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The Novelty of Shehu Transform on Variable Coefficient Ordinary Differential Equations

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Abstract

Variable-coefficient ordinary differential equations (ODEs) are generally more difficult to solve using classical integral transforms, which are primarily suited to constant-coefficient problems. This study develops a generalized analytical method based on the Shehu transform for solving variable-coefficient ODEs. As part of the analytical framework, the Shehu transforms of $ty(t)$, $t^2y(t)$, $ty'(t)$, $ty''(t)$, $t^2y'(t)$, and $t^2y''(t)$ are derived to facilitate the treatment of variable coefficients and derivative-product terms. The proposed method is subsequently applied to several diverse examples, including nonlinear ODEs, to demonstrate its applicability to different classes of variable-coefficient problems. The resulting solutions are consistent with the corresponding exact solutions available in the existing literature, thereby confirming the accuracy and validity of the analytical procedure. These findings demonstrate that the Shehu transform provides an effective and viable framework for solving complex variable-coefficient ODEs. The study contributes generalized operational relations that extend the applicability of the Shehu transform beyond conventional constant-coefficient equations and provide an analytical foundation for addressing broader classes of variable-coefficient differential equations.

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Keywords: Analytical Solutions; Nonlinear ODEs; Ordinary Differential Equations; Shehu Transform; Variable Coefficients

Introduction

Over the years, integral transforms have played a significant role in the analytical solution of differential equations, particularly ordinary differential equations (ODEs). Classical integral transforms such as the Laplace and Fourier transforms (Yisa & Baruwa, 2022), introduced in the eighteenth and nineteenth centuries respectively, have been extensively employed in solving a wide range of mathematical and engineering problems (Yisa & Baruwa, 2022). Their applicability spans linear systems, boundary value problems, and initial value problems, making them indispensable tools in applied mathematics. In recent years, the complexity of mathematical models has necessitated the development of new integral transforms capable of addressing emerging challenges in mathematical analysis. Consequently, several modern transforms have been introduced, including the Elzaki transform (Elzaki & Elzaki, 2011), Sumudu transform (Aggarwal & Gupta, 2019), Aboodh transform (Mohand *et al*, 2016), Mohand transform (Aggarwal, *et al*, 2019), the widely recognized Shehu transform (Maitama & Zhao, 2019), and more recently, the HK transform (Hilmi & Jwamer, 2024). These transforms have demonstrated remarkable effectiveness in constructing analytical and semi-analytical solutions to various classes of differential equations. Notably, integral transforms have proven effective in solving problems involving ordinary and partial differential equations, such problems include; linear and nonlinear problems (Amnah, 2023), integer and fractional order problems (Yisa *et al, 2025)*, as well as equations with constant and variable coefficients (Mohand, 2016). Their versatility has made them valuable mathematical tools for tackling increasingly complex problems arising in science, engineering, and applied mathematics. Of particular concern is the variable coefficient ordinary differential equations, which often pose greater analytical difficulties than the constant coefficient problems. Several integral transforms, including the Elzaki, Aboodh, and HK, have shown promising capability in handling such equations (Abbas *et al*, 2026; Mohand, 2016; Elzaki & Elzaki, 2011). Furthermore, the combination of integral transforms with semi-analytical techniques has significantly enhanced their applicability. Methods such as the Variational Iteration Method (He, 1999), Homotopy Analysis Method

(Liao, 1992), and Adomian Decomposition Method (Yisa & Tiamiyu, 2024) have been successfully integrated with integral transforms to solve complex problems, including those involving fractional-order derivatives (Yisa, Onah & Aselebe, 2025). This study is primarily focused on the application of the Shehu transform to the solution of variable coefficient ordinary differential equations. Specifically, this study investigates variable coefficients of different powers and establishes solution procedures through illustrative examples.

Definition: Shehu Transform (Maitama & Zhao, 2019)

The Shehu transform of the function $v(x)$ of exponential order is defined over the set of functions.

$$P = \left\{v(x): \exists N, \varepsilon_1, \varepsilon_2 > 0, |v(x)| < N\, exp\left(\frac{|x|}{\varepsilon_i}\right), if\ x\epsilon(-1)^i[0,\infty)\right\},$$

By the integral

$$\mathrm{S}\{v(x)\} = V(s,u) = \int_0^\infty exp\left(\frac{-sx}{u}\right) v(x)dx$$

$$\mathrm{S}\{v(x)\} = \lim_{\alpha\to\infty}\int_0^\alpha exp\left(\frac{-sx}{u}\right) v(x)dx; \qquad s > 0, u > 0$$

The integral above converges provided the limit of the integral exists, and diverges otherwise.

The inverse of Shehu transform is given by

$$\mathrm{S}^{-1}\{V(s,u)\} = v(x), for\ x \geq 0.$$

This can as well be stated as

$$v(x) = \mathrm{S}^{-1}\{V(s,u)\} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}exp\left(\frac{sx}{u}\right)V(s,u)ds,$$

Where s and u are the Shehu transform variables, and α is a real constant and the integral is taken along $s = \alpha$ in the complex plane $s = x + iy$.

Shehu Transform of Derivatives (Maitama & Zhao, 2019)

The Shehu transform of derivatives of a given function $f(x)$ with Shehu transform $F(s,u)$ is defined as;

$$S\{f'(x)\} = \frac{s}{u}F(s,u) - f(0)$$

The nth order derivative is given as

$$S\{f^n(x)\} = \frac{s^n}{u^n}F(s,u) - \sum_{i=0}^{n-1}\left(\frac{s}{u}\right)^{n-(i+1)} f^{(i)}(0),$$

Where $F(s,u)$ is the Shehu transform of the function$f(x)$.

Methodology

Consider the general form of a variable coefficient ordinary differential equation

$$t^m\frac{d^ny}{dt^n} + t^{m-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + t\frac{dy}{dt} + t^0y = f(t) \qquad (1)$$

Where,

$y = y(t)$ Is the unknown function of the independent variable x.

$t^m, t^{m-1}, \ldots, t^0$ Are the variable coefficients, which are the functions of t.

The $f(t)$ is a function called the non-homogenous term. (If $f(t) = 0$, the equation (1) becomes homogenous.

m, n Represents the power of the variables and the order of the differential equation respectively.

Taking the Shehu transform of (1), we have

$$S\left\{t^m\frac{d^ny}{dt^n} + t^{m-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + t\frac{dy}{dt} + t^0y\right\} = S\{f(t)\} \quad (2)$$

$$t^m\frac{d^ny}{dt^n} = -u^m\frac{d^m}{ds^m}[S\{y^n(t)\}] \qquad (3)$$

Substituting (3) into (2), we have;

$$-u^m\frac{d^m}{ds^m}[S\{y^n(t)\}] - u^{m-1}\frac{d^{m-1}}{ds^{m-1}}[S\{y^{n-1}(t)\}] + \cdots + -u\frac{d}{ds}(S\{y'(t)\}) + Y(s,u) = F(s,u) \qquad (4)$$

Applying the Shehu transform of derivatives to $y^{(n)}(t), y^{(n-1)}(t)$ and $y(t)$, also applying the initial conditions and performing the product rule of differentiation on the Shehu transform of derivatives afterwards simplifying , we get and ODE of the form

$$\frac{d}{ds}Y(s,u) + K\,Y(s,u) = F(s,u) \qquad (5)$$

Where K is a variable in terms of s and u. Finding the integrating factor of (5) and multiplying (5) by the integrating factor afterwards simplifying, we obtain a differential equation of the form;

$$\frac{d}{ds}\,P\,Y(s,u) = P\,F(s,u) \qquad (6)$$

P is the integrating factor obtained above. Let $P\,F(s,u) = Q$. Integrating (6), we have

$$PY(s,u) = Q + C$$

$$Y(s,u) = \frac{Q+C}{P}$$

Finding the value of the constant C and applying the inverse Shehu transform, we obtain

$$y(t) = \mathrm{S}^{-1}\left\{\frac{Q+C}{P}\right\}$$

Shehu Transform of Variable Coefficient

This section is concern with the derivation of Shehu transform of variable coefficient ordinary differential equation. The goal is to prove their formation for future potential research using Shehu transform for variable coefficient ordinary differential equations.

Theorem : Let $\mathrm{S}\{y(t)\} = Y(s,u)$. Then

a. $\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}Y(s,u)$

b. $\mathrm{S}\{ty'(t)\} = -s\frac{d}{ds}Y(s,u) - Y(s,u)$

c. $\mathrm{S}\{ty'(t)\} = -\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + y(0)$

d. $\mathrm{S}\{t^2y(t)\} = u^2\frac{d^2}{ds^2}Y(s,u)$

e. $\mathrm{S}\{t^2y'(t)\} = us\frac{d^2}{ds^2}Y(s,u) + u\frac{d}{ds}Y(s,u)$

f. $\mathrm{S}\{t^2y''(t)\} = s^2\frac{d^2}{ds^2}Y(s,u) + 2s\frac{d}{ds}Y(s,u) - u\frac{d}{ds}y(0)$

Proof

a. $\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}Y(s,u)$

Recall that

$$\mathrm{S}\{y(t)\} = \int_0^\infty e^{-\frac{st}{u}}y(t)dt = Y(s,u)$$

Taking the derivative of both sides, we get

$$\frac{d}{ds}Y(s,u) = \frac{d}{ds}\left[\int_0^\infty e^{-\frac{st}{u}}y(t)dt\right]$$

$$\frac{d}{ds}Y(s,u) = \left[\int_0^\infty \frac{d}{ds}\left(e^{-\frac{st}{u}}\right)y(t)dt\right]$$

$$\frac{d}{ds}Y(s,u) = -\frac{1}{u}\int_0^\infty t\left(e^{-\frac{st}{u}}\right)y(t)dt$$

$$\frac{d}{ds}Y(s,u) = -\frac{1}{u}\mathrm{S}\{ty(t)\}$$

$$\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}Y(s,u)$$

b. $\mathrm{S}\{ty'(t)\} = -s\frac{d}{ds}Y(s,u) - Y(s,u)$

From (a), we have

$$\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}Y(s,u)$$

This can also be written as;

$$\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}\mathrm{S}\{y(t)\}$$

Substituting $y(t)$ for $y'(t)$, we have

$$\mathrm{S}\{ty'(t)\} = -u\frac{d}{ds}\mathrm{S}\{y'(t)\}$$

$$\mathrm{S}\{ty'(t)\} = -u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right)$$

Hence,

$$\mathrm{S}\{ty'(t)\} = -s\frac{d}{ds}Y(s,u) - Y(s,u)$$

c. $\mathrm{S}\{ty'^{(t)}\} = -\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + y(0)$

Also from (a),

$$\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}Y(s,u)$$

This can also be written as;

$$\mathrm{S}\{ty(t)\} = -u\frac{d}{ds}\mathrm{S}\{y(t)\}$$

Substituting $y(t)$ for $y''(t)$, we have

$$\mathrm{S}\{ty''(t)\} = -u\frac{d}{ds}\mathrm{S}\{y''(t)\}$$

$$\mathrm{S}\{ty''(t)\} = -u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right)$$

d. $\mathrm{S}\{t^2y(t)\} = u^2\frac{d^2}{ds^2}Y(s,u)$

Recall that

$$\mathrm{S}\{y(t)\} = \int_0^\infty e^{-\frac{st}{u}}y(t)dt = Y(s,u)$$

Taking the first derivative with respect to s on both sides, we get

$$\frac{d}{ds}Y(s,u) = \frac{d}{ds}\left[\int_0^\infty e^{-\frac{st}{u}}y(t)dt\right]$$

$$\frac{d}{ds}Y(s,u) = -\frac{1}{u}\int_0^\infty t\left(e^{-\frac{st}{u}}\right)y(t)dt$$

Taking the second derivative we get

$$\frac{d^2}{ds^2}Y(s,u) = -\frac{1}{u}\frac{d}{ds}\left[\int_0^\infty t\left(e^{-\frac{st}{u}}\right)y(t)dt\right]$$

This can also be written as;

$$\frac{d^2}{ds^2}Y(s,u) = -\frac{1}{u}\left[\int_0^\infty \frac{d}{ds}\left(te^{-\frac{st}{u}}\right)y(t)dt\right]$$

$$\frac{d^2}{ds^2}Y(s,u) = \frac{1}{u^2}\left[\int_0^\infty t^2 e^{-\frac{st}{u}} y(t)dt\right]$$

$$\frac{d^2}{ds^2}Y(s,u) = \frac{1}{u^2}\mathrm{S}\{t^2 y(t)\}$$

Hence,

$$\mathrm{S}\{t^2 y(t)\} = u^2 \frac{d^2}{ds^2}Y(s,u)$$

e. $\mathrm{S}\{t^2 y'(t)\} = us\frac{d^2}{ds^2}Y(s,u) + u\frac{d}{ds}Y(s,u)$

From (d)

$$\mathrm{S}\{t^2 y(t)\} = u^2 \frac{d^2}{ds^2}Y(s,u)$$

This can also be written as;

$$\mathrm{S}\{t^2 y(t)\} = u^2 \frac{d^2}{ds^2}\mathrm{S}\{y(t)\}$$

Substituting $y(t)$ for $y'(t)$, we get

$$\mathrm{S}\{t^2 y'(t)\} = u^2 \frac{d^2}{ds^2}\mathrm{S}\{y'(t)\}$$

$$\mathrm{S}\{t^2 y'(t)\} = u^2 \frac{d^2}{ds^2}\left(\frac{s}{u}Y(s,u) - y(0)\right)$$

Applying product rule of differentiation, we get

$$\mathrm{S}\{t^2 y'(t)\} = us\frac{d^2}{ds^2}Y(s,u) + u\frac{d}{ds}Y(s,u)$$

f. $\mathrm{S}\{t^2 y''(t)\} = s^2\frac{d^2}{ds^2}Y(s,u) + 2s\frac{d}{ds}Y(s,u) - u\frac{d}{ds}y(0)$

From (d)

$$\mathrm{S}\{t^2 y(t)\} = u^2 \frac{d^2}{ds^2}\mathrm{S}\{y(t)\}$$

Substituting $y(t)$ for $y''(t)$, we have

$$\mathrm{S}\{t^2 y''(t)\} = u^2 \frac{d^2}{ds^2}\mathrm{S}\{y''(t)\}$$

$$\mathrm{S}\{t^2 y''(t)\} = u^2 \frac{d^2}{ds^2}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right)$$

Applying product rule of differentiation, we get

$$S\{t^2y''(t)\} = s^2\frac{d^2}{ds^2}Y(s,u) + 2s\frac{d}{ds}Y(s,u) - u\frac{d}{ds}y(0)$$

Numerical Examples

Numerical Example 1 (Attaweel & Almassry, 2020)

Solve the differential equation:

$$y'' + ty' - y = 0, y(0) = 0, y'(0) = 1$$

Taking the Shehu transform of both sides, we get

$$S\{y'' + ty' - y\} = S\{0\}$$

Applying the Shehu transform of derivatives and the variable coefficient derivative, we have;

$$\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right) - u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right) - Y(s,u) = 0$$

Applying initial conditions and simplifying, we get

$$\left(\frac{s^2}{u^2}Y(s,u) - 1\right) - u\frac{d}{ds}\left(\frac{s}{u}Y(s,u)\right) - Y(s,u) = 0$$

$$\frac{s^2}{u^2}Y(s,u) - 1 - s\frac{d}{ds}Y(s,u) - Y(s,u) - Y(s,u) = 0$$

$$\frac{s^2}{u^2}Y(s,u) - 1 - s\frac{d}{ds}Y(s,u) - 2Y(s,u) = 0$$

$$\left(\frac{s^2}{u^2} - 2\right)Y(s,u) - s\frac{d}{ds}Y(s,u) = 1$$

$$\frac{d}{ds} - \left(\frac{s^2 - 2u^2}{u^2s}\right)Y(s,u) = -\frac{1}{s}$$

Applying the integrating factor method

I.F= $e^{-\int\left(\frac{s^2-2u^2}{u^2s}\right)ds} = s^2e^{\frac{-s^2}{2u^2}}$

Multiplying through by I.F, and simplifying, we get

$$\frac{d}{ds}\left(s^2e^{\frac{-s^2}{2u^2}}Y(s,u)\right) = -se^{\frac{-s^2}{2u^2}}$$

Integrating both sides, we get

$$s^2 e^{\frac{-s^2}{2u^2}} Y(s,u) = -\int s e^{\frac{-s^2}{2u^2}} ds$$

$$s^2 e^{\frac{-s^2}{2u^2}} Y(s,u) = u^2 e^{\frac{-s^2}{2u^2}} + c$$

$$Y(s,u) = \frac{u^2}{s^2} + \frac{c}{s^2 e^{\frac{-s^2}{2u^2}}}$$

$$at\ c = 0, we\ get$$

$$Y(s,u) = \frac{u^2}{s^2}$$

Applying the inverse Shehu transform, we get

$$\mathrm{S}^{-1}\{Y(s,u)\} = \mathrm{S}^{-1}\left\{\frac{u^2}{s^2}\right\}$$

$$y(t) = t$$

Numerical Example 2 (Attaweel & Almassry, 2020)

Consider the ordinary differential equation:

$$y'' + 2ty' - 4y = 1,$$

With initial conditions $y(0) = 0, y'(0) = 0$.

Taking the Shehu transform of both sides, we get

$$\mathrm{S}\{y'' + 2ty' - 4y\} = \mathrm{S}\{1\}$$

Applying the Shehu transform of derivatives and the variable coefficient derivative, we have;

$$\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right) + 2\left[-u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right)\right] - 4Y(s,u) = \frac{u}{s}$$

Applying initial conditions, we get

$$\left(\frac{s^2}{u^2}Y(s,u)\right) + 2\left[-u\frac{d}{ds}\left(\frac{s}{u}Y(s,u)\right)\right] - 4Y(s,u) = \frac{u}{s}$$

$$\frac{s^2}{u^2}Y(s,u) - 2s\frac{d}{ds}Y(s,u) - 2Y(s,u) - 4Y(s,u) = \frac{u}{s}$$

$$\frac{s^2}{u^2}Y(s,u) - 2s\frac{d}{ds}Y(s,u) - 6Y(s,u) = \frac{u}{s}$$

$$\left(\frac{s^2}{u^2} - 6\right)Y(s,u) - 2s\frac{d}{ds}Y(s,u) = \frac{u}{s}$$

$$\frac{d}{ds} + \left(\frac{6u^2 - s^2}{2u^2 s}\right)Y(s,u) = -\frac{u}{2s^2}$$

Applying integrating factor to the differential equation, we get

$$I.F = e^{\int\left(\frac{6u^2 - s^2}{2u^2 s}\right)ds} = s^3 e^{\frac{-s^2}{4u^2}}$$

Multiplying through by I.F, and simplifying, we get

$$\frac{d}{ds}\left(s^3 e^{\frac{-s^2}{4u^2}}Y(s,u)\right) = -\frac{us}{2}e^{\frac{-s^2}{4u^2}}$$

Integrating both sides, we get

$$s^3 e^{\frac{-s^2}{4u^2}}Y(s,u) = -\frac{u}{2}\int se^{\frac{-s^2}{4u^2}}ds$$

Integrating the right hand side, we get

$$s^3 e^{\frac{-s^2}{4u^2}}Y(s,u) = u^3 e^{\frac{-s^2}{4u^2}} + c$$

$$Y(s,u) = \frac{u^3}{s^3} \; at \; \; c = 0$$

Applying the inverse Shehu transform, we get

$$\mathrm{S}^{-1}\{Y(s,u)\} = \mathrm{S}^{-1}\left\{\frac{u^3}{s^3}\right\}$$

$$y(t) = \frac{1}{2}t^2$$

Numerical Example 3 (Attaweel & Almassry, 2020)

Solve the differential equation:

$$ty'' - y' = t^2,$$

With initial conditions $y(0) = 0, y'(0) = 0$

Taking the Shehu transform of both sides, we get

$$\mathrm{S}\{ty'' - y'\} = \mathrm{S}\{t^2\}$$

$$\left[-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u)-\frac{s}{u}y(0)-y'(0)\right)\right]-\left(\frac{s}{u}Y(s,u)-y(0)\right)=2\frac{u^3}{s^3}$$

Applying the initial conditions we have

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u)\right)-\left(\frac{s}{u}Y(s,u)\right)=2\frac{u^3}{s^3}$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u)-\frac{2s}{u}Y(s,u)-\frac{s}{u}Y(s,u)=2\frac{u^2}{s^2}$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u)-\frac{3s}{s}Y(s,u)=2\frac{u^3}{s^3}$$

$$\frac{d}{ds}Y(s,u)+\frac{3}{s}Y(s,u)=-2\frac{u^4}{s^5}$$

Applying the use of integrating factor, we get

$$I.F=e^{\int\frac{3}{s}ds}$$

$$I.F=s^3$$

Multiplying through by the integrating factor and simplifying, we get

$$\frac{d}{ds}\left(s^3Y(s,u)\right)=-2\frac{u^4}{s^2}$$

Integrating both sides, we get

$$s^3Y(s,u)=-2u^4\int\frac{1}{s^2}\,ds$$

$$s^3Y(s,u)=2\frac{U^4}{s}+C$$

At $C=0$, we have

$$Y(s,u)=2\frac{u^4}{s^4}$$

Applying the inverse Shehu transform, we have the solution as

$$\mathrm{S}^{-1}\{Y(s,u)\}=\mathrm{S}^{-1}\left\{2\frac{u^4}{s^2}\right\}$$

$$y(t)=\frac{t^3}{3}$$

Numerical Example 4 (Attaweel & Almassry, 2020)

Solve the differential equation:

$$ty'' + y' + 4ty = 0,$$

With initial conditions $y(0) = 3, y'(0) = 0$.

Taking the Shehu transform of both sides, we have

$$S\{ ty'' + y' + 4ty\} = S\{0\}$$

$$\left[-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right)\right] + \left(\frac{s}{u}Y(s,u) - y(0)\right) + 4\left[-u\frac{d}{ds}\left(Y(s,u)\right)\right] = 0$$

Applying initial conditions and simplifying, we get

$$\left[-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - 3\frac{s}{u}\right)\right] + \left(\frac{s}{u}Y(s,u) - 3\right) + 4\left[-u\frac{d}{ds}\left(Y(s,u)\right)\right] = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + 3 + \frac{s}{u}Y(s,u) - 3 - 4u\frac{d}{ds}Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + \frac{s}{u}Y(s,u) - 4u\frac{d}{ds}Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{s}{u}Y(s,u) - 4u\frac{d}{ds}Y(s,u) = 0$$

$$-s^2\frac{d}{ds}Y(s,u) - sY(s,u) - 4u^2\frac{d}{ds}Y(s,u) = 0$$

$$(s^2 + 4u^2)\frac{d}{ds}Y(s,u) - sY(s,u) = 0$$

$$\frac{d}{ds}Y(s,u) - \frac{s}{s^2 + 4u^2}Y(s,u) = 0$$

Applying the use of integrating factor method, we have

$$I.F = e^{\int \frac{s}{s^2+4u^2} ds}$$

On integrating and afterwards simplifying,

$$I.F = (s^2 + 4u^2)^{\frac{1}{2}}$$

Multiplying through by the integrating factor and simplifying, we get

$$\frac{d}{ds}(s^2 + 4u^2)^{\frac{1}{2}} Y(s,u) = 0$$

Taking the integral of both sides, we obtain

$$(s^2 + 4u^2)^{\frac{1}{2}} Y(s,u) = C$$

$$Y(s,u) = \frac{C}{(s^2 + 4u^2)^{\frac{1}{2}}}$$

To determine the value of C

$$\lim_{s\to\infty} sY(s,u) = uy(0)$$

$$Y(s,u) = \frac{C}{(s^2 + 4u^2)^{\frac{1}{2}}}$$

Multiplying through by s, we have

$$sY(s,u) = \frac{Cs}{(s^2 + 4u^2)^{\frac{1}{2}}}$$

Simplifying the limit, we ger

$$sY(s,u) = \frac{Cs}{s} = c$$

Hence

$$sY(s,u) = c$$

$$c - uy(0)$$

$$c = 3u$$

Therefore,

$$Y(s,u) = \frac{3u}{(s^2 + 4u^2)^{\frac{1}{2}}}$$

Applying the inverse Shehu transform, we get

$$\mathrm{S}^{-1}\{Y(s,u)\} = 3\mathrm{S}^{-1}\left\{\frac{u}{(s^2 + 4u^2)^{\frac{1}{2}}}\right\}$$

$$y(t) = 3J_0(2t)$$

Numerical Example 5 (Abbas *et al*, 2026)

Consider the higher order differential equations:

$$ty'' + (1 - 2t)y' - 2y = 0, y(0) = 1, y'(0) = 2$$

$$ty'' + y' - 2ty' - 2y = 0, y(0) = 1, y'(0) = 2$$

Taking the Shehu transform of both sides, we get

$$\mathrm{S}\{ty'' + y' - 2ty' - 2y\} = \mathrm{S}\{0\}$$

Applying the Shehu transform of derivatives, we obtain

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right) + \frac{s}{u}Y(s,u) - y(0)$$

$$- 2\left[-u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right)\right] - 2Y(s,u) = 0$$

Applying the initial conditions, we have

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u} - 2\right) + \frac{s}{u}Y(s,u) - 1 - 2\left[-u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - 1\right)\right] - 2Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + 1 + \frac{s}{u}Y(s,u) - 1 - 2\left(-s\frac{d}{ds}Y(s,u) - Y(s,u)\right)$$

$$- 2Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + \frac{s}{u}Y(s,u) + 2s\frac{d}{ds}Y(s,u) + 2Y(s,u) - 2Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) + \frac{s}{u}Y(s,u) + 2s\frac{d}{ds}Y(s,u) = 0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{s}{u}Y(s,u) + 2s\frac{d}{ds}Y(s,u) = 0$$

$$-s^2\frac{d}{ds}Y(s,u) - sY(s,u) + 2us\frac{d}{ds}Y(s,u) = 0$$

$$(2us - s^2)\frac{d}{ds}Y(s,u) - sY(s,u) = 0$$

$$\frac{d}{ds}Y(s,u) + \frac{1}{s - 2u}Y(s,u) = 0$$

Applying the use of integrating factor,

$$I.F = e^{\int \frac{1}{s-2u} ds}$$

$$I.F = s - 2u$$

Multiplying through by $I.F$ and simplifying, we get

$$\frac{d}{ds}(s - 2u)Y(s,u) = 0$$

Taking the integral of both sides, we have

$$(s - 2u)Y(s,u) = C$$

$$Y(s,u) = \frac{C}{s - 2u}$$

From

$$\lim_{s\to\infty} sY(s,u) = uy(0), C = u$$

Hence,

$$Y(s,u) = \frac{u}{s - 2u}$$

Applying the inverse Shehu transform we get

$$\mathrm{S}^{-1}\{Y(s,u)\} = \mathrm{S}^{-1}\left\{\frac{u}{s - 2u}\right\}$$

$$y(t) = e^{2t}$$

Nuimerical Example 6 (Abbas *et al*, 2026)

Consider the higher order differential equation boundary value problem:

$$ty'' - ty' - y = 0, y(0) = 0, y'(0) = 2$$

Taking the Shehu transform of both sides, we have

$$\mathrm{S}\{ty'' - ty' - y\} = \mathrm{S}\{0\}$$

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right) + u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right) - Y(s,u) = 0$$

Applying the initial conditions, we have

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u)-2\right)+u\frac{d}{ds}\left(\frac{s}{u}Y(s,u)\right)-Y(s,u)=0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u)-\frac{2s}{u}Y(s,u)+s\frac{d}{ds}Y(s,u)+Y(s,u)-Y(s,u)=0$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u)-\frac{2s}{u}Y(s,u)+s\frac{d}{ds}Y(s,u)=0$$

$$-s^2\frac{d}{ds}Y(s,u)-2sY(s,u)+us\frac{d}{ds}Y(s,u)=0$$

$$(us-s^2)\frac{d}{ds}Y(s,u)-2sY(s,u)=0$$

$$\frac{d}{ds}Y(s,u)+\frac{2}{s-u}Y(s,u)=0$$

The integrating factor $I.F=(s-u)^2$

Multiplying through by the integrating factor and simplifying, we get

$$\frac{d}{ds}(s-u)^2Y(s,u)=0$$

Taking the integral of both sides, we get

$$(s-u)^2Y(s,u)=C$$

$$Y(s,u)=\frac{C}{(s-u)^2}$$

From

$$\lim_{s\to\infty}s^2Y(s,u)=u^2y'(0)\,,C=2u^2$$

Hence,

$$Y(s,u)=\frac{2u^2}{(s-u)^2}$$

Applying the inverse Shehu transform, we obtain

$$\mathrm{S}^{-1}\{Y(s,u)\}=2\mathrm{S}^{-1}\left\{\frac{u^2}{(s-u)^2}\right\}$$

$$y(t)=2te^t$$

Numerical Example 7 (Abbas *et al*, 2026)

Consider the second order ordinary differential equation

$$ty'' - y' = t^2, y(0) = 0, y'(0) = 0$$

Taking the Shehu transform of both sides, we get

$$S\{ty'' - y'\} = S\{t^2\}$$

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0)\right) - \left(\frac{s}{u}Y(s,u) - y(0)\right) = 2\frac{u^3}{s^3}$$

Applying the initial conditions, we get

$$-u\frac{d}{ds}\left(\frac{s^2}{u^2}Y(s,u)\right) - \left(\frac{s}{u}Y(s,u)\right) = 2\frac{u^3}{s^3}$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{2s}{u}Y(s,u) - \frac{s}{u}Y(s,u) = 2\frac{u^3}{s^3}$$

$$-\frac{s^2}{u}\frac{d}{ds}Y(s,u) - \frac{3s}{u}Y(s,u) = 2\frac{u^3}{s^3}$$

$$s^2\frac{d}{ds}Y(s,u) + 3sY(s,u) = -2\frac{u^4}{s^3}$$

$$\frac{d}{ds}Y(s,u) + \frac{3}{s}Y(s,u) = -2\frac{u^4}{s^5}$$

The integrating factor $I.F = s^3$

Multiplying through by the integrating factor and simplifying, we get

$$\frac{d}{ds}\, s^3Y(s,u) = -2\frac{u^4}{s^2}$$

Integrating both sides, we get

$$s^3Y(s,u) = 2\frac{u^4}{s} + C$$

At $C = 0$

$$Y(s,u) = 2\frac{u^4}{s^4}$$

Applying the inverse Shehu transform, we get

$$S^{-1}\{Y(s,u)\} = 2S^{-1}\left\{\frac{u^4}{s^4}\right\}$$

$$y(t) = \frac{t^3}{3}$$

Conclusion

This study has clearly demonstrated the effectiveness of the Shehu transform in solving variable coefficient ordinary differential equations through the establishment and proof of Shehu transform properties for certain derivatives involving variable coefficients. To validate the proposed proofs, seven numerical examples sourced from existing literature were considered and successfully solved. The analytical solutions obtained were found to be consistent with those reported in the literature, particularly in **(**Abbas *et al*, 2026) and **(**Attaweel & Almassry, 2020), thereby confirming the accuracy and reliability of the proposed approach. Future research may extend the application of the Shehu transform to other classes of differential equations, including partial differential equations, fractional order differential equations, and systems of nonlinear differential equations with variable coefficients.

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