmental experiment,

$(\omega\theta)_{rx}=$ Interaction between r^{th} Treatment type (2) and x^{th} environmental experiment,

$(\pi\theta)_{tx}=$ Interaction between t^{th} Treatment type (3) and x^{th} environmental experiment,

$\varepsilon_{ij(klpqrt)}=$ Effect of random error

$$TSS=\sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2}Y_{ijklpqrtx}-\frac{G^2}{gm^4}, \qquad SST_k=\sum_{k}^{m^2}\frac{y^2_{..k......}}{gm^2}-\frac{G^2}{gm^4},$$

$$SST_r=\sum_{r}^{m^2}\frac{y^2_{...r.....}}{gm^2}-\frac{G^2}{gm^4}\quad SST_t=\sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2}-\frac{G^2}{gm^4},\quad SSRB=\sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4}-\frac{G^2}{gm^4},$$

$$SSCB=\sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4}-\frac{G^2}{gm^4}\quad SSRRb=\sum_{i}^{m}\sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda(\alpha)^2_{i.....l.x}}{gm^2}-\sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4},$$

$$SSRRC=\sum_{j}^{m}\sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma(\beta)^2_{.j.....p.x}}{gm^2}-\sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4}$$

$$SSS=\sum_{q}^{m^2}\sum_{x}^{g}\frac{\varphi^2_{........qx}}{gm^4}-\frac{G^2}{gm^4},\ SSEE=\sum_{x}^{g}\frac{\theta^2_{.........x}}{gm^4}-\frac{G^2}{gm^4}$$

$$SSI_{(1)}=\sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2}-\sum_{k}^{m^2}\frac{y^2_{...k......}}{gm^2}-\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}+\frac{G^2}{gm^4}$$

$$SSI_{(2)}=\sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2}-\sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2}-\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4}+\frac{G^2}{gm^4}$$

Table 4: ANOVA table for (SEHBGLSS) designs for model Type II of order 16

Source of Variations	**Sum of squares**	**Degrees of Freedom**	**Mean squares**
Environments	$SSEE$	$g-1$	$MSEE=\frac{SSEE}{g-1}$
Treatment type (1)	SSt_1	m^2-1	$MSt_1=\frac{SSt_1}{m^2-1}$
Treatment type (2)	SSt_2	m^2-1	$MSt_2=\frac{SSt_2}{m^2-1}$
Treatment type (3)	SSt_3	m^2-1	$MSt_3=\frac{SSt_3}{m^2-1}$
Rows within row-block	$SSrrb$	$gm(m-1)$	$MSrrb=\frac{SSrrb}{gm(m-1)}$
Column within column-blocks	$SSccb$	$gm(m-1)$	$MSccb=\frac{SSccb}{gm(m-1)}$
Sub-Squares	SSS	gm^2-1	$MSS=\frac{SSS}{gm^2-1}$
Row-block	$SSbr$	$gm-1$	$MSbr=\frac{SSbr}{gm-1}$
Columns-block	$SSbc$	$gm-1$	$MSbc=\frac{SSbc}{gm-1}$
Environment x treatment type (1)	$SSI_{(1)}$	$(g-1)(m^2-1)$	$MSI_{(1)}=\frac{SSI_{(1)}}{(g-1)(m^2-1)}$
Environment x treatment type (2)	$SSI_{(2)}$	$(g-1)(m^2-1)$	$MSI_{(2)}=\frac{SSI_{(2)}}{(g-1)(m^2-1)}$
Environment x treatment type (3)	$SSI_{(3)}$	$(g-1)(m^2-1)$	$MSI_{(3)}=\frac{SSI_{(3)}}{(g-1)(m^2-1)}$
Error	SSE	$gm^2(m^2-6)+(2g+3)$	$MSE=\frac{SSE}{gm^2(m^2-6)+(2g+3)}$
TOTAL	SST	gm^4-1	

$$SSI_{(3)} = \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} - \sum_{t}^{m^2}\frac{y^2_{.....t....}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSE = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{.....l...x}}{gm^4} - \sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{......p..x}}{gm^4}$$
$$-\sum_{i}^{m}\sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda(\alpha)^2_{i.....l.x}}{gm^2} - \sum_{j}^{m}\sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma(\beta)^2_{.j.....p.x}}{gm^2} - \sum_{q}^{m^2}\sum_{x}^{g}\frac{\varphi^2_{........qx}}{gm^4} -$$
$$\sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k......x}}{gm^2} - \sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2} - \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} + 2\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} - 12\frac{G^2}{gm^4} \quad (4)$$

Model 3: *The modification of* Sahib et al. (2025) model to Type III is as follows:

In this model it is assumed that the horizontal square effects are nested in the row block effects and the vertical square effects are nested in the column block effects.

$$Y_{ij(klpqrst)x} = \mu + \theta_x + \alpha_{ix} + \beta_{jx} + \tau_k + \omega_r + \pi_t + \lambda_{lx} + \gamma_{px} + \varphi(\alpha)_{q(i)x} + \delta(\beta)_{s(j)x} + (\tau\theta)_{kx} + (\omega\theta)_{rx} + (\pi\theta)_{tx} + \varepsilon_{ij(klpqrst)} \quad (5)$$

Where $i,j,k,l,p,q,r,s,t = 1,2,...,m^2$; $m = 4; k = m^2$; $x = 1,2,...,g$.

Where; $\mu =$ Overall mean,

$\theta_x = x^{th}$ Environment-experiment,

$\alpha_{ix} = i^{th}$ Row block effect in x^{th} environment experiment,

$\beta_{jx} = j^{th}$ Column block effect in x^{th} environment experiment,

$\tau_k = k^{th}$ Treatments effect of type (1),

$\omega_r =$ Effect of the r^{th} treatments of type (2),

$\pi_t =$ Effect of the t^{th} treatments type (3),

$\lambda_{lx} = l^{th}$ Row effect in x^{th} environment experiment,

$\gamma_{px} = p^{th}$ Column effect in x^{th} environment experiment,

$\varphi_{qx} = q^{th}$ Sub-Square effect in x^{th} environment experiment,

$\varphi(\alpha)_{q(i)x} = q^{th}$ Horizontal square effect nested in the row-block in x^{th} environmental experiment,

$\delta(\beta)_{s(j)x} = s^{th}$ Vertical square effect nested in the column-block in x^{th} environmental experiment, $(\tau\theta)_{kx} =$ Interaction between k^{th} Treatment type (1) and x^{th} environment experiment,

$(\omega\theta)_{rx} =$ Interaction between r^{th} Treatment type (2) and x^{th} environment experiment,

$(\pi\theta)_{tx} =$ Interaction between t^{th} Treatment type (3) and x^{th} environment experiment,

$\varepsilon_{ij(klpqrt)}$ = Effect of random error

$$TSS = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \frac{G^2}{gm^4}\ , SST_k = \sum_{k}^{m^2}\frac{y^2_{..k......}}{gm^2} - \frac{G^2}{gm^4},$$

$$SST_r = \sum_{r}^{m^2}\frac{y^2_{...r.....}}{gm^2} - \frac{G^2}{gm^4} \,.\, SST_t = \sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2} - \frac{G^2}{gm^4}, \qquad SSRB = \sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4} - \frac{G^2}{gm^4},$$

$$SSCB = \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4} - \frac{G^2}{gm^4}$$

$$SSR = \sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{.....l...x}}{gm^4} - \frac{G^2}{gm^4}, \qquad SSC = \sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{......p..x}}{gm^4} - \frac{G^2}{gm^4},$$

$$SSVS = \sum_{q}^{m^2}\sum_{j}^{m}\sum_{x}^{g}\frac{\varphi\beta^2_{.j......qx}}{gm^2} - \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4}$$

$$SSEE = \sum_{x}^{g}\frac{\theta^2_{.........x}}{gm^4} - \frac{G^2}{gm^4}, \qquad SSI_{(1)} = \sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2} - \sum_{k}^{m^2}\frac{y^2_{...k......}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

Table 5: ANOVA table for (SEHBGLSS) designs for model Type III of order 16

Source of Variations	Sum of squares	Degrees of Freedom	Mean squares
Environments	$SSEE$	$g-1$	$MSEE = \frac{SSEE}{g-1}$
Treatment type (1)	$SSt_{(1)}$	m^2-1	$MSt_1 = \frac{MSt_{(1)}}{m^2-1}$
Treatment type (2)	$SSt_{(2)}$	m^2-1	$MSt_2 = \frac{MSt_{(2)}}{m^2-1}$
Treatment type (3)	$SSt_{(3)}$	m^2-1	$MSt_3 = \frac{MSt_{(3)}}{m^2-1}$
Rows	SSr	gm^2-1	$MSr = \frac{SSr}{gm^2-1}$
Columns	SSc	gm^2-1	$MSc = \frac{SSc}{gm^2-1}$
Row-block	$SSbr$	$gm-1$	$MSbr = \frac{SSbr}{gm\ -1}$
Columns-block	$SSbc$	$gm-1$	$MSbc = \frac{SSbc}{gm\ -1}$
Horizontal sq. within row-block	$SShrb$	$gm(m-1)$	$MShrb = \frac{SShrb}{gm\ (m-1)}$
Vertical sq. within column-block	$SSvcb$	$gm(m-1)$	$MSvcb = \frac{SSvcb}{gm\ (m-1)}$

Env. Experiment x treatment type (1)	$SSI_{(1)}$	$(g-1)(m^2-1)$	$MSI_{(1)} = \frac{SSI_{(1)}}{(g-1)(m^2-1)}$
Env. Experiment x treatment type (2)	$SSI_{(2)}$	$(g-1)(m^2-1)$	$MSI_{(2)} = \frac{SSI_{(2)}}{(g-1)(m^2-1)}$
Env. Experiment x treatment type (3)	$SSI_{(3)}$	$(g-1)(m^2-1)$	$MSI_{(3)} = \frac{SSI_{(3)}}{(g-1)(m^2-1)}$
Error	SSE	$gm^2(m^2-7)+2(g+2)$	$MSE = \frac{SSE}{gm^2(m^2-7)+2(g+2)}$
TOTAL	SST	gm^4-1	

$$SSI_{(2)} = \sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2} - \sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSI_{(3)} = \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} - \sum_{t}^{m^2}\frac{y^2_{.....t....}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSE = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{.....l...x}}{gm^4} - \sum_{q}^{m^2}\sum_{i}^{m}\sum_{x}^{g}\frac{\varphi\alpha^2_{i.......qx}}{gm^2} - \sum_{q}^{m^2}\sum_{j}^{m}\sum_{x}^{g}\frac{\varphi\beta^2_{.j......qx}}{gm^2}$$
$$-\sum_{q}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{........qx}}{gm^4} - \sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2} - \sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2} - \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} + 2\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} - 12\frac{G^2}{gm^4} \quad (6)$$

Model 4: *The modification of* Sahib et al. (2025) model to type IV is as follows:

In this model it is assumed that the row effects and the horizontal square effects are nested in the row block effects and the column effects and the vertical square effects are nested in the column block effects.

$$Y_{ij(klpqrst)x} = \mu + \theta_x + \alpha_{ix} + \beta_{jx} + \tau_k + \omega_r + \pi_t + \lambda(\alpha)_{l(i)x} + \gamma(\beta)_{p(j)x} + \varphi(\alpha)_{q(i)x} + \delta(\beta)_{s(j)x} + (\omega\theta)_{rx} + (\pi\theta)_{tx} + \varepsilon_{ij(klpqrst)} \quad (7)$$

Where $i,j,k,l,p,q,r,s,t = 1,2,...,m^2;\ m=4; k=m^2;\ x=1,2,...,g.$

Where;

$\mu =$ Overall mean,

$\theta_x = x^{th}$ Environment-experiment,

$\alpha_{ix} = i^{th}$ Row block effect in x^{th} environmental experiment,

$\beta_{jx} = j^{th}$ Column block effect in x^{th} environmental experiment,

$\tau_k = k^{th}$ Treatments effect of type (1),

$\omega_r =$ Effect of the r^{th} treatments of type (2),

π_t = Effect of the t^{th} treatments type (3),
$\lambda(\alpha)_{l(i)x} = l^{th}$ Row effect nested in i^{th} block (row) in x^{th} environmental experiment, $\gamma(\beta)_{p(j)x} = p^{th}$ Column effect nested in j^{th} block (column) in x^{th} environmental experiment, $\varphi(\alpha)_{q(i)x} = q^{th}$ Horizontal square effect nested in the row-block in x^{th} environmental experiment, $\delta(\beta)_{s(j)x} = s^{th}$ Vertical square effect nested in the column-block in x^{th} environmental experiment, $(\tau\theta)_{kx}$ = Interaction between k^{th} Treatment type (1) and x^{th} environmental experiment,
$(\omega\theta)_{rx}$ = Interaction between r^{th} Treatment type (2) and x^{th} environmental experiment,
$(\pi\theta)_{tx}$ = Interaction between t^{th} Treatment type (3) and x^{th} environmental experiment,
$\varepsilon_{ij(klpqrt)}$ = Effect of random error

$$TSS = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \frac{G^2}{gm^4},\ SST_k = \sum_{k}^{m^2}\frac{y^2_{..k......}}{gm^2} - \frac{G^2}{gm^4},\ SST_r = \sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2} - \frac{G^2}{gm^4}$$

$$SST_t = \sum_{t}^{m^2}\frac{y^2_{....t....}}{gm^2} - \frac{G^2}{gm^4},\ SSRB = \sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4} - \frac{G^2}{gm^4},\ SSCB = \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4} - \frac{G^2}{gm^4}$$

$$SSRRC = \sum_{j}^{m}\sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma(\beta)^2_{.j.....p.x}}{gm^2} - \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4},\ SSHS = \sum_{q}^{m^2}\sum_{i}^{m}\sum_{x}^{g}\frac{\varphi\alpha^2_{i.......qx}}{gm^2} - \sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4}$$

$$SSVS = \sum_{q}^{m^2}\sum_{j}^{m}\sum_{x}^{g}\frac{\varphi\beta^2_{.j......qx}}{gm^2} - \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4},\ SSEE = \sum_{x}^{g}\frac{\theta^2_{.........x}}{gm^4} - \frac{G^2}{gm^4}$$

Table 6: ANOVA table for (SEHBGLSS) designs for model Type IV of order 16

Source of Variations	**Sum of squares**	**Degrees of Freedom**	**Mean squares**
Environments	$SSEE$	$g-1$	$MSEE = \frac{SSEE}{g-1}$
Treatment type (1)	SSt_1	m^2-1	$MSt_{(1)} = \frac{SSt_{(1)}}{m^2-1}$
Treatment type (2)	SSt_2	m^2-1	$MSt_{(2)} = \frac{SSt_{(2)}}{m^2-1}$
Treatment type (3)	SSt_3	m^2-1	$MSt_{(3)} = \frac{SSt_{(3)}}{m^2-1}$
Rows within row-block	$SSrrb$	$gm(m-1)$	$MSrrb = \frac{SSrrb}{gm(m-1)}$
Column within column-block	$SSccb$	$gm(m-1)$	$MSccb = \frac{SSccb}{gm(m-1)}$
Horizontal sq. within row-block	$SShrb$	$gm(m-1)$	$MShrb = \frac{SShrb}{gm(m-1)}$

Vertical sq. within column-block	$SSvcb$	$gm(m-1)$	$MSvcb = \frac{SSvcb}{gm(m-1)}$
Row-block	$SSbr$	$gm-1$	$MSbr = \frac{SSbr}{gm-1}$
Column-block	$SSbc$	$gm-1$	$MSbc = \frac{SSbc}{gm-1}$
Env. Experiment x treatment type (1)	$SSI_{(1)}$	$(g-1)(m^2-1)$	$MSI_{(1)} = \frac{SSI_{(1)}}{(g-1)(m^2-1)}$
Env. Experiment x treatment type (2)	$SSI_{(2)}$	$(g-1)(m^2-1)$	$MSI_{(2)} = \frac{SSI_{(2)}}{(g-1)(m^2-1)}$
Env. Experiment x treatment type (3)	$SSI_{(3)}$	$(g-1)(m^2-1)$	$MSI_{(3)} = \frac{SSI_{(3)}}{(g-1)(m^2-1)}$
Error	SSE	$gm^2(m^2-7)+2(gm+g+1)$	$MSE = \frac{SSE}{gm^2(m^2-7)+2(gm+g+1)}$
TOTAL	SST	gm^4-1	

$$SSI_{(1)} = \sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2} - \sum_{k}^{m^2}\frac{y^2_{...k......}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSI_{(2)} = \sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2} - \sum_{r}^{m^2}\frac{y^2_{....r.....}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSI_{(3)} = \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} - \sum_{t}^{m^2}\frac{y^2_{.....t....}}{gm^2} - \sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} + \frac{G^2}{gm^4}$$

$$SSE = \sum_{i}^{m}\sum_{j}^{m}\sum_{x}^{g}\sum_{k,l,p,q,r,t}^{m^2} Y_{ijklpqrtx} - \sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda^2_{.....l...x}}{gm^4} - \sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma^2_{......p..x}}{gm^4}$$
$$-\sum_{i}^{m}\sum_{l}^{m^2}\sum_{x}^{g}\frac{\lambda(\alpha)^2_{i.....l.x}}{gm^2} - \sum_{q}^{m^2}\sum_{i}^{m}\sum_{x}^{g}\frac{\varphi\alpha^2_{i.......qx}}{gm^2} + \sum_{i}^{m}\sum_{x}^{g}\frac{\alpha^2_{i.......x}}{gm^4}$$
$$-\sum_{j}^{m}\sum_{p}^{m^2}\sum_{x}^{g}\frac{\gamma(\beta)^2_{.j.....p.x}}{gm^2} - \sum_{q}^{m^2}\sum_{j}^{m}\sum_{x}^{g}\frac{\varphi\beta^2_{.j......qx}}{gm^2} + \sum_{j}^{m}\sum_{x}^{g}\frac{\beta^2_{.j......x}}{gm^4}$$
$$\sum_{k}^{m^2}\sum_{x}^{g}\frac{\tau\theta^2_{...k.....x}}{gm^2} - \sum_{r}^{m^2}\sum_{x}^{g}\frac{\omega\theta^2_{....r....x}}{gm^2} - \sum_{t}^{m^2}\sum_{x}^{g}\frac{\pi\theta^2_{.....t...x}}{gm^2} + 2\sum_{x}^{g}\frac{\theta^2_{.........x}}{m^4} - 12\frac{G^2}{gm^4} \quad (8)$$

Measure of Efficiency

Relative Efficiency (RE) is a measure used to compare the efficiency of units by calculating their efficiency scores relative to each other. It helps to determine which method/design is more efficient.

1. Relative Efficiency is calculated as the ratio of the variances (or mean squared errors) i.e.

$$RE_{(HBGLSS:SEHBGLSS)} = \frac{MSE_{(HBGLSS)}}{MSE_{(SEHBGLSS)}} \tag{9}$$

$RE > 1$: $SEHBGLSSD$ is more efficient than $HBGLSSD$ (better precision and accuracy)

$RE < 1$: $SEHBGLSSD$ is less efficient than $HBGLSSD$ (poorer precision and accuracy)

$RE = 1$: $SEHBGLSSD$ and $HBGLSSD$ have the same efficiency (similar precision and accuracy).

2. The power of the F-test can be interpreted much the same way so that;

$$RE_{(HBGLSS:SEHBGLSS)} = \frac{Pow_{(HBGLSS)}}{pow_{(SEHBGLSS)}} \tag{10}$$

Then:

$RE > 1$: $SEHBGLSS$D is more efficient than $HBGLSSD$ (better precision and accuracy)

$RE < 1$: $SEHBGLSSD$ is less efficient than $HBGLSSD$ (poorer precision and accuracy)

$RE = 1$: $SEHBGLSSD$ and $HBGLSSD$ have the same efficiency (similar precision and accuracy).

Illustration: Tables 7 and 8 give hypothetical datasets for two environments ($E1$ and $E2$) obtained using HBGLSSD of order 16.

Table 7: Hypothetical datasets for HBGLSSD of order 16 (ENVIRONMENT I)

	Column block 1				Column block 2				Column block 3				Column block 4			
	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15	C16
Row Block-I	20	25	29	25	23	28	24	19	21	16	10	14	16	20	15	10
	15	21	17	20	21	23	18	15	16	21	15	19	21	26	20	25
	19	26	22	24	23	17	25	20	21	25	20	23	22	18	19	29
	26	20	15	10	13	22	18	15	16	19	24	18	20	24	29	33
Row Block-II	28	19	14	9	12	21	16	10	18	22	19	22	21	23	27	30
	24	15	20	15	16	24	11	16	14	27	24	27	26	19	23	26
	27	24	17	20	22	19	19	20	14	27	24	27	26	19	23	26
	23	17	22	25	28	23	25	18	16	19	25	29	27	20	24	19
Row Block-III	22	19	25	22	25	20	23	20	19	24	18	27	26	19	25	20
	18	14	19	15	17	24	28	24	23	29	22	21	22	24	20	16
	23	20	15	20	16	20	25	20	24	23	19	15	17	20	15	21
	19	25	26	17	19	25	19	25	20	27	23	10	11	15	10	15
Row Block-IV	22	24	27	20	21	26	21	26	24	26	22	12	13	17	11	17
	26	28	22	16	17	22	26	20	19	22	27	17	18	22	17	22
	12	25	29	33	32	27	31	25	23	27	33	25	24	27	24	19
	16	20	25	29	27	33	29	21	20	24	29	20	21	32	29	24

Table 8: Hypothetical datasets for SEHBGLSSD of order 16 (ENVIRONMENT II)

	Column block 1				Column block 2				Column block 3				Column block 4			
	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$	$C8$	$C9$	$C10$	$C11$	$C12$	$C13$	$C14$	$C15$	$C16$
Row Block-I	20	19	25	21	17	22	26	20	19	22	27	17	25	19	24	21
	15	18	19	21	17	24	28	24	23	29	22	21	20	21	20	15
	24	21	15	19	32	27	31	25	23	27	33	25	14	23	16	20
	21	23	19	18	27	33	29	21	20	24	29	20	12	15	17	16
Row Block-II	20	24	25	21	21	26	21	26	24	26	22	12	13	17	11	17
	27	23	22	15	17	22	26	20	19	22	27	17	18	22	17	22
	16	24	26	32	32	27	31	25	23	27	33	25	24	27	24	19
	18	20	24	27	27	33	29	21	20	24	29	20	21	32	29	24
Row Block-III	21	23	27	30	24	15	20	15	16	24	11	16	16	20	15	10
	26	19	23	26	27	24	17	20	16	21	15	19	21	26	20	25
	27	20	24	19	23	17	25	20	21	25	20	23	22	18	19	29
	26	19	23	26	23	17	22	25	28	23	25	18	20	24	29	33
Row Block-IV	26	19	14	15	12	21	16	10	18	22	19	22	24	22	27	33
	20	17	20	29	16	24	11	16	14	27	24	27	22	14	21	29
	25	25	15	21	22	19	19	20	14	27	24	27	22	16	27	25
	24	16	21	26	27	23	24	16	18	14	25	27	29	22	20	18

Table 9: ANOVA Result for (HBGLSS) Design

Source	D.F	SS	MS	F	P
Rows	15	997.6250	66.5083	3.5035	0.2146
Columns	15	293.7501	19.5833	1.03162	0.3209
Type (1) Treatments	15	311.0000	20.7330	1.0921	0.3114
Block (Square)	15	913.6250	60.9083	3.2085	0.2302
Type (2) Treatments	15	196.5001	13.1003	0.69	0.7721
Type (3) Treatments	15	359.7503	23.9833	1.2634	0.3371
Row-Blocks	3	370.3281	123.4427	2.0100	0.1456
Column-Blocks	3	320.0000	106.666	1.320	0.3014
Error	159	2401.1875	15.1018		
TOTAL	255	6164.0000			

Table 10: Combined Analysis for SEHBGLSS Designs of Type I

Source	D.F	SS	MS	F	P-Value
Environments	1	317.9753	317.9753	23.2382	0.0008
Treatment type (1)	15	311.0000	20.7330	1.5152	0.2901
Treatment type (2)	15	196.5001	13.1003	0.9574	0.6722
Treatment type (3)	15	359.7503	23.9833	1.7527	0.2818
Row-block	7	637.5906	91.0844	6.6566	0.0518
Column-block	7	318.0017	45.4288	3.3200	0.1587
Rows	31	1935.3925	62.4320	4.5626	0.1195
Columns	31	1954.7627	63.0569	4.6083	0.1107
Sub-Squares	31	1178.1569	38.0051	2.7775	0.1463
Environment x treatment type (1)	15	248.5104	16.5674	1.2108	0.3521
Environment x treatment type (2)	15	241.1979	16.0799	1.1751	0.3755
Environment x treatment type (3)	15	253.3014	16.8868	1.2341	0.3425
Error	313	4282.8602	13.6833		
TOTAL	511	12235.000			

From table 9 and 10 above, the Mean Squares Error (MSE) for each of the design is given as below;

$$MSE_{SEHBGLSSD} = 13.6833 \qquad MSE_{HBGLSSD} = 15.1018$$

So that relative efficiency (RE) is thus;

$$RE = \frac{MSE_{HBGLSSD}}{MSE_{SEHBGLSSD}} = \frac{15.1018}{13.6833} = 1.1037 \tag{11}$$

The P-value for type (1) treatments effect in each design is given in Table 9 and 10 as;

$$P_{SEHBGLSSD} = 0.2901 \qquad P_{HBGLSSD} = 0.3114$$

$$RE = \frac{P_{(HBGLSSD)}}{P_{(SEHBGLSSD)}} = \frac{0.3114}{0.2901} = 1.0734 \tag{12}$$

Table 11: Power of test and its Parameters for *SEHBGLSSD* and *HBGLSSD*

Design	Sample Size	Effect Size	Power	Group
SEHGBLSSD	512	0.3681	0.8976	32
HGBLSSD	256	0.1996	0.6844	16

Table 11: displays the non-centrality parameters and the power of test for each of the design. The level of significance used is 0.05.

Similarly, as captured in Table4.3 above, the power of test for each of the design stated are used to obtain the relative efficiency as follows;

$$RE = \frac{Pow_{SEHBGLSSD}}{Pow_{HBGLSSD}} = \frac{0.8976}{0.6844} = 1.3115 \qquad (13)$$

Table 12: Combined Analysis for SEHBGLSS Designs of Type II

Source	D.F	SS	MS	F	P-Value
Environments	1	317.9753	317.9753	22.2491	0.0010
Treatment type (1)	15	311.0000	20.7330	1.4507	0.3002
Treatment type (2)	15	196.5001	13.1003	0.9166	0.6921
Treatment type (3)	15	359.7503	23.9833	1.6781	0.2991
Row-block	7	637.5906	91.0844	6.3733	0.0321
Column-block	7	318.0017	45.4288	3.1787	0.1668
Rows within RB	24	1714.6425	71.4434	4.9990	0.1075
Columns within CB	24	1785.0127	74.3755	5.2041	0.1055
Sub-Squares	31	1178.1569	38.0051	2.6593	0.1593
Environment x treatment type (1)	15	248.5104	16.5674	1.1592	0.3702
Environment x treatment type (2)	15	241.1979	16.0799	1.1251	0.3855
Environment x treatment type (3)	15	253.3014	16.8868	1.1816	0.3312
Error	327	4673.3602	14.2916		
TOTAL	511	12235.000			

Table 13: Combined Analysis for SEHBGLSS Designs of Type III

Source	D.F	SS	MS	F	P-Value
Environments	1	317.9753	317.9753	26.6365	0.0007
Treatment type (1)	15	311.0000	20.7330	1.7368	0.2702
Treatment type (2)	15	196.5001	13.1003	1.0974	0.3552
Treatment type (3)	15	359.7503	23.9833	2.0091	0.2818
Row-block	7	637.5906	91.0844	7.6300	0.0511
Column-block	7	318.0017	45.4288	3.8055	0.1557
Rows	31	1935.3925	62.4320	5.2299	0.1865
Columns	31	1954.7627	63.0569	5.2822	0.1101
H-Square within RB	24	513.3255	21.3886	1.7917	0.2463
V-Square within CB	24	236.0006	9.8334	0.8237	0.6422
Env. Experiment x treatment type (1)	15	248.5104	16.5674	1.3878	0.3432
Env. Experiment x treatment type (2)	15	241.1979	16.0799	1.3470	0.3665
Env. Experiment x treatment type (3)	15	253.3014	16.8868	1.4146	0.3225
Error	296	3533.5341	11.9376		
TOTAL	511	12235.000			

Table 14: Combined Analysis for SEHBGLSS Designs of Type IV

Source	D.F	SS	MS	F	P-Value
Environments	1	317.9753	317.9753	19.3196	0.0015
Treatment type (1)	15	311.0000	20.7330	1.2597	0.3015
Treatment type (2)	15	196.5001	13.1003	0.7959	0.6991
Treatment type (3)	15	359.7503	23.9833	1.4572	0.3218
Row-block	7	637.5906	91.0844	5.5341	0.1655
Column-block	7	318.0017	45.4288	2.7602	0.1668
Rows within RB	24	1714.6425	71.4434	4.3408	0.1569
Columns within CB	24	1785.0127	74.3755	4.5189	0.1555
H-Square within RB	24	513.3255	21.3886	1.2995	0.2563
V-Square within CB	24	236.0006	9.8334	0.5975	0.8022
Env. Experiment x treatment type (1)	15	248.5104	16.5674	1.0066	0.3744
Env. Experiment x treatment type (2)	15	241.1979	16.0799	0.9770	0.3935
Env. Experiment x treatment type (3)	15	253.3014	16.8868	1.0260	0.3681
Error	310	5102.1910	16.4587		
TOTAL	511	12235.000			

Discussion

Table 3: presents analysis of variance for the Hyper Block Graeco-Latin Sudoku square design *(HBGLSSD)* dataset to test whether the factors are statistically significant. With all factors having p-values greater than 0.05, it is evident to conclude that all the factor effects are not significant. Similarly, Table 4: shows the ANOVA results for the combined analysis of several experiments of Hyper Block Graeco-Latin Sudoku square design *(SEHBGLSSD)* with the indication that the treatment factor effects are also not significant at a significant probability of 0.05 except for the environment factor. Although the interaction effects were not statistically significant, their inclusion improved model efficiency by reducing unexplained variability.

Equation 11: provides the relative efficiency between the *SEHBGLSSD* and *HBGLSSD* in the context of the mean squared error *(MSE)*, which is given as 1.1037 With the $RE > 1$, it is then an evidence that combined analysis of several experiments of Hyper Block Graeco-Latin Sudoku square design *(SEHBGLSSD)* is more efficient than that of Hyper Block Graeco-Latin Sudoku square design *(HBGLSSD)* because of the environment factor. That means *SEHBGLSSD* has lower MSE denoting that it has higher precision and efficiency than the *HBGLSSD*.

In terms of the P-value approach, the p-values associated with the F-statistic for the treatment effect were computed for both designs. Equation 12: demonstrates the relative efficiency (RE) between the *SEHBGLSSD* and *HBGLSSD* with a value of 1.0734. However, as the $RE > 1$, it offers that *SEHBGLSSD* has a lower p-value, suggesting that it is more robust in detecting treatment effects compared to *HBGLSSD*.

Likewise, the power of the F-test was calculated to assess the ability of each design to detect treatment effects and the relative efficiency between the two designs in terms of their power, was computed as captured in equation 13, with a value of 1.3115.

The RE for the power values of both designs is equaling to 1, indicating that *SEHBGLSSD* has a higher statistical power and therefore more effective in detecting treatment effects compared to that of *HBGLSSD*.

Finally, Table 10 – 14, presents the results of model type I – IV of combined analysis of Hyper Block Graeco-Latin Sudoku square designs with more efficiency (except model type IV) compared to the existing models. Overall, the results of the analysis from both the hypothetical experimental data have shown a remarkable efficiency and robustness for the combined analysis of several experiments under the new developed models on Hyper Block Graeco-Latin Sudoku square design in their usage and application to experimentation.

Conclusion

The combined analysis of several experiments of Hyper Block Graeco-Latin Sudoku square Designs *(SEHBGLSSD)* models except for type IV outperformed that of Hyper Block Graeco-Latin Sudoku square designs *(HBGLSSD)* with respect to MSE, p-values, and statistical power. This indicates that *SEHBGLSSD* is more efficient in estimating and detecting treatment effects than HBGLSSD.

These results have practical value for researchers working in various fields, including agriculture, industrial experiments, and clinical trials. The selection of design should depend on the specific objectives of the study. *SEHBGLSSD* is recommended when accuracy and reliability are essential, whereas *HBGLSSD* remains a suitable option when greater flexibility in design is required.

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