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Comparative Study of Shifted Chebyshev Polynomials on the Solution of Nonlinear Boundary Value Problems

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Abstract

The usefulness of orthogonal polynomials has increasingly been extended to the solution of initial and boundary value problems in recent years. Among these, Chebyshev polynomials—classified into four distinct kinds—are widely employed; however, trial functions in numerical schemes have predominantly relied on polynomials of the second kind, with limited attention to the others. This study applies all four kinds of Chebyshev polynomials as trial functions within the collocation method. Shifted forms of each kind of Chebyshev polynomial were used as trial functions and substituted into the governing differential equations. The resulting equations were then evaluated at selected collocation points within the domain, converting the differential equations into systems of linear equations, which were solved simultaneously using Maple 18.0 software. For each kind of Chebyshev polynomial, approximations of sixth, tenth, and twelfth order were constructed, and the corresponding results were compared with available exact solutions and, where exact solutions were not available, with results from other established numerical methods. Three mathematical problems were considered to validate the effectiveness of the four kinds of Chebyshev polynomials in this framework. Residual equations for each kind of polynomial were obtained at different orders, and the associated constants were also determined for each order, thereby providing a systematic assessment of their performance as trial functions in the collocation technique.

Keywords: Orthogonal Polynomials; Chebyshev Polynomials; Collocation Method; Initial and Boundary Value Problems; Shifted Polynomials; Numerical Approximation

INTRODUCTION

In various fields of science and engineering, nonlinear evolution equations, as well as their analytical and numerical solutions, are fundamentally important but difficult to arrive at its exact solution. Non-linear differential equations are indispensable tools for modelling many physical phenomenons such as chemical reactions, spring–mass systems, and beam bending. These equations are also useful in ecology and economics. The most popular of these types of equations are Riccati, Emden–Fowler, Duffing, Van der Pol, Rayleigh and Yermakov's equations. Most nonlinear differential equations do not have exact solutions, so approximation and numerical techniques has to be applied [1].

There are various methods essential for solving non-linear differential equations; among them are Elzaki integral transform, Adomial decomposition methods and Chebyshev collocation methods.

The Chebyshev polynomials are polynomials with the largest possible leading coefficient, whose absolute value on the interval [-1,1] is bounded above by 1. The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds are defined, respectively, on [- 1, l] according to the following trigonometric Formulae [2]:

$$T_n(x) = \cos n\theta, \tag{1}$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{2}$$

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta} \tag{3}$$

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \tag{4}$$

where $x = \cos\theta,\ 0 \le \theta \le \pi,\ n = 1, 2, 3, \ldots$

The nomenclature of "third- and fourth-kind Chebyshev polynomials" appears to have been first used by Gautschi. Since

$$\sin\theta = (1-x^2)^{1/2},\ \cos\frac{1}{2}\theta = (\frac{1}{2}(1+x))^{1/2},\ \sin\frac{1}{2}\theta = (\frac{1}{2}(1-x))^{1/2} \tag{5}$$

it follows that;

$$T_n(x),\ (1-x^2)^{1/2}U_n(x),\ (1+x)^{1/2}V_n(x),\ (1-x)^{1/2}W_n(x) \tag{6}$$

. Therefore the following minimum properties are deduced,

$$2^{n-1}T_n(x),\ 2^{-n}U_n(x), 2^{-n}V_n(x)\ \mathit{and}\ 2^{-n}W_n(x). \tag{7}$$

We only have space to give a few of the formulae that hold for this polynomial. All four polynomials share the same recurrence relation;

$$p_n = 2xp_{n-1} - p_{n-2},\ p_0 = 1, \tag{8}$$

But with different starting polynomials, namely:

$$p_1 = x,\ 2x,\ 2x-1,\ 2x+1 \tag{9}$$

for first, second, third and fourth kinds.

Hence, it is normally sufficient to establish properties for third-kind polynomials, and then deduce analogous properties for fourth kind (by replacing x by –x).

A key pair of formulae, for the third and fourth polynomials, establishes a strong link with first and second kinds:

$$V_n(x) = u^{-1}T_{2n+1}(u) \tag{10}$$

$$W_n(x) = U_{2n}(u) \tag{11}$$

$$\mathit{where}\ u = [\frac{1}{2}(1+x)]^{1/2} = \cos\frac{1}{2}\theta$$
$$\mathit{for}\ x = \cos\theta$$

$$T_n(x) = T_{2n}(u) \tag{12}$$

$$U_n(x) = \frac{1}{2}u^{-}U_{2n+1}(u) \tag{13}$$

It is clear from these formulae that $T_n, U_n, V_n, W_n,$ all together formed the first- and second-kind polynomials in the new variable u (weighted by U^{-1} in two cases).

Analysis of the method:

Suppose we have a differential equation

$$L(U(x)) = \eta \tag{14}$$

In the domain η

$$B\mu(u) = \eta \ on \ \ \partial\eta \tag{15}$$

Where L(U) denotes a general differential operator in solving derivatives of dependent variables

The following:

1. Shifted polynomial for different kinds at different orders are assumed to be the trial function.

$$y = \sum_{i=0}^{n} a_i T_i^* \tag{16}$$

2. Substitute the assumed trial function into the differential equation given to generate the residual.

3. The domain within η has chosen as collocation points.

4. Imposing the boundary condition on the trial function was used.

5. Evaluating the residual at each arbitrary collocation points to give set of algebraic equations.

6. Solve the system of algebraic equation to obtain the constants.

7. Substituting the constant generated into the trial functions, order sixth, tenth and twelfth of all kinds were obtained.

METHODOLOGY

Illustration 1: Consider Lane-Emden equation [2, 3] given by:

$$y''(x)+\frac{\alpha}{x}y'(x)+g(x,y)=h(x) \tag{17}$$

With boundary coundition:

$$y(0)=\alpha_0, y'(0)=\alpha_1 \tag{18}$$

when g(x, y) = $4(2e^{y}+e^{y/2})$, $h(x)=0,\ \alpha=2\ and\ \ y(0)=0,\ y'(0)=0$. This case has the exact solution $y(x)=-2In(1+x^2)$.

For first kind:

$$\begin{aligned} y_6(x)=&\,0.35399603607503-1.6001963994848x-0.24020230747739(2x-1)^2\\ &+0.11747338348942(2x-1)^3-0.010572799203801(2x-1)^4\\ &-0.010496962067605(2x-1)^5+0.0037554920279834(2x-1)^6 \end{aligned} \tag{19}$$

$$\begin{aligned} y_{10}(x)=&-0.23999960752379(2x-1)^2+0.11733441705348(2x-1)^3\\ &-0.011200556859500(2x-1)^4-0.010513097356161(2x-1)^5\\ &+0.0049977860487975(2x-1)^6-0.00014778371593175(2x-1)^7\\ &-0.00069078912052328(2x-1)^8+0.00017604118610235(2x-1)^9\\ &+0.000032991009094783(2x-1)^{10}-1.5999975562181x+0.35370\\ &975361341 \end{aligned} \tag{20}$$

$$\begin{aligned} y_{12}(x)=&-0.24000007212040(2x-1)^2+0.11733345110647(2x-1)^3\\ &-0.011200038511439(2x-1)^4-0.010496124401621(2x-1)^5\\ &+0.0049917645110663(2x-1)^6-0.00021082041205366(2x-1)^7\\ &0.00067422301240988(2x-1)^8+0.00026833088992989(2x-1)^9\\ &+0.000011401616629409(2x-1)^{10}-0.000041712905963776(2x-1)^{11}\\ &+0.000010780201219949(2x-1)^{12}-1.6000004824184x+0.3537135\\ &1159209 \end{aligned} \tag{21}$$

For the second kind:

$$\begin{aligned} y_6(x)=&\,0.35399603607505-1.6001963994647x-0.24020230747738\\ &(2x-1)^2+0.11747338348944(2x-1)^3-0.010572799203831(2x-1)^4\\ &-0.010496962067615(2x-1)^5+0.0037554920279974(2x-1)^6 \end{aligned} \tag{22}$$

$$
\begin{aligned}
y_{10}(x) = & -0.23999960752380(2x-1)^2 + 0.11733441705352(2x-1)^3 \\
& -0.011200556859430(2x-1)^4 - 0.010513097356339(2x-1)^5 \\
& +0.0049977860485079(2x-1)^6 - 0.00014778371569247(2x-1)^7 \\
& -0.00069078912010119(2x-1)^8 + 0.00017604118601168(2x-1)^9 \\
& +0.000032991008913847(2x-1)^{10} - 1.5999975562182x + 0.35370 \\
& 975361341
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
y_{12}(x) = & -0.23999960752380(2x-1)^2 + 0.11733441705352(2x-1)^3 \\
& -0.011200556859430(2x-1)^4 - 0.010513097356339(2x-1)^5 \\
& +0.0049977860485079(2x-1)^6 - 0.00014778371569247(2x-1)^7 \\
& -0.00069078912010119(2x-1)^8 + 0.00021082041068695(2x-1)^9 \\
& +0.000011401617397815(2x-1)^{10} - 0.000041712905197654(2x-1)^{11} \\
& +0.000010780200765722(2x-1)^{12} - 1.6000004824183x + 0.35371 \\
& 351159200
\end{aligned}
\tag{24}
$$

For the third kind:

$$
\begin{aligned}
y_6(x) = & \, 0.3599603607503 - 1.6001963994646x \\
& -0.2402023747739(2x-1)^2 + 0.11747338348944(2x-1)^3 \\
& -0.01057279920383(2x-1)^4 - 0.010496962067614(2x-1)^5 \\
& +0.0037554920279966(2x-1)^6
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
y_{10}(x) = & -0.23999960752377(2x-1)^2 + 0.11733441705353(2x-1)^3 \\
& -0.011200556859431(2x-1)^4 - 0.010513097356340(2x-1)^5 \\
& +0.0049977860483673(2x-1)^6 - 0.00014778371570206(2x-1)^7 \\
& -0.00069078911983714(2x-1)^8 + 0.00017604118601596(2x-1)^9 \\
& +0.000032991008780289(2x-1)^{10} - 1.5999975562181x + 0.35370 \\
& 975361343
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
y_{12}(x) = & -0.24000007212039(2x-1)^2 + 0.11733345110653(2x-1)^3 \\
& -0.011200038511387(2x-1)^4 - 0.010496124402074(2x-1)^5 \\
& +0.0049917645109852(2x-1)^6 - 0.00021082041078560(2x-1)^7 \\
& -0.00067422301263984(2x-1)^8 + 0.00026833088831822(2x-1)^9 \\
& +0.000011401617344406(2x-1)^{10} - 0.000041712905262749(2x-1)^{11} \\
& +0.000010780200801184(2x-1)^{12} - 1.6000004824182x + 0.3537135 \\
& 1159196
\end{aligned}
\tag{27}
$$

For the fourth kind:

$$y_6(x) = -0.50000000092200 + 1.7500000012136x + 0.62499999885840 \\ (2x-1)^2 + 0.12499999812640(2x-1)^3 + 2.2848000008\times10^{-9}(2x-1)^4 \\ + 1.05599999968\times10^{-9}(2x-1)^5 - 1.0880000064\times10^{-9}(2x-1)^6 \tag{28}$$

$$y_{10}(x) = 0.6249999966(2x-1)^2 + 0.1249999984(2x-1)^3 \\ + 8.86831499\times10^{-9}(2x-1)^4 + 1.396220502\times10^{-9}(2x-1)^5 \\ + 1.457567240\times10^{-9}(2x-1)^6 - 2.302240847\times10^{-8}(2x-1)^7 \\ - 5.376183321\times10^{-9}(2x-1)^8 + 8.32201936\times10^{-9}(2x-1)^9 \\ + 2.926112941\times10^{-9}(2x*1)^{10} + 1.75x - 0.4999999979 \tag{29}$$

$$y_{12}(x) = 0.62499999999985(2x-1)^2 + 0.12500000000076(2x-1)^3 \\ + 7.275810831251\times10^{-13}(2x-1)^4 - 5.1195986401701\times10^{-12}(2x-1)^5 \\ - 7.407045207743\times10^{-13}(2x-1)^6 + 1.4408802371551\times10^{-11}(2x-1)^7 \\ - 2.5187981339135\times10^{-12}(2x-1)^8 - 1.8519381886035\times10^{-11}(2x-1)^9 \\ + 7.524697435197\times10^{-12}(2x-1)^{10} + 8.0948750032916\times10^{-12}(2x-1)^{11} \\ - 4.4455282407371\times10^{-12}(2x-1)^{12} + 1.75x - 0.50000000000078 \tag{30}$$

Illustration 2: Consider the nonlinear differential equation [4, 5] given by:

$$f^{iv}(\eta) + S\left[-\eta f'''(\eta) - 3f''(\eta) - \beta f'(\eta)f''(\eta) + f(\eta)f'''(\eta)\right] = 0 \\ f(0) = 0, f''(0) = 0, f(1) = 1, f'(1) = 0 \\ \beta = \sum\nolimits_{1}^{0} \begin{matrix} axsymmentric \\ two-\dim entional \end{matrix} \tag{31}$$

For the first Kind

N = 12   S = -1.5 $\beta = 0.5$

$$f(x) = -0.257667(2x-1)^2 - 0.0478851(2x-1)^3 + 0.0264578(2x-1)^4 + 0.00319419 \\ (2x-1)^5 - 0.00109736(2x-1)^6 - 0.000175565(2x-1)^7 + 0.00000666920(2x-1)^8 \\ + 0.00000809104(2x-1)^9 + 0.00000184842(2x-1)^{10} - 1.61560\times10^{-7}(2x-1)^{11} \\ - 1.21439\times10^{-7}(2x-1)^{12} + 1.08972x + 0.187440 \tag{32}$$

N = 12   S = -0.5 $\beta = 0.5$

$$f(x) = -0.206447(2x-1)^2 - 0.0593188(2x-1)^3 + 0.00691164(2x-1)^4 + 0.00116743 (2x-1)^5 - 0.000111542(2x-1)^6 - 0.0000184724(2x-1)^7 - 6.42768\times 10^{-7}(2x-1)^8 + 1.22825\times 10^{-7}(2x-1)^9 + 8.84238\times 10^{-8}(2x-1)^{10} + 6.58732\times 10^{-9}(2x-1)^{11} - 7.67396\times 10^{-9}(2x-1)^{12} + 1.11634x + 0.141479 \quad (33)$$

For the second Kind

N = 12 S = -1.5 $\beta = 0.5$

$$f(x) = -0.257667(2x-1)^2 - 0.0478851(2x-1)^3 + 0.0264580(2x-1)^4 + 0.00319416 (2x-1)^5 - 0.00109729(2x-1)^6 - 0.000175534(2x-1)^7 + 0.00000657376(2x-1)^8 + 0.00000807203(2x-1)^9 + 0.00000191358(2x-1)^{10} - 1.57119\times 10^{-7}(2x-1)^{11} - 1.37649\times 10^{-7}(2x-1)^{12} + 1.08972x + 0.187438 \quad (34)$$

N = 12 S = -0.5 $\beta = 0.5$

$$f(x) = -0.206447(2x-1)^2 - 0.0593188(2x-1)^3 + 0.00691165(2x-1)^4 + 0.00116745 (2x-1)^5 - 0.000111583(2x-1)^6 - 0.0000184997(2x-1)^7 - 5.7825\times 10^{-7}(2x-1)^8 + 1.43286\times 10^{-7}(2x-1)^9 + 4.39178\times 10^{-8}(2x-1)^{10} + 1.44827\times 10^{-9}(2x-1)^{11} + 3.32744\times 10^{-9}(2x-1)^{12} + 1.11634x + 0.141477 \quad (35)$$

For the third Kind

N = 12 S = -1.5 $\beta = 0.5$

$$f(x) = -0.257667(2x-1)^2 - 0.0478852(2x-1)^3 + 0.02464578(2x-1)^4 + 0.00319422 (2x-1)^5 - 0.00109743(2x-1)^6 - 0.000175617(2x-1)^7 + 0.00000679115(2x-1)^8 + 0.00000813150(2x-1)^9 + 0.00000176694(2x-1)^{10} - 1.71940\times 10^{-7}(2x-1)^{11} - 1.01771\times 10^{-7}(2x-1)^{12} + 1.08972x + 0.187441 \quad (36)$$

N = 12 S = -0.5 $\beta = 0.5$

$$f(x) = -0.206447(2x-1)^2 - 0.0593188(2x-1)^3 + 0.00691164(2x-1)^4 + 0.00116744 (2x-1)^5 - 0.000111569(2x-1)^6 - 0.00001848490(2x-1)^7 - 6.02153\times 10^{-7}(2x-1)^8 + 1.32495\times 10^{-7}(2x-1)^9 + 6.11451\times 10^{-8}(2x-1)^{10} + 4.10249\times 10^{-9}(2x-1)^{11} - 1.00610\times 10^{-9}(2x-1)^{12} + 1.11634x + 0.141479 \quad (37)$$

For the fourth Kind

N = 12 S = -1.5 $\beta = 0.5$

$$f(x) = -0.257667(2x-1)^2 - 0.0478851(2x-1)^3 + 0.02464578(2x-1)^4 + 0.00319419 (2x-1)^5 - 0.00109736(2x-1)^6 - 0.000175565(2x-1)^7 + 0.00000666920(2x-1)^8 + 0.00000809104(2x-1)^9 + 0.00000184842(2x-1)^{10} - 1.61560\times10^{-7}(2x-1)^{11} - 1.21439\times10^{-7}(2x-1)^{12} + 1.08972x + 0.187440 \tag{38}$$

N = 12 S = -0.5 $\beta = 0.5$

$$f(x) = -0.206447(2x-1)^2 - 0.0593188(2x-1)^3 + 0.00691164(2x-1)^4 + 0.00116744 (2x-1)^5 - 0.000111569(2x-1)^6 - 0.0000184849(2x-1)^7 - 6.02153\times10^{-7}(2x-1)^8 + 1.32495\times10^{-7}(2x-1)^9 + 6.11451\times10^{-8}(2x-1)^{10} + 4.10249\times10^{-9}(2x-1)^{11} - 1.00610\times10^{-9}(2x-1)^{12} + 1.11634x + 0.141479 \tag{39}$$

Illustration 3: Consider the nonlinear differential equation [6] given by:

$$(1+\Gamma)f^{iv} - S(\vartheta f^{iii} + 3f^{ii} + f^{i}f^{ii} - ff^{iii}) - \Gamma\delta(2f^{ii}(f^{iii})^2 + (f^{ii})^2 f^{iv}) - Ha^2 f^{ii} = 0 \tag{40}$$

For the first Kind

N = 12, Ha=9, S= 3.5, δ =0.07, Γ =9

$$f(x) = -0.11236055(2x-1)^2 - 0.064265443(2x-1)^3 - 0.023347541(2x-1)^4 - 0.0082362469 (2x-1)^5 - 0.0022178304(2x-1)^6 - 0.00070354203(2x-1)^7 - 0.00022485288(2x-1)^8 - 0.000082965798(2x-1)^9 - 0.000029139171(2x-1)^{10} - 0.000010175088(2x-1)^{11} - 0.0000024119056(2x-1)^{12} + 1.1465967x + 0.064883950 \tag{41}$$

For the second Kind

N = 12, Ha=9, S= 3.5, δ =0.07, Γ =9

$$f(x) = -0.115236044(2x-1)^2 - 0.064417811(2x-1)^3 - 0.022542920(2x-1)^4 - 0.0078530086 (2x-1)^5 - 0.0020862438(2x-1)^6 - 0.00066489812(2x-1)^7 - 0.00021091270(2x-1)^8 - 0.000075656336(2x-1)^9 - 0.000025559086(2x-1)^{10} - 0.0000087604056(2x-1)^{11} - 0.0000020984830(2x-1)^{12} + 1.1460403x + 0.067054032 \tag{42}$$

For the third Kind

N = 12, Ha=9, S= 3.5, δ =0.07, Γ =9

$$f(x) = -0.11520644(2x-1)^2 - 0.064417813(2x-1)^3 - 0.022542919(2x-1)^4 - 0.0078530082 (2x-1)^5 - 0.0020862439(2x-1)^6 - 0.00066489840(2x-1)^7 - 0.00021091250(2x-1)^8 - 0.000075655914(2x-1)^9 - 0.000025559511(2x-1)^{10} - 0.0000087606006(2x-1)^{11} - 0.0000020982644(2x-1)^{12} + 1.1460403x + 0.067054055 \quad (43)$$

For the fourth Kind

N = 12, Ha=9, S= 3.5, δ =0.07, Γ =9

$$f(x) = -0.11236055(2x-1)^2 - 0.064265443(2x-1)^3 - 0.023347541(2x-1)^4 - 0.0082362469 (2x-1)^5 - 0.0022178304(2x-1)^6 - 0.00070354203(2x-1)^7 - 0.00022485288(2x-1)^8 - 0.000082965798(2x-1)^9 - 0.000029139171(2x-1)^{10} - 0.000010175088(2x-1)^{11} - 0.0000024119056(2x-1)^{12} + 1.1465967x + 0.064883950 \quad (44)$$

RESULTS AND DISCUSSION

Tables 1, 2, 3 and 4, Show the error obtained for Lane Emden equation in case 1 when shifted Chebyshev polynomials of first, second third and fourth kind were used as trial functions respectively. The maximum error found for sixth, tenth, and twelfth order shifted Chebyshev polynomial (N=6, N=10 and N=12) are: 0.00005080449890, 0.000002067027410, 0.000000127255011 and minimum error obtained with sixth, tenth, and twelfth order shifted Chebyshev polynomial (N=6, N=10 and N=12) are: 0.000009471438142, 0.000001975527394, 0.000000377110366 for all the four kinds of the shifted Chebyshev polynomial. In all the four kinds of shifted Chebyshev polynomials, it was observed that increase in the order of the polynomial reduced the error to a bearable minimal. This implies that in order to improve the accuracy of the result of the higher order Chebyshev polynomial is required. The results obtained were compared with that of [2] and were found to be in agreement.

Table 1: Error values of the first kind of the Lane-Emden

SN	N = 6	N = 10	N = 12
0.0	0	0	0
0.1	0.00009471438142	0.000001975527394	0.000000377110366
0.2	0.00017232619270	0.000002381340378	0.000000456268092
0.3	0.00019405070530	0.000000232314771	0.000000448867960
0.4	0.00019629005931	0.000002155079770	0.000000417375840

SN	N = 6	N = 10	N = 12
0.5	0.00018493897110	0.000001921867270	0.000000373011270
0.6	0.00015925550590	0.000001658376480	0.000000322725710
0.7	0.00013395365560	0.000000138938016	0.000000270814340
0.8	0.00012155800620	0.000001112256770	0.000000220197220
0.9	0.00010407929141	0.000001141360411	0.000000172682401
1.0	0.00005080449890	0.000002067027410	0.000000127255011

Table 2: Error values of the second kind of the Lane-Emden

SN	N = 6	N = 10	N = 12
0.0	0	0	0
0.1	0.00009471438142	0.000001975527394	0.000000377110366
0.2	0.00017232619270	0.000002381340378	0.000000456268092
0.3	0.00019405070530	0.000000232314771	0.000000448867960
0.4	0.00019629005931	0.000002155079770	0.000000417375840
0.5	0.00018493897110	0.000001921867270	0.000000373011270
0.6	0.00015925550590	0.000001658376480	0.000000322725710
0.7	0.00013395365560	0.000000138938016	0.000000270814340
0.8	0.00012155800620	0.000001112256770	0.000000220197220
0.9	0.00010407929141	0.000001141360411	0.000000172682401
1.0	0.00005080449890	0.000002067027410	0.000000127255011

Table 3: Error values of the third kind of the Lane-Emden

SN	N = 6	N = 10	N = 12
0.0	0	0	0
0.1	0.00009471438142	0.000001975527364	0.000000377110366
0.2	0.00017232619270	0.000002381340338	0.000000456268092
0.3	0.00019405070540	0.000000232314765	0.000000448867960
0.4	0.00019629005931	0.000002155079713	0.000000417375840
0.5	0.00018493897120	0.000001921867270	0.000000373011270
0.6	0.00015925550590	0.000001658376390	0.000000322725710
0.7	0.00013395365570	0.000000138938016	0.000000270814340
0.8	0.00012155800630	0.000001112256770	0.000000220197220
0.9	0.00010407929151	0.000001141360411	0.000000172682401
1.0	0.00005080449890	0.000002067026320	0.000000127255011

Table 4: Error values of the fourth kind of the Lane-Emden

SN	N = 6	N = 10	N = 12
0.0	0	0	0
0.1	0.00009471438142	0.000001975527364	0.000000377110366
0.2	0.00017232619270	0.000002381340338	0.000000456268092
0.3	0.00019405070540	0.000000232314765	0.000000448867960
0.4	0.00019629005931	0.000002155079713	0.000000417375840
0.5	0.00018493897120	0.000001921867270	0.000000373011270
0.6	0.00015925550590	0.000001658376390	0.000000322725710
0.7	0.00013395365570	0.000000138938016	0.000000270814340
0.8	0.00012155800630	0.000001112256770	0.000000220197220
0.9	0.00010407929151	0.000001141360411	0.000000172682401
1.0	0.00005080449890	0.000002067026320	0.000000127255011

Tables 5, 6, 7 and 8, Showed the results of Illustration 2 when the four different kinds of the Chebyshev polynomials were used at only one orders, which is order twelfth. Due to non-existence of the exact solution to this problem, the solution obtained was compared with that of Runge Kutta method of order four [4] and weighted residual method [5].There is an agreement between the result obtained with Chebyshev polynomial and that of the referenced solution. However, the higher the order of the polynomial considered, the closer the result to the referenced solution. And it was compared with all the four kinds of the shifted Chebyshev polynomials.

Tables 9, 10, 11 and 12, showed the results of Illustration 3 when the four different kinds of the Chebyshev polynomials were used at only one order, which is order twelfth. Due to non-existence of the exact solution to this problem, comparison has been made for functions between Runge-Kutta Fehlberg , AGM solution[7] and shifted Chebyshev polynomial of all four kinds, respectively. Obtained results show that shifted Chebyshev polynomial of all four kinds error is very little compare to Runge-Kutta Fehlberg *and* AGM so that shifted Chebyshev polynomial is a high accuracy method in nonlinear di*ff*erential equations solution, especially in the *fi*eld of heat and mass transfer (Eyring-Powell*)*. There is an agreement between the result obtained with Chebyshev polynomial and that of the referenced solution. However, the higher the order of the polynomial considered, the closer the result to the referenced solution. And it was compared with all the four kinds of the shifted Chebyshev polynomials.

Table 5: shows the result of shifted chebyshev polynomial of the first kind of Squeezing flow

S	X	Runge-kutta	WRM	N=12
-1.5	0.2	0.319526	0.319727	0.326101
	0.4	0.603830	0.603994	0.613446
	0.6	0.822876	0.822736	0.830628
	0.8	0.956801	0.956886	0.959734
-0.5	0.2	0.302582	0.302582	0.304039
	0.4	0.578082	0.578082	0.580242
	0.6	0.800780	0.800780	0.802562
	0.8	0.947702	0.947702	0.948398

Table 6: shows the result of shifted chebyshev polynomial of the second kind Squeezing flow

S	x	Runge-kutta	WRM	N=12
-1.5	0.2	0.319526	0.319727	0.326099
	0.4	0.603830	0.603994	0.613444
	0.6	0.822876	0.822736	0.830623
	0.8	0.956801	0.956886	0.959732
-0.5	0.2	0.302582	0.302582	0.304037
	0.4	0.578082	0.578082	0.580240
	0.6	0.800780	0.800780	0.802560
	0.8	0.947702	0.947702	0.948396

Table 7: shows the result of shifted chebyshev polynomial of the third kind Squeezing flow

S	x	Runge-kutta	WRM	N=12
-1.5	0.2	0.319526	0.319727	0.326102
	0.4	0.603830	0.603994	0.613447
	0.6	0.822876	0.822736	0.830626
	0.8	0.956801	0.956886	0.959735

S	x	Runge-kutta	WRM	N=12
-0.5	0.2	0.302582	0.302582	0.304040
	0.4	0.578082	0.578082	0.580243
	0.6	0.800780	0.800780	0.802563
	0.8	0.947702	0.947702	0.948398

Table 8: shows the result of shifted chebyshev polynomial of the fourth kind Squeezing flow

S	x	Runge-kutta	WRM	N=12
-1.5	0.2	0.319526	0.319727	0.326101
	0.4	0.603830	0.603994	0.613446
	0.6	0.822876	0.822736	0.830625
	0.8	0.956801	0.956886	0.959734
-0.5	0.2	0.302582	0.302582	0.304039
	0.4	0.578082	0.578082	0.580242
	0.6	0.800780	0.800780	0.802562
	0.8	0.947702	0.947702	0.948398

Table 9: shows the result of shifted chebyshev polynomial of the first kind Eyring-Powell

θ	Runge-Kutta Fehlberg	AGM	N=12
0.0	0.00000000	0.00000000	0.00000000
0.1	0.13570349	0.13586036	0.13381274
0.2	0.26993918	0.27024429	0.26630948
0.3	0.40111409	0.40154944	0.39605158
0.4	0.52737667	0.52791339	0.52134356
0.5	0.64646613	0.64706222	0.64007419
0.6	0.75553451	0.75613223	0.74951590
0.7	0.85092861	0.85145329	0.84605917
0.8	0.92791395	0.92828058	0.92484506
0.9	0.98031233	0.98045878	0.97923394
1.0	1.00000000	1.00000000	1.00000000

Table 10: shows the result of shifted chebyshev polynomial of the second kind Eyring-Powell

ϑ	Runge-Kutta Fehlberg	AGM	N=12
0.0	0.00000000	0.00000000	0.00000000
0.1	0.13570349	0.13586036	0.13381274
0.2	0.26993918	0.27024429	0.26630948
0.3	0.40111409	0.40154944	0.39605158
0.4	0.52737667	0.52791339	0.52134356
0.5	0.64646613	0.64706222	0.64007419
0.6	0.75553451	0.75613223	0.74951590
0.7	0.85092861	0.85145329	0.84605917
0.8	0.92791395	0.92828058	0.92484506
0.9	0.98031233	0.98045878	0.97923394
1.0	1.00000000	1.00000000	1.00000000

Table 11: shows the result of shifted chebyshev polynomial of the third kind Eyring-Powell

ϑ	Runge-Kutta Fehlberg	AGM	N=12
0.0	0.00000000	0.00000000	0.00000000
0.1	0.13570349	0.13586036	0.13381274
0.2	0.26993918	0.27024429	0.26630948
0.3	0.40111409	0.40154944	0.39605158
0.4	0.52737667	0.52791339	0.52134356
0.5	0.64646613	0.64706222	0.64007419
0.6	0.75553451	0.75613223	0.74951590
0.7	0.85092861	0.85145329	0.84605917
0.8	0.92791395	0.92828058	0.92484506
0.9	0.98031233	0.98045878	0.97923394
1.0	1.00000000	1.00000000	1.00000000

Table 12: shows the result of shifted chebyshev polynomial of the fourth kind Eyring-Powell

ϑ	Runge-Kutta Fehlberg	AGM	N=12
0.0	0.00000000	0.00000000	0.00000000
0.1	0.13570349	0.13586036	0.13381274
0.2	0.26993918	0.27024429	0.26630948
0.3	0.40111409	0.40154944	0.39605158
0.4	0.52737667	0.52791339	0.52134356
0.5	0.64646613	0.64706222	0.64007419
0.6	0.75553451	0.75613223	0.74951590
0.7	0.85092861	0.85145329	0.84605917
0.8	0.92791395	0.92828058	0.92484506
0.9	0.98031233	0.98045878	0.97923394
1.0	1.00000000	1.00000000	1.00000000

Figures 1 and 2 show the residual generated while solving Lane Emden equation and squeezing flow of shifted Chebyshev polynomials using twelfth order of first kind. it was observed that from the two figures that the residual has been minimised to as closed to zero as possible which justifies the accuracy of the shifted Chebyshev collocation procedure.

The Shifted Chebyshev polynomial Method is used to obtain the Lane-Emden of the sixth, tenth and twelfth order value problems**,** Runge-Kutta, Weighted Residual Method,and Eyring-Powell of the twelfth order value problems. The residual was minimized to acquire different solution in the literature the result was compared to check the efficiency and effectiveness of four kinds of Shifted Chebyshev Polynomials to show the effect of Lane-Emden, squeezing flow of the negatives, and Eyring-Powell. Also there is improvement in the results obtained to the previous results in the journal and there are slight different in all kinds of Chebyshev polynomials.

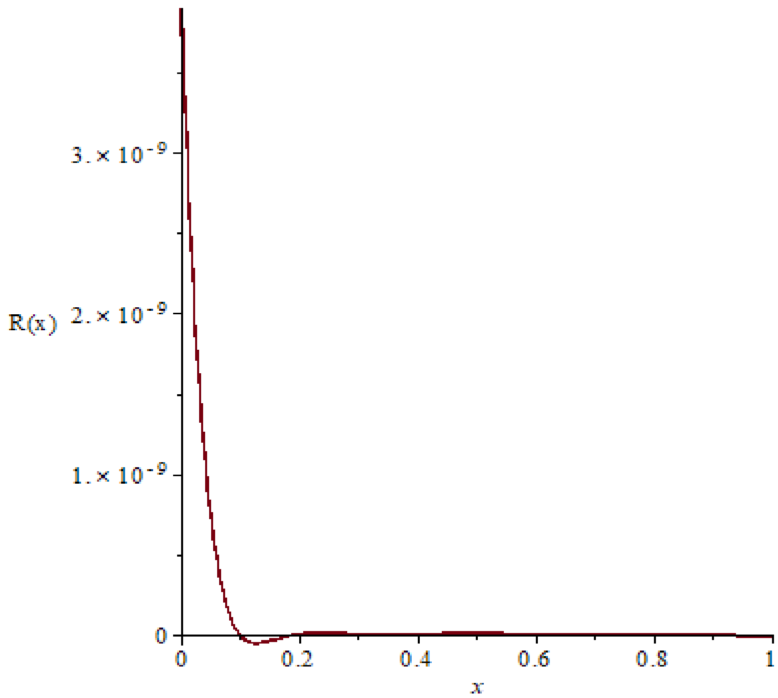

Figure 1: Residual of a Lane Emden equation of the first kind of order twelfth

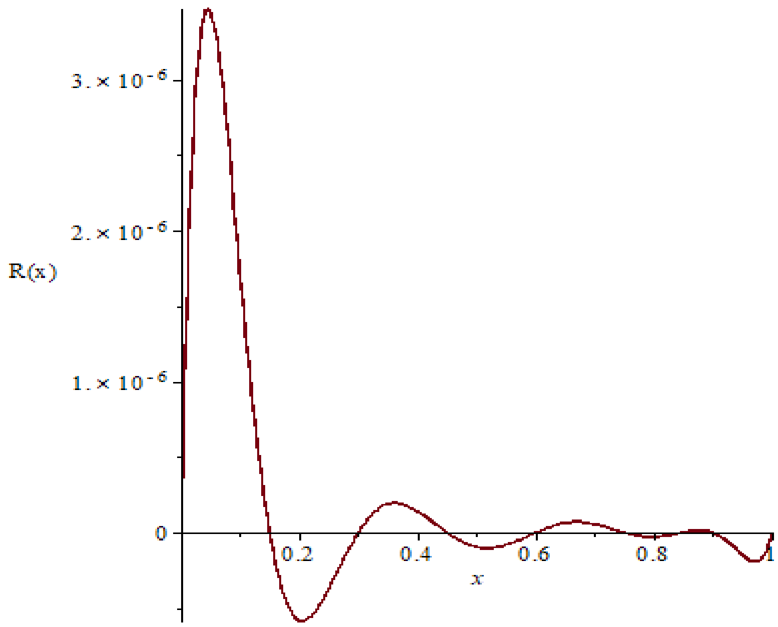

Figure 2: Residual of squeezing flow equation of the first kind of order twelfth

CONCLUSION

In this paper Shifted Chebyshev Polynomial of all kinds were used efficiently to solve the sixth, tenth and twelfth orders boundary value fluid problem. Using sixth, tenth and twelfth orders of Shifted Chebyshev Polynomial of all kinds give a better result and using them all is also good.

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