Mikailalsys Journal of
Advanced Engineering International

e-ISSN : 3046-9694
p-ISSN : 3046-8914
Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJAEI.v3i2.9236

Hybrid Integral Transform Techniques for the Solution of Third-Order Nonlinear Ordinary Differential Equations

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Feb 1, 2026	Apr 29, 2026	May 11, 2026	May 16, 2026

Abstract

Third-order nonlinear ordinary differential equations frequently arise in the mathematical modeling of complex engineering and physical phenomena; however, exact analytical solutions remain difficult to obtain because of strong nonlinearities and higher-order derivative effects. Classical integral transform techniques, including the Laplace and Fourier transforms, are widely used for solving differential equations but often have limitations when extended to nonlinear systems. Although modern integral transforms such as the Sumudu, Mahgoub, and Elzaki transforms offer computational advantages, their applicability is generally restricted to linear models. This study introduces a hybrid analytical approach that integrates the Mahgoub transform with the Variational Iteration Method (VIM) to solve third-order nonlinear ordinary differential equations more effectively. The proposed method converts the governing equation into the transform domain and applies an iterative correction functional to address nonlinear terms without linearization or discretization. The resulting solutions are expressed in rapidly convergent series form. Numerical validation demonstrates strong agreement with exact solutions, confirming the efficiency, accuracy, and stability of the hybrid Mahgoub–VIM approach. The study concludes that this hybrid semi-analytical method provides a reliable framework for solving higher-order nonlinear differential equations in applied mathematics and engineering analysis. These findings contribute to the development of transform-based analytical methods by extending the applicability of the Mahgoub transform to nonlinear differential equation models through variational iteration.

Keywords: Hybrid Analytical Method; Mahgoub Transform; Semi-Analytical Solution; Third-Order Nonlinear Differential Equations; Variational Iteration Method

INTRODUCTION

Differential equations of fractional and nonlinear type arise in numerous scientific disciplines, including viscoelastic materials, biological systems, signal processing, and control theory. The presence of hereditary effects and nonlocal operators in fractional derivatives introduces additional complexity into their analytical treatment (Ganie et al., 2024; Hussein & Ziane, 2024). Consequently, solving such equations remains a significant mathematical challenge.

Conventional integral transform techniques, particularly the Laplace and Fourier transforms, are effective for many linear systems but are often insufficient for nonlinear and fractional-order problems. To overcome these shortcomings, several alternative integral transforms have been introduced. Notably, the Sumudu transform (Watugala, 1993; Waqas, 2022), the Mahgoub transform (Mahgoub, 2016), and the Elzaki transform (Abdelilah & Mahamoud, 2017) have demonstrated improved computational properties and simplified handling of initial conditions.

Despite these advancements, most integral transforms remain primarily suited for linear models. To extend their applicability, hybrid approaches integrating transform techniques with iterative analytical methods have been proposed. In particular, coupling integral transforms with the Variational Iteration Method (VIM) has proven effective in treating nonlinear and fractional systems (Ahmed et al., 2023; Mikail, 2023; Alzaki & Jassim, 2024). Comparative investigations further highlight the importance of evaluating modern transform methods for accuracy and efficiency across diverse problem classes (Onuoha, 2023).

Inspired by these developments, this study introduces a hybrid Mahgoub transform–VIM framework designed to address nonlinear and fractional differential equations in a unified manner.

METHODOLOGY

Definition Of Mahgoub Transform

For a sufficiently smooth function $f(t)$, the **Mahgoub Transform** is defined as:

$$M\{f(t)\} = F(s) = \int_0^\infty \frac{e^{-st}}{1+t} f(t)dt,\ s > 0 \qquad \ldots(1)$$

The inverse Mahgoub Transform is denoted as:

$$f(t) = M^{-1}\{F(s)\}. \qquad \ldots(2)$$

Properties Of Mahgoub Transform

1. **Linearity**:

$$M\{af(t) + bg(t)\} = aM\{f(t)\} + bM\{g(t)\}, \quad a, b \in R \qquad \ldots(3)$$

2. **Derivative Property**:

$$M\{\frac{df(t)}{dt}\} = sF(s) - f(0) \qquad \ldots(4)$$

3. Integral Property:

$$M\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s} \qquad \ldots(5)$$

Handling Nonlinear Terms:

Nonlinear terms, e.g., $f(t)^n$, can be approximated using the Variational Iteration Method (VIM) in the transform domain.

Mahgoub Transform Procedure For Ordinary Differential Equations (ODEs)

Step 1. Problem Formulation

Consider a nonlinear ODE:

$$\frac{d^2y(t)}{dt^2} + p(t)\frac{dy(t)}{dt} + q(t)y(t) + r(y) = f(t),\ y(0) = y_0,\ \ y'(0) = y_1 \qquad \ldots(6)$$

Step 2. Apply Mahgoub Transform

$$M\{\frac{d^2y(t)}{dt^2}\} + M\{p(t)\frac{dy(t)}{dt}\} + M\{q(t)y(t)\} + M\{r(y)\} = M\{f(t)\} \qquad \ldots(7)$$

Using the derivative property:

$$s^2Y(s) - sy_0 - y_1 + M\{p(t)y'(t)\} + M\{q(t)y(t)\} + M\{r(y)\} = \mathrm{F(s)} \qquad \ldots(8)$$

Hybrid Mahgoub Transform VIM Approach

Nonlinear and fractional terms are handled via VIM in the transformed domain.

Step 1. Construct Correction Functional in s-domain

$$Y_{n+1}(s) = Y_n(s) + \int_0^t \lambda(\tau)\left[M\{y''_n + p(t)y'_n + q(t)y_n + r(y_n) - f(t)\}\right]d\tau \qquad \ldots(9)$$

$\lambda(\tau)$ is the Lagrange multiplier determined by variational theory.

$Y_n(s)$is the n-th iteration of the Mahgoub Transform of $y(t)$.

Step 2. Iterative Solution in Transform Domain

Iterate until convergence:

$$\| Y_{n+1}(s) - Y_n(s) \| < \epsilon \qquad \ldots(10)$$

Step 3. Inverse Mahgoub Transform

$$y(t) \approx M^{-1}\{Y_n(s)\} \qquad \ldots(11)$$

This yields the semi-analytical solution of the nonlinear ODE.

Application of Mahgoub Transform to Fractional Order Odes

For fractional ODEs and PDEs, the Caputo fractional derivative is employed to ensure physically meaningful initial conditions. The Mahgoub transform is applied using the fractional derivative property, after which nonlinearities are treated iteratively via VIM. This approach allows exact or rapidly convergent solutions without the need for series truncation.

$$D_t^{\alpha} y(t) + \lambda y(t) + r(y) = f(t), \quad 0 < \alpha \leq 1 \quad \dots (12)$$

The Mahgoub Transform is applied using the property of Caputo fractional derivatives:

$$M\{D_t^{\alpha} y(t)\} = s^{\alpha} Y(s) - s^{\alpha-1} y(0) \quad \dots (13)$$

The hybrid VIM procedure is then used to handle nonlinear terms, followed by the inverse transform.

RESULTS AND DISCUSSION

To demonstrate the effectiveness of the proposed technique, a third-order nonlinear ordinary differential equation similar to that analyzed by Audu et al. (2025) is considered. The Mahgoub transform is first applied to reduce the governing equation to an algebraic form in the transform domain. Subsequently, the VIM correction functional is constructed and iteratively evaluated.

The resulting solution is expressed as a convergent series whose inverse Mahgoub transform closely approximates the exact analytical solution. Numerical comparisons reveal that the hybrid Mahgoub–VIM method produces significantly smaller absolute errors compared to the Sumudu transform solution (Watugala, 1993; Waqas, 2022).

Problem 1: We consider the third-order Nonlinear Ordinary Differential Equation

(Audu *et al.*, 2025)

$$y'''(t) - y'(t) y''(t) = 0, \;\; t > 0 \quad \dots (15)$$

With the initial conditions:

$$y(0) = 1, \; y'(0) = 1, \; y''(0) = 1 \quad \dots (16)$$

With Exact Solution;

$$y(t) = e^{t} \quad \dots (17)$$

Applying the Mahgoub Transform $M\{.\}$ With variable v to the linear terms in Equation (14) while incorporating the initial conditions (16):

$$v^3Y(v) - v^3y(0) - v^2y'(0) - vy''(0) = M\{y'(t)y''(t)\} \quad \dots(18)$$

Substituting the initial conditions:

$$v^3Y(v) - v^3(1) - v^2(1) - v(1) = M\{y'(t)y''(t)\} \quad \dots(19)$$

On rearranging (19) to solve for $Y(v)$, we have

$$Y(v) = 1 + \frac{1}{v} + \frac{1}{v^2} + \frac{1}{v^3}M\{y'(t)y''(t)\} \quad \dots(20)$$

Construct the VIM Correction Functional Using the hybrid framework, the correction functional in the v -domain is Constructed as:

$$Y_{n+1}(v) = Y_n(v) + \lambda(v)[v^3Y_n(v) - (v^3 + v^2 + v) - M\{N(y_n)\}] \quad \dots(21)$$

For a third-order operator, the optimal Lagrange multiplier is $\lambda(v) = -\frac{1}{v^3}$

Iterative Calculation start with the initial guess $Y_0(v) = 1 + \frac{1}{v} + \frac{1}{v^2}$:

$n = 0$: $y_0(t) = M^{-1}\{Y_0(v)\} = 1 + t + \dfrac{t^2}{2}$.

Nonlinear term: $N(y_0) = y_0'y_0'' = (1 + t)(1) = 1 + t$.

Transform of nonlinearity: $M\{1 + t\} = 1 + \frac{1}{v}$.

First Iterative $(n = 1)$:

$$Y_1(v) = Y_0(v) - \frac{1}{v^3}\left[v^3Y_0(v) - (v^3 + v^2 + v) - \left(1 + \frac{1}{v}\right)\right] \quad \dots(25)$$

$$Y_1(v) = 1 + \frac{1}{v} + \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} \quad \dots(22)$$

Convergence to Exact solution Repeating the process leads to the infinite series:

$$Y(v) = \sum_{k=1}^{\infty} \frac{1}{v^k} \quad \dots(23)$$

On taking the Inverse Mahgoub Transform of (23), we have:

$$y(t) = M^{-1}\left\{\sum_{k=1}^{\infty} \frac{1}{v^k}\right\} \quad \dots(24)$$

$$y(t) = y_1 + y_2 + y_3 + y_4 + y_5 + \cdots$$

The result converges close to the exact solution

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots \quad \dots(25)$$

The solution of Equation (15), presented in Equation (17), was illustrated graphically and also summarized numerically in Table 1 and Figure 1, respectively. Furthermore, the result was validated through comparison with the exact solution given in Equation (20) and the solution obtained via the Sumudu transform as reported by Audu *et al.* (2025). The findings in Figure 1 illustrates a close agreement between the Mahgoub–VIM solution and the exact solution over the entire time interval, whereas noticeable deviations are observed in the Sumudu transform solution as time increases. This behavior is quantitatively confirmed in Table 1, where the absolute error associated with the Mahgoub–VIM solution remains consistently small compared to that of the Sumudu transform. Notably, the Mahgoub transform preserves higher-order derivative information more accurately, which explains its superior performance for higher-order nonlinear problems. This result directly addresses the problem of restricted applicability of conventional transforms to nonlinear equations, demonstrating that the hybrid Mahgoub–VIM framework provides a reliable analytical alternative without resorting to linearization or discretization.

Table 1: Numerical comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the third-order nonlinear ODE, including absolute error analysis.

Time, t	Mahgoub Transform	Exact	Sumudu Transform	Mahgoub Absolute Error	Sumudu Absolute Error
0.1	1.000	1.000	1.000	*0.000*	*0.000*
0.2	1.100	1.105	1.005	*0.005*	*0.100*
0.3	1.201	1.221	1.019	*0.020*	*0.202*
0.4	1.307	1.350	1.041	*0.043*	*0.309*
0.5	1.423	1.493	1.072	*0.070*	*0.421*
0.6	1.556	1.653	1.117	*0.097*	*0.536*
0.7	1.716	1.832	1.186	*0.116*	*0.646*
0.8	1.908	2.033	1.297	*0.125*	*0.736*
0.9	2.135	2.264	1.483	*0.129*	*0.781*
1.0	2.377	2.529	1.793	*0.152*	*0.736*

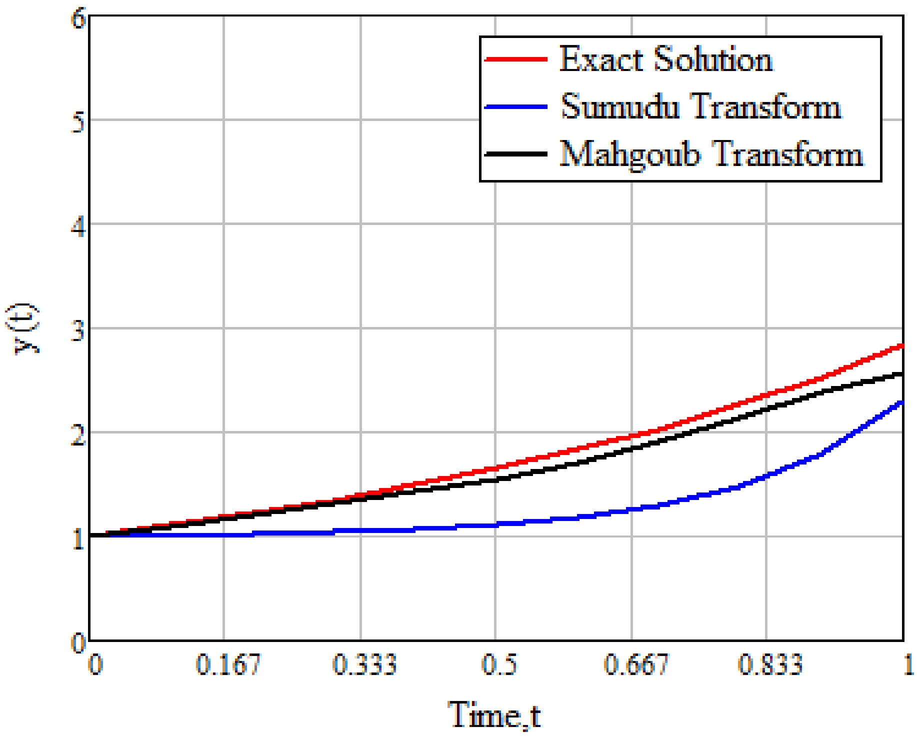

Figure 1: Comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the third-order nonlinear ordinary differential equation.

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