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Elzaki Transform Approach for Solving Linear Proportional Delay Differential Equations

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Abstract

Proportional delay differential equations (PDDEs) arise naturally in physics, economics, population dynamics, epidemiology, and viscoelasticity due to delays that scale proportionally with the independent variable, yet they remain analytically challenging because the delayed argument disrupts the classical structure of ordinary differential equations. This paper presents a human-centered, simplified, and computationally friendly method for solving linear PDDEs using a hybrid approach that combines the Elzaki Transform with established decomposition techniques. Within this framework, the Elzaki Transform is used to convert the original PDDE into an associated functional equation, which is then handled through a systematic decomposition process that avoids excessive algebraic complexity. Two illustrative examples are worked out in detail to demonstrate the step-by-step implementation of the method, showing that the proposed approach yields solutions efficiently while preserving mathematical rigor and interpretability. The analysis highlights that the hybrid Elzaki–decomposition technique offers conceptual transparency, reduces computational overhead, and provides a practical alternative to classical transform-based and purely numerical schemes for linear PDDEs. The study concludes that this approach can serve as an accessible yet robust tool for applied researchers who routinely encounter PDDEs, and it opens pathways for future extensions to more general classes of delay and functional differential equations.

Keywords: Elzaki Transform; Proportional Delay Differential Equations; Linear PDDEs; Semi-Analytical Methods; Decomposition Techniques.

Introduction

Delay differential equations (DDEs) are mathematical models in which the rate of change of a quantity depends not only on its current state but also on its values at past times. When such delays grow proportionally with the independent variable, the resulting models are known as proportional delay differential equations (PDDEs) [1, 2]. These equations take forms such as

$$\frac{dy}{dt} = f\left(t, y(t), y(\alpha t)\right), 0 < \alpha < 1, \qquad \ldots(1)$$

and are also called pantograph equations, a term introduced by Ockendon and Tayler to model electric locomotives [3].

In recent years, PDDEs have become increasingly important because they capture self-similar memory effects in systems such as epidemic spread [4], viscoelastic wave propagation [5], thermal processes [6], and population dynamics [7]. Their nonlocal nature makes them harder to solve using standard ODE tools, prompting the development of new transform-based and iterative tools.

Transform methods have been widely used to solve various classes of DDEs. The Laplace Transform and Sumudu Transform have been applied with some success [8, 9], but their application to proportional delays is limited because of difficulties in handling scaled arguments. The Elzaki Transform, introduced as an alternative integral transform with favorable convergence properties [10], has shown strong potential for solving differential equations involving nonlocal and delayed structures [11–13].

The main goal of this work is to construct an accessible and efficient hybrid technique that uses the Elzaki Transform to simplify the differential structure and then employs series-based decomposition for the proportional delay term. This combination keeps the method both analytic and computationally light. This human-centered perspective explaining not only what is done but also why is intended to make the method more intuitive for researchers.

Preliminaries

Proportional Delay Differential Equations.

A linear PDDE of order n typically has the structure

$$y^{(n)}(t) = ay(t) + by(\alpha t) + g(t),\ 0 < \alpha < 1, \qquad \ldots(2)$$

Subject to $y^{(i)}(0) = \delta_i, i = 0,1,2,\ldots$

Where:

- $y(t)$ is the unknown function,
- $y(\alpha t)$ introduces the proportional delay,
- a and b are constants,
- $g(t)$ is a known forcing term.

Such equations can exhibit unusual dynamic features such as oscillations even when the associated ODE does not [14].

Elzaki Transform

Definition 2.1. [10, 11] Elzaki Transform denoted by the operator $E[.]$, is defined by the integral equation

$$E[y(t)] = r\int_0^\infty y(t)e^{-\frac{t}{r}}dt,\ k_1 \le r \le k_2,\ t \ge 0 \qquad \ldots(3)$$

Some key ET properties used in this work include:

Common Properties:

$1.E[1] = r^2$

$2.E[t^n] = n!r^{n+2}$

$3.E^{-1}[r^{n+2}] = \dfrac{t^n}{n!}, n \in N$

The last property is called inverse Elzaki operator.

Derivative Properties:

Let $E[y(t)]$ be the Elzaki transform of $y(t)$, then

$$E\left[y^{(n)}(t)\right]=\frac{E[y(t)]}{r^{n}}-\sum_{m=0}^{n-1} r^{2-n+m} y^{(m)}(0),\ n=1,2,3,\ldots \qquad \ldots(4)$$

Linearity:

1. $E[ay_1(t)+by_2(t)]=aE[y_1(t)]+bE[y_2(t)]$

2. $E^{-1}[ay_1(t)+by_2(t)]=aE^{-1}[y_1(t)]+bE^{-1}[y_2(t)]$

These properties make the Elzaki Transform particularly suitable for PDDEs.

Methodology: Elzaki Transform Scheme.

To solve the linear PDDE (2):

We apply the Elzaki Transform to both sides;

$$E\left[y^{(n)}(t)\right]=E[g(t)+ay(t)+by(\alpha t)]$$

Using transform properties, we obtain:

$$E\left[y^{(n)}(t)\right]=E[g(t)]+aE[y(t)]+bE[y(\alpha t)]$$
$$\frac{E[y(t)]}{r^{n}}-\sum_{m=0}^{n-1} s^{2-n+m} y^{(m)}(0)=E[g(t)]+aE[y(t)]+bE[y(\alpha t)] \qquad \ldots(5)$$
$$\frac{E[y(t)]}{r^{n}}-C=E[g(t)]+aE[y(t)]+bE[y(\alpha t)], C=\sum s^{2-n+m} y^{(m)}(0)$$

Rearranging (5), we obtain:

$$E[y(t)]=r^{n}C+r^{n}E[g(t)]+ar^{n}E[y(t)]+br^{n}E[y(\alpha t)] \qquad \ldots(6)$$

Applying inverse Elzaki transform on both sides of (5), we obtain:

$$y(t)=E^{-1}\left[r^{n}C+r^{n}E[g(t)]\right]+aE^{-1}\left[r^{n}E[y(t)]\right]+bE^{-1}\left[r^{n}E[y(\alpha t)]\right] \qquad \ldots(7)$$

The presence of $y(\alpha t)$ prevents direct inversion. To handle this, many authors recommend recursive decomposition in transform space [14–16]. We follow the same idea:

We define recurrence relation:

$$\begin{cases} y_0(t) = E^{-1}\left[r^n C + r^n E[g(t)]\right] \\ y_1(t) = aE^{-1}\left[r^n E[y_0(t)]\right] + bE^{-1}\left[r^n E[y_0(\alpha t)]\right] \\ y_{n+1}(t) = bE^{-1}\left[r^n E[y_n(\alpha t)]\right], n \geq 1 \end{cases} \quad \ldots(8)$$

In truncated series form, the approximate analytic solution of (2) is given by,

$$y(t) = y_0(t) + \sum_{n=1}^{\infty} y_n(t) = y_1(t) + y_2(t) + y_3(t) + \ldots \quad \ldots(9)$$

The solution is expressed as the convergent series.

This approach keeps the core structure transform-based but handles the proportional delay through controlled series expansion.

Illustrative Examples

In this section, we provide two PDDEs to show how this technique works.

Example 4.1 Consider the following linear PDDE[Quitaiba W.I., 2014, updated to integer oder]:

$$\begin{cases} y'(t) = y\left(\dfrac{t}{2}\right) \\ y(0) = 0 \end{cases} \quad \ldots(10)$$

The exact solution is $y(t) = \sum_{n=0}^{\infty} \dfrac{\left(\frac{1}{2}\right)^{\frac{1}{2}n(n-1)}}{n!} t^n$.

Solution:

Applying Elzaki transform on both sides of (10) yields:

$$E[y'(t)] = E[y(t)]$$

Equivalently,

$$\frac{E[y(t)]}{r} - ry(0) = E\left[y\left(\frac{t}{2}\right)\right] \quad \ldots(11)$$

On substituting initial value in (11) and re-arranging we obtain:

$$E[y(t)] = r^2 + rE\left[y\left(\frac{t}{2}\right)\right],\ y(0) = 1 \qquad \ldots(12)$$

Applying inverse Elzaki transform on both sides of (12). We obtain:

$$y(t) = E^{-1}\left[r^2 + rE\left[y\left(\frac{t}{2}\right)\right]\right]$$

Equivalently,

$$y(t) = 1 + E^{-1}\left[rE\left[y\left(\frac{t}{2}\right)\right]\right] \qquad \ldots(13)$$

From (13), we obtain the following recurrence relations:

$$\begin{cases} y_0(t) = 1 \\ y_1(t) = E^{-1}\left[rE\left[y_0\left(\frac{t}{2}\right)\right]\right] = t \\ y_2(t) = E^{-1}\left[rE\left[y_1\left(\frac{t}{2}\right)\right]\right] = \frac{\left(\frac{1}{2}\right)t^2}{2!} \\ y_3(t) = E^{-1}\left[rE\left[y_2\left(\frac{t}{2}\right)\right]\right] = \frac{\left(\frac{1}{8}\right)t^3}{3!} \\ \vdots \\ y_k(t) = E^{-1}\left[rE\left[y_{k-1}\left(\frac{t}{2}\right)\right]\right] = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}k(k-1)}}{k!} t^k \end{cases}$$

$$y(t) = y_0(t) + \sum_{n=0}^{\infty} y_n(t)$$

Therefore, $y(t) = 1 + t + \frac{\left(\frac{1}{2}\right)t^2}{2!} + \frac{\left(\frac{1}{8}\right)t^3}{3!} + \ldots + \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}k(k-1)}}{k!} t^k + \ldots = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}n(n-1)}}{n!} t^n$

This matches the structure of the exact solution and converges rapidly.

Example 4.2

Example 4.1 Consider the following linear PDDE[Ibrahim M.D., Okai J.O., Michael Cornelius, 2024]:

$$\begin{cases} y''(t) = 2 - t^2 + \frac{3}{4} y(t) + y\left(\frac{t}{2}\right) \\ y(0) = y'(0) = 0 \end{cases} \qquad ...(14)$$

The exact solution is $y(t) = t^2$.

Solution:

Applying Elzaki Transform on both sides of (14), we obtain:

$$E[y''(t)] = E\left[2 - t^2 + \frac{3}{4} y(t) + y\left(\frac{t}{2}\right)\right]$$

Equivalently,

$$\frac{E[y(t)]}{r^2} - y(0) - ry(0) = 2r^2 - 2r^4 + \frac{3}{4} E[y(t)] + E\left[y\left(\frac{t}{2}\right)\right] \qquad ...(15)$$

Substituting the initial values in (15) and re-arranging, we obtain:

$$E[y(t)] = 2r^4 - 2r^6 + \frac{3}{4} r^2 E[y(t)] + r^2 E\left[y\left(\frac{t}{2}\right)\right] \qquad ...(16)$$

Applying inverse Elzaki transform on both sides of (16), yields:

$$y(t) = E^{-1}\left[2r^4 - 2r^6 + \frac{3}{4} r^2 E[y(t)] + r^2 E\left[y\left(\frac{t}{2}\right)\right]\right]$$

Equivalently,

$$y(t) = t^2 - \frac{t^4}{12} + \frac{3}{4} E^{-1}\left[r^2 E[y(t)]\right] + E^{-1}\left[r^2 E\left[y\left(\frac{t}{2}\right)\right]\right]$$

We now define recurrence relations as:

$$\begin{cases} y_0(t) = t^2 \\ y_1(t) = -\frac{t^4}{12} + \frac{3}{4}E^{-1}\left[r^2 E[y_0(t)]\right] + E^{-1}\left[r^2 E\left[y_0\left(\frac{t}{2}\right)\right]\right] \qquad = 0 \\ y_{n+1}(t) = \frac{3}{4}E^{-1}\left[r^2 E[y_n(t)]\right] + E^{-1}\left[r^2 E\left[y_n\left(\frac{t}{2}\right)\right]\right] = 0, n \geq 1 \end{cases} \quad \ldots(17)$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots..$$

$$y(t) = t^2 + 0 + 0 + 0 + \ldots.$$

Therefore, $y(t) = t^2$, which is exact.

Discussion

The examples demonstrate that while proportional delay equations often defy closed-form solutions, the Elzaki Transform–based strategy systematically reduces them to an infinite but rapidly convergent series. The transform handles differential terms cleanly and explains the influence of each proportional delay recursively.

Compared with classical Laplace-based methods, the Elzaki approach avoids the significant algebraic complications that arise when handling scaled arguments. Several researchers have reported similar advantages for nonlinear or multi-delay systems [11–13, 16–18].

Conclusion

This work presented a simple yet effective method for solving linear proportional delay differential equations using the Elzaki Transform. The method is intuitive, computationally friendly, and well-suited for equations in which delay scales proportionally with time. Illustrative examples confirmed that the technique yields accurate results comparable with existing literature. Future work may explore extending the approach to fractional-order PDDEs or stochastic delay systems.

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