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Analytical Properties of the Uniform Exponential Distribution

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Abstract

This study introduces a new probability distribution, the Uniform–Exponential Distribution, constructed by combining the Uniform and Exponential distributions to capture phenomena characterized by both constant and exponentially decaying behavior. The distribution is developed within the T-X family framework, and its fundamental properties are derived, including the probability density function (PDF), cumulative distribution function (CDF), hazard function, reverse hazard function, and moment-generating function (MGF). Analytical expressions for key moments—mean, variance, skewness, and kurtosis—are presented to describe the distribution's shape and behavior. Parameter estimation is conducted using maximum likelihood estimation (MLE), ensuring statistical rigor in model fitting. Theoretical examples are provided to illustrate the distribution's practical relevance. Potential applications are identified in reliability analysis, survival modeling, and environmental science, underscoring the distribution's flexibility and utility in modeling diverse real-world processes.

Keywords: Uniform–Exponential Distribution; T-X Family; Hazard Function; Moment-Generating Function; Maximum Likelihood Estimation; Reliability Analysis

Introduction

Probability distributions form the cornerstone of statistical modeling, providing the formal structure for representing and analyzing random variability. Among these, the Uniform and Exponential distributions are widely recognized for their conceptual simplicity and broad practical relevance. The Uniform distribution assumes equal probability across all values within a defined interval, making it suitable for modeling phenomena where outcomes are equally likely. In contrast, the Exponential distribution is characterized by its memoryless property is extensively used in survival analysis, reliability engineering, and queuing theory to describe waiting times and decay patterns. While each distribution is valuable on its own, empirical data often exhibit behaviors too complex to be captured by a single distribution. This has led to the advancement of mixture distributions, which combine two or more probability models to improve representational accuracy (Lee *et al.*, 2021).

This study introduces the Uniform–Exponential Distribution (UE), a novel hybrid model that integrates the defining features of both the Uniform and Exponential distributions. This formulation is particularly relevant for cases where an initial phase of constant behavior is followed by an exponential decline. For example, in reliability engineering, certain systems may exhibit a constant failure rate during early operation before experiencing an accelerated failure risk. In environmental science, pollutant levels may remain stable within a certain range before decreasing exponentially. Such hybrid patterns lie outside the scope of standard single-component models, making the Uniform–Exponential Distribution a valuable extension (Smith *et al.*, 2020).

The construction of the distribution employs the T–X family framework (Alzaatreh *et al.*, 2013), a general methodology for generating new probability laws through the transformation of existing ones. This ensures mathematical rigor while allowing flexibility in modeling diverse data behaviors. The Uniform–Exponential form is parameterized by **λ**, **a** and **b**, representing the exponential decay rate and uniform interval bounds, respectively. Previous studies demonstrate that similar hybrid structures can capture complex statistical patterns in both engineering and environmental applications (Zhang *et al.*, 2022).

The applicability of the Uniform–Exponential model is illustrated across multiple disciplines. In reliability engineering, it models systems with uniform and exponential phases of hazard. In survival analysis, it represents time-to-event data characterized by a constant baseline hazard that transitions to an accelerated hazard phase. In environmental monitoring, it is applied to pollutant concentration datasets where values remain bounded for a period before decaying exponentially. These examples are consistent with broader research trends emphasizing flexible probabilistic frameworks (Wang *et al.*, 2021; Ayenigba *et al.,* 2025a; Ayenigba *et al.*, 2025b; Ayenigba *et al.*, 2025c).

The analytical contributions of this work are presented last for completeness. Specifically, closed-form expressions are derived for the probability density function (PDF), cumulative distribution function (CDF), hazard and reverse hazard functions, and the moment-generating function (MGF). These results provide insight into the distribution's central tendency, variability, and tail behaviour. Parameter estimation is addressed using maximum likelihood estimation (MLE) and the distribution's applicability is demonstrated through theoretical examples. Potential applications in reliability analysis, survival modeling, and environmental science are examined.

Methodology

Using the T-X family framework is a powerful method for deriving new probability distributions by combining existing ones. The T-X family was introduced by Alzaatreh et al. (2013).

The T-X family is defined by the cumulative distribution function (CDF):

$$F(x) = \int_{0}^{W(G(x))} r(t)dt, \qquad (1)$$

Where:

$G(x)$ is the CDF of the baseline distribution(e.g., Uniform distribution),

$W(.)$ is a non-decreasing function that maps [0,1] to the support of T,

$R(t)$ is the probability density function (PDF) of the transformer distribution T

The CDF of the baseline distribution is given by:

$$G(x) = \frac{x-a}{b-a}, \quad a \le x \le b. \qquad (2)$$

The PDF of the transformer (T) is given by:

$$r(t) = \lambda e^{-\lambda t}, \;\; t \ge 0. \qquad (3)$$

Let the transformation function $W(.)$ be defined as:

$$W(u) = -\ln(1-u), \qquad (4)$$

which maps $[0,1]\ to\ [0,\infty)$.

This ensures that the support of the new distribution aligns with the Exponential distribution.

Substitute equation (2) into (4) and together with (3) into (1), we have:

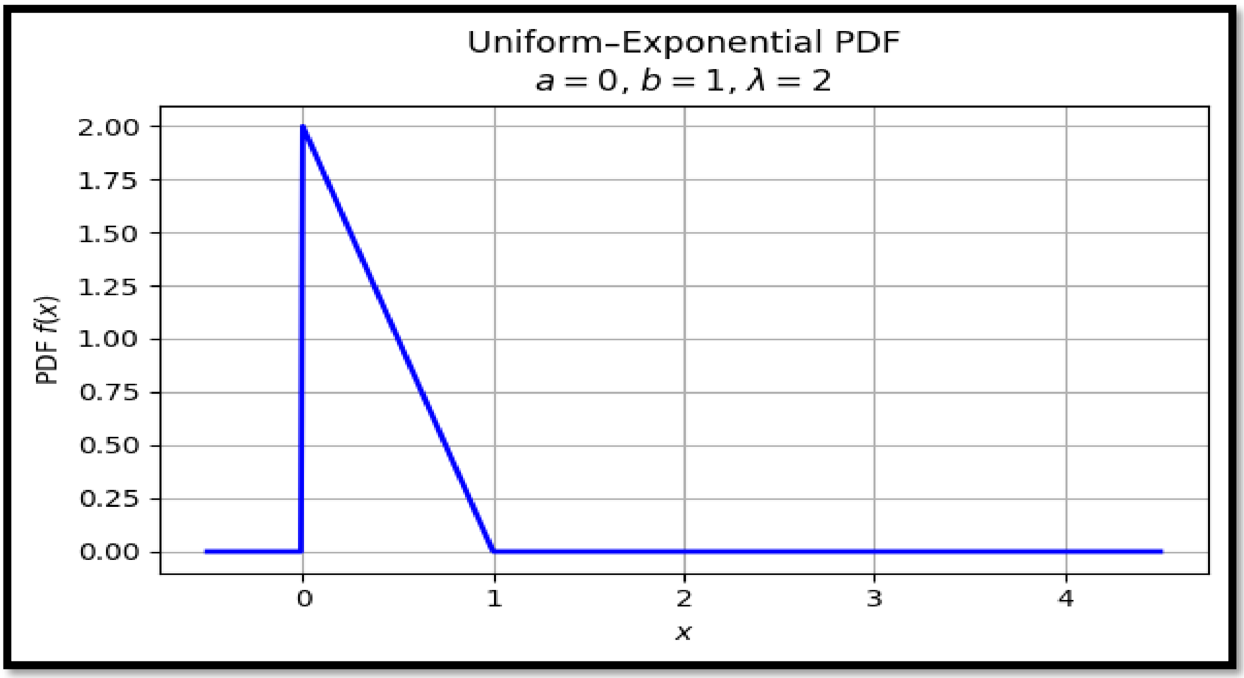
$$F(x) = \int_{0}^{-ln\left(\frac{b-x}{b-a}\right)} \lambda e^{-\lambda t} dt \qquad (5)$$

$$F(x) = 1 - exp\left(\lambda\left(-ln\left(\frac{b-x}{b-a}\right)\right)\right)$$

$$F(x) = 1 - \left(\frac{b-x}{b-a}\right)^{\lambda}, a \leq x \leq b. \qquad (6)$$

Differentiating equation (6), we have the PDF:

$$f(x) = \frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1}, \quad a \leq x \leq b. \qquad (7)$$



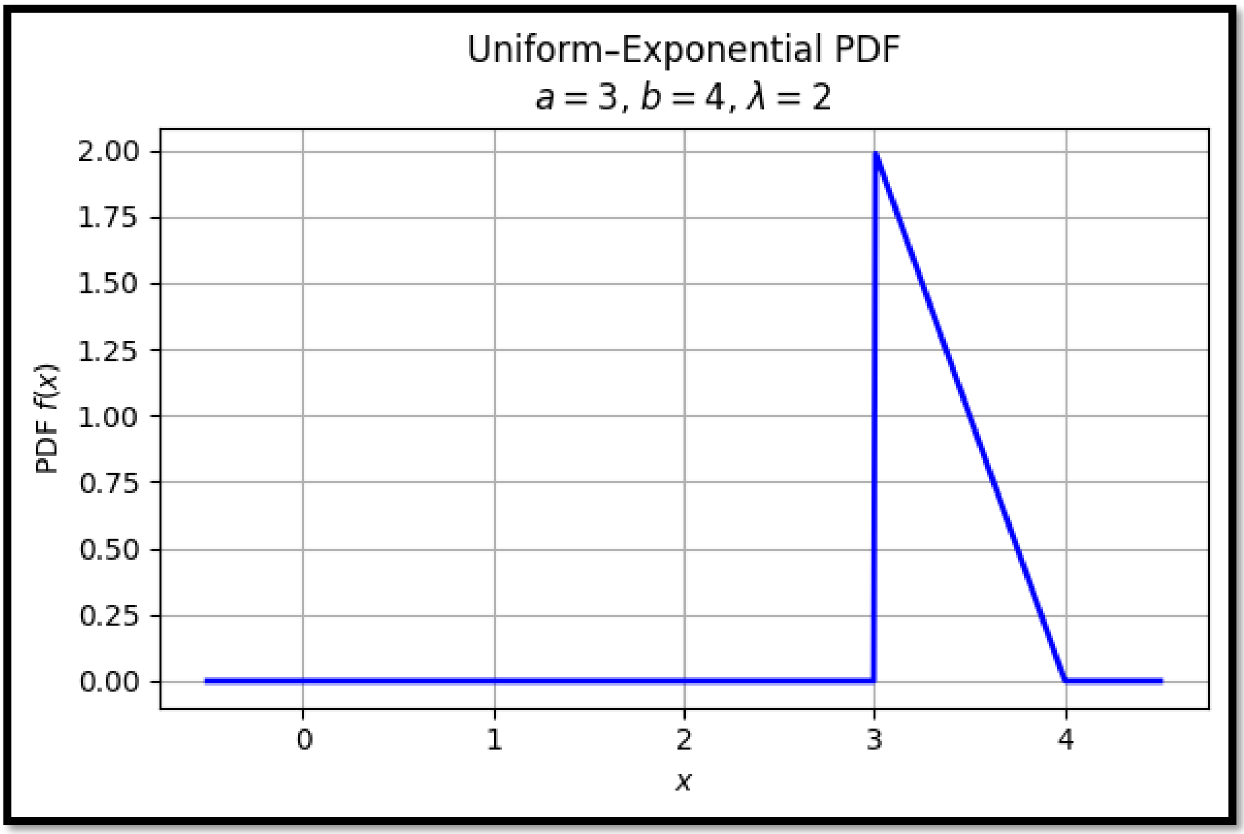


Figure 1: *Plot of PDF of UE*

Validity of the Distribution

We need to show that:

$$\int_a^b f(x)dx = 1$$

$$\int_a^b \frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1} dx$$

Let $u = \frac{b-x}{b-a}$, $when\ x = a, u = 1\ and\ when\ x = b, u = 0.\ Then\ dx = -(b-a)du$

$$\int_a^b \frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1} dx = \int_1^0 \frac{\lambda}{b-a} u^{\lambda-1}(-(b-a)du)$$

$$= \lambda \int_{0}^{1} u^{\lambda-1} du$$

$$= \lambda \left[\frac{u^{\lambda}}{\lambda}\right]_{0}^{1}$$

$$= \lambda \left(\frac{1}{\lambda} - 0\right)$$

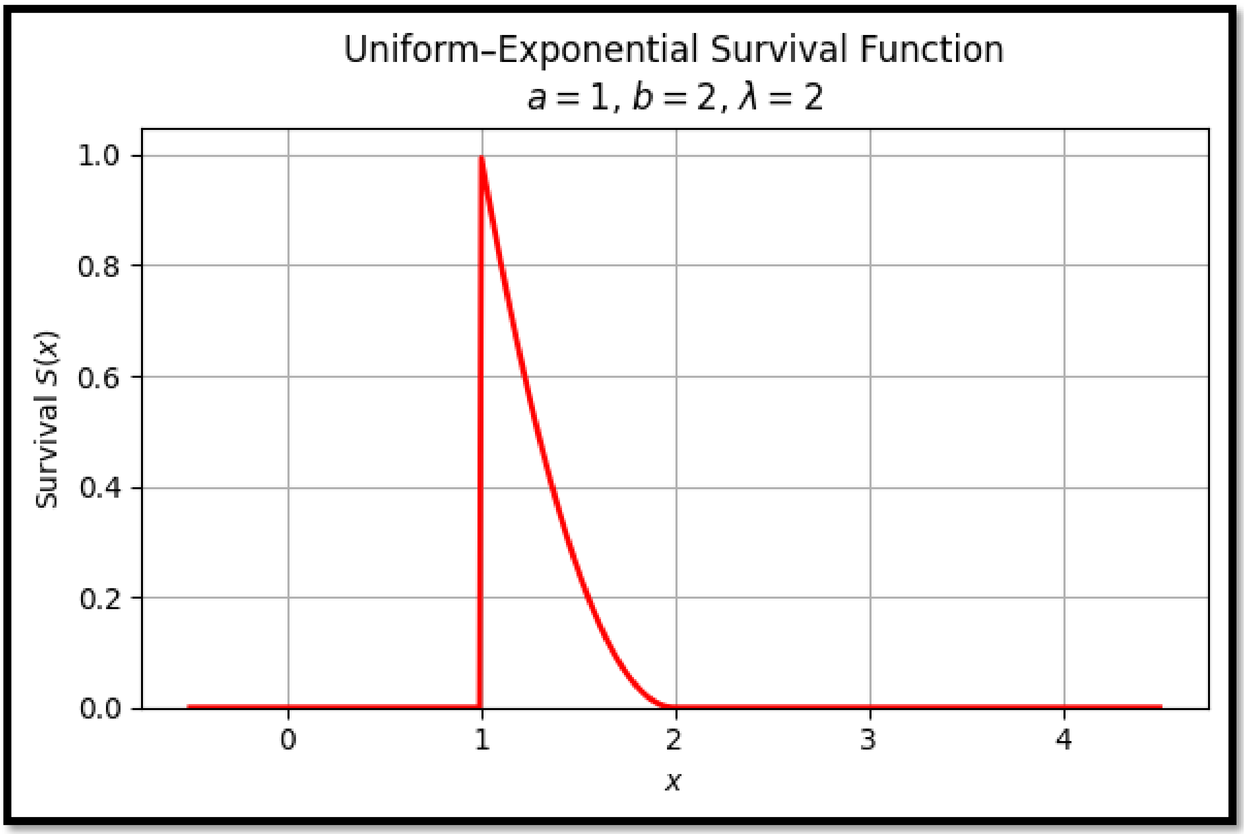
$$= 1$$

Survival Function

The survival function S(x) is given by

$$S(x) = 1 - F(x)$$

$$S(x) = \left(\frac{b-x}{b-a}\right)^{\lambda}, a \le x \le b. \qquad (8)$$



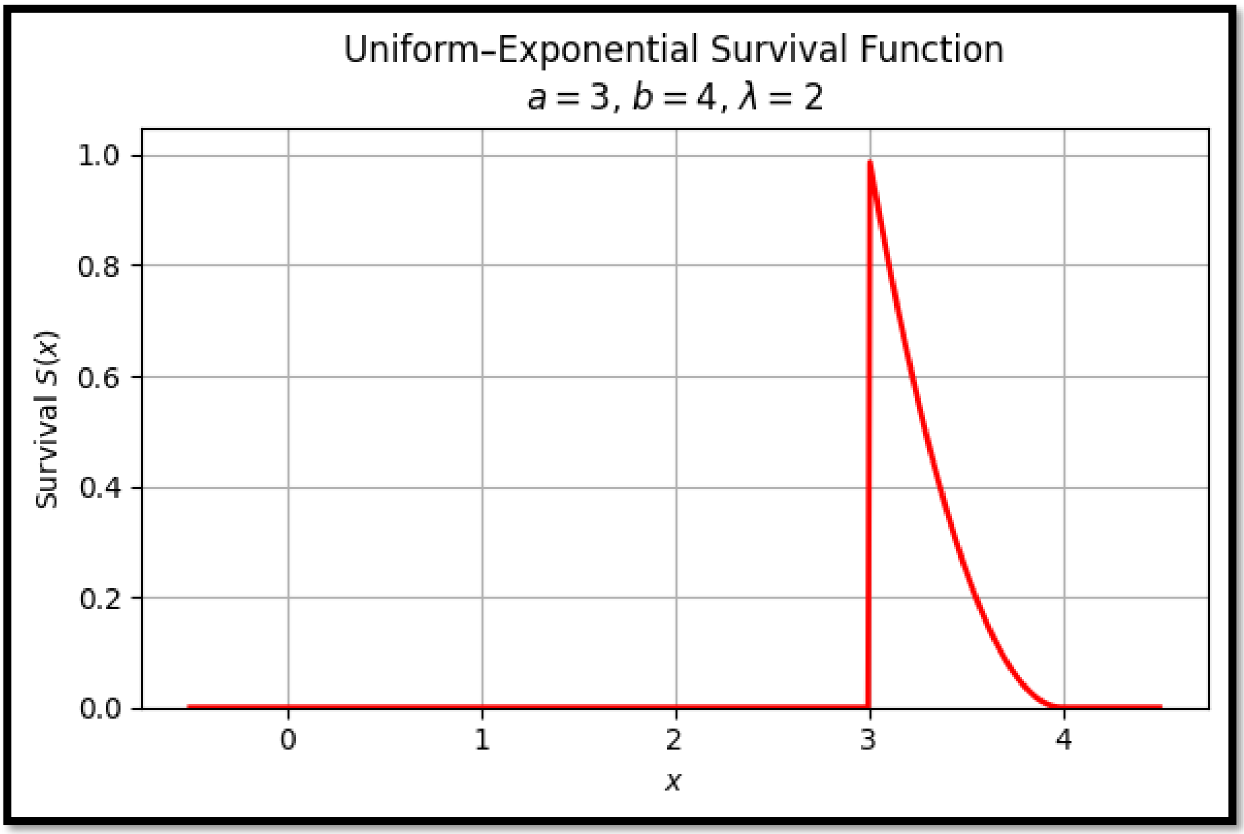


Figure2: *Graphs of Survival function*

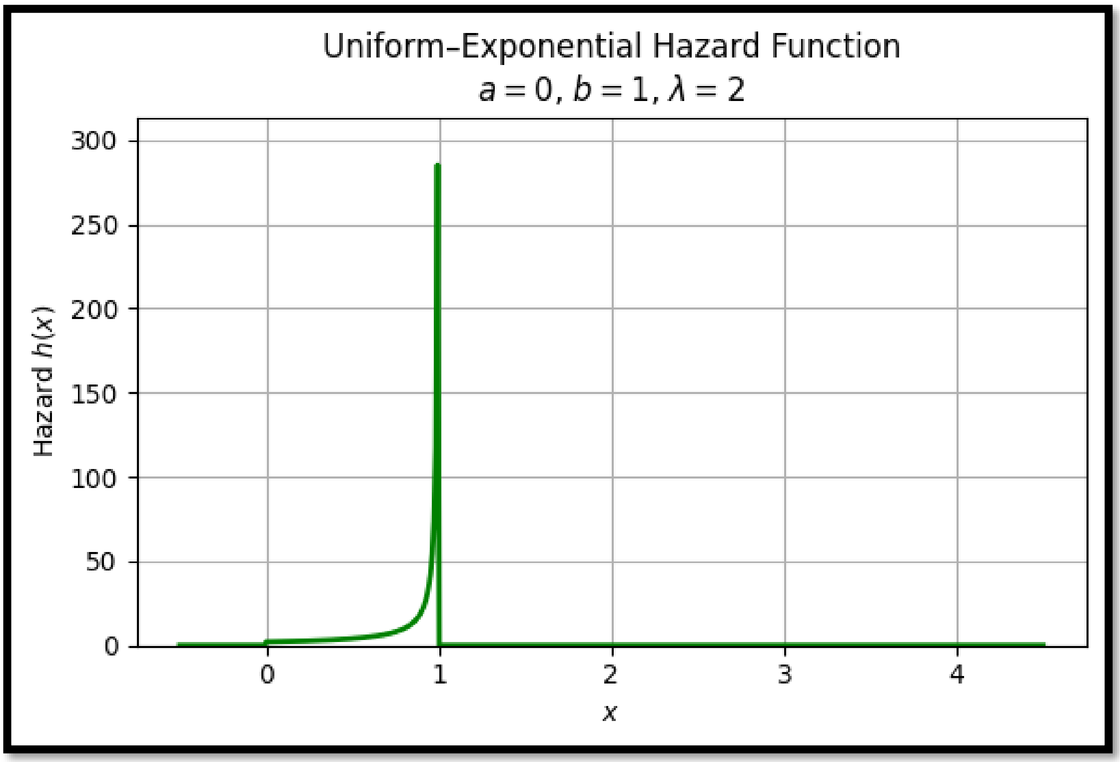
Hazard Function

The hazard function h(x) is defined as the ratio of the PDF to the survival function.

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1}}{\left(\frac{b-x}{b-a}\right)^{\lambda}} \qquad (9)$$

$$h(x) = \frac{\lambda}{b-x}, \qquad a \le x \le b. \qquad (10)$$



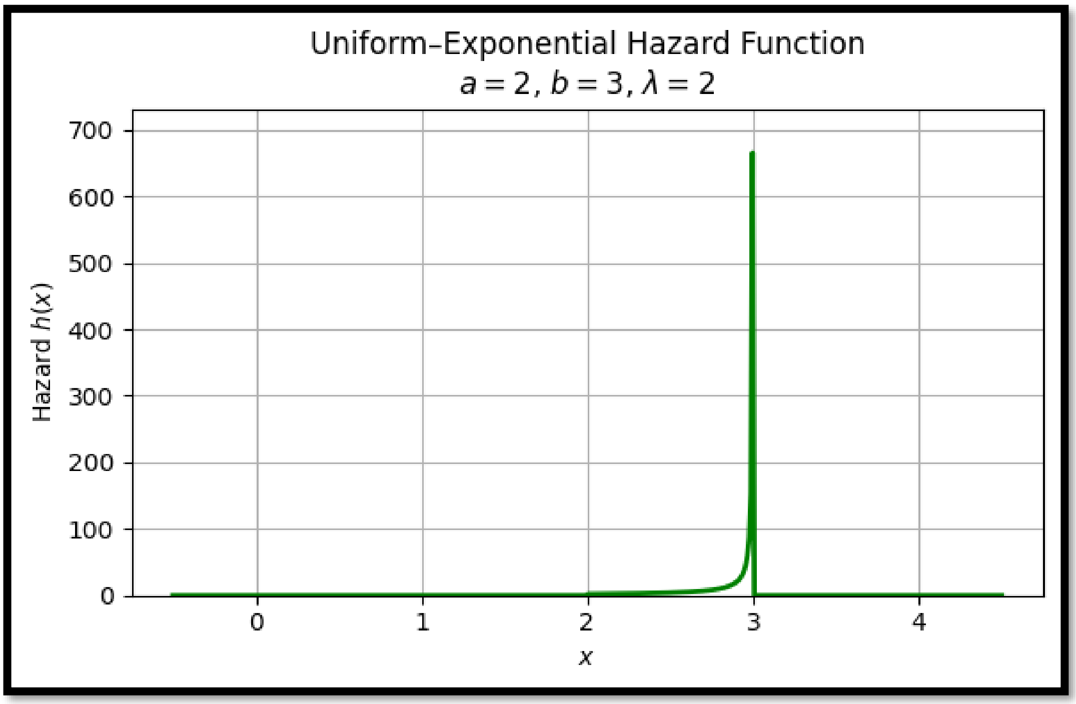


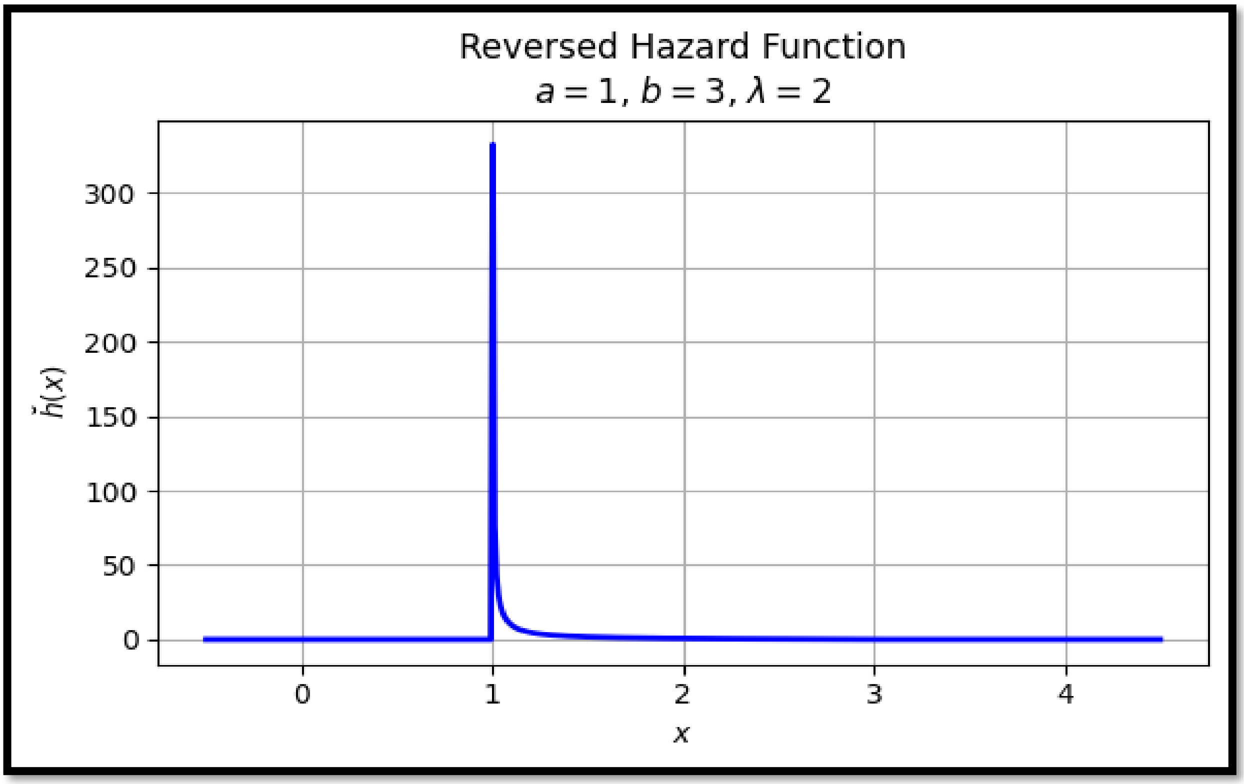
Figure 3: ***Graphs of Hazard function***

Reverse Hazard Function

The reverse hazard function $\breve{h}(x)$ is defined as the ratio of the PDF to the CDF

$$\breve{h}(x) = \frac{f(x)}{F(x)}$$

$$\breve{h}(x) = \frac{\frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1}}{1-\left(\frac{b-x}{b-a}\right)^{\lambda}}, a \leq x \leq b. \qquad (11)$$



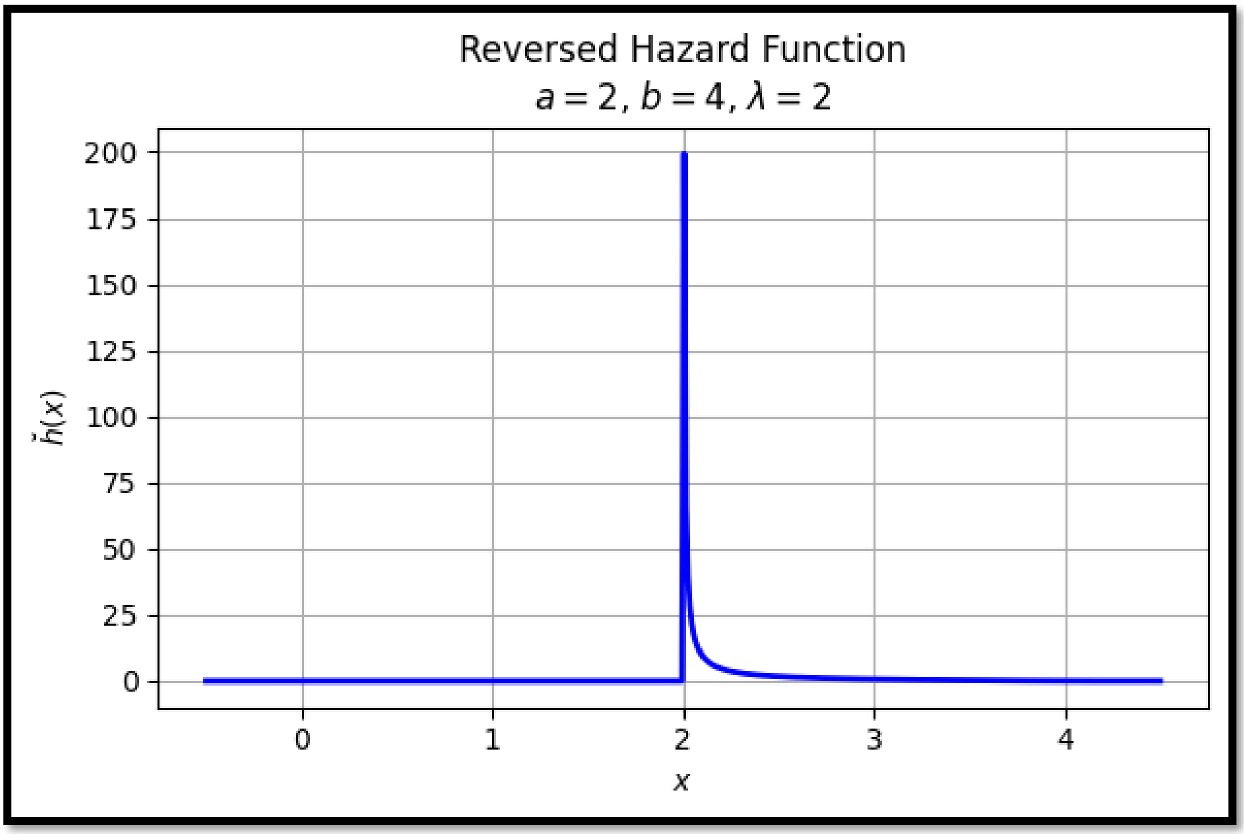


Figure 4: *Graphs of Reverse Hazard Function*

Parameter Estimation

Let Let $X_1,\ X_2, X_3, \dots, X_n$ denote a random sample of size *n*, where each X is an independent and identically distributed (i.i.d.) random variable following PDF defined in Equation (5). The likelihood function $L(\lambda, a, b)$ is presented below:

$$L(\lambda, a, b;\ x_1,\ x_2, x_3, \dots, x_n) = \prod_{i=1}^{n} f(x_i; \lambda, a, b)$$

$$L(\lambda, a, b) = \prod_{i=1}^{n}\left(\frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1}\right) \qquad (12)$$

$$L(\lambda, a, b) = \left(\frac{\lambda}{b-a}\right)^{n} \prod_{i=1}^{n}\left(\frac{b-x}{b-a}\right)^{\lambda-1} \qquad (13)$$

Let $l(\lambda, a, b) = lnL(\lambda, a, b)$, Then,

$$l(\lambda, a, b) = nln\lambda - nln(b-a) + (\lambda - 1)\left[\sum_{i=1}^{n} \ln(b - x_i) - nln(b-a)\right] \qquad (14)$$

$$\frac{\partial y}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} ln\left(\frac{b - x_i}{b - a}\right) = 0 \qquad (15)$$

$$\lambda = \frac{n}{-\sum_{i=1}^{n} ln\left(\frac{b-x_i}{b-a}\right)} \qquad (16)$$

$$\frac{\partial y}{\partial a} = \frac{n}{b-a} - (\lambda - 1)\sum_{i=1}^{n} \frac{1}{b - x_i} . \frac{1}{b-a} = 0 \qquad (17)$$

$$\frac{\partial y}{\partial b} = \frac{-n}{b-a} + (\lambda - 1)\sum_{i=1}^{n} \frac{1}{b - x_i} . \frac{1}{b-a} = 0 \qquad (18)$$

The MLE equations are nonlinear and typically require numerical methods Newton-Raphson to solve with the constrains: $a \leq \min(x_1,\ x_2, x_3, \dots, x_n)\ and\ b \geq \max\ (x_1,\ x_2, x_3, \dots, x_n)$

Moment Generating Function

The MGF of a random variable X is defined as:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{a}^{b} e^{tx} \frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda - 1} dx \qquad (19)$$

Let $u = \frac{b-x}{b-a},\ when\ x = a, u = 1\ and\ when\ x = b, u = 0.\ Then\ dx = -(b-a)du$

$$M_X(t) \ = \int_{0}^{1} e^{t(b-u(b-a))} \frac{\lambda}{b-a} u^{\lambda - 1}(-(b-a)du) \ (20)$$

$$M_X(t) \ = \lambda e^{tb} \int_{0}^{1} e^{-tu(b-a))} u^{\lambda - 1} du \qquad (21)$$

Note: $\int_0^1 e^{-tu(b-a))}u^{\lambda-1}du$ *is* the **lower incomplete gamma function.**

Verification of the MGF

One of the key properties of MGF is $M_X(0) = 1$

$$M_X(0) = \lambda e^0 \int_0^1 e^0 u^{\lambda-1}du = \lambda.\frac{1}{\lambda} = 1$$

The mean and Variance of the distribution

$$E(X) = \int_a^b xf(x)dx$$

$$E(X) = \int_a^b x\frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1} dx \qquad (22)$$

Let $u = \frac{b-x}{b-a}$, *when* $x = a, u = 1$ *and when* $x = b, u = 0$. *Then* $x = b - u(b-a)$.

$dx = -(b-a)du$

$$E(X) = \int_1^0 (b - u(b-a).\frac{\lambda}{b-a}.u^{\lambda-1}(-(b-a)du) \qquad (23)$$

$$E(X) = \lambda b\int_1^0 u^{\lambda-1}du - \lambda(b-a)\int_1^0 u^{\lambda}du \qquad (24)$$

$$E(X) = \lambda b.\frac{1}{\lambda} - \lambda(b-a)\frac{1}{\lambda+1}$$

$$E(X) = b - \frac{\lambda(b-a)}{\lambda+1} \qquad (25)$$

The variance of X is given by:

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_a^b x^2\frac{\lambda}{b-a}\left(\frac{b-x}{b-a}\right)^{\lambda-1} dx \qquad (26)$$

This is simplified to

$$E[X^2] = b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2} \qquad (27)$$

Then, the variance becomes:

$$Var(X) = \left(b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2}\right) - \left(b - \frac{\lambda(b-a)}{\lambda+1}\right)^2 \qquad (28)$$

The Moment

The first 4 moment of the distribution are:

$$E(X) = b - \frac{\lambda(b-a)}{\lambda+1} \qquad (29)$$

$$E[X^2] = b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2} \qquad (30)$$

$$E[X^3] = b^3 - \frac{3b^2(b-a)\lambda}{\lambda+1} + \frac{3b(b-a)^2\lambda}{\lambda+2} - \frac{(b-a)^3\lambda}{\lambda+3} \quad (31)$$

$$E[X^4] = b^4 - \frac{4b^3(b-a)\lambda}{\lambda+1} + \frac{6b^2(b-a)^2\lambda}{\lambda+2} - \frac{4b(b-a)^3\lambda}{\lambda+3} + \frac{(b-a)^4\lambda}{\lambda+4} \qquad (32)$$

The Coefficient of Skewness

The coefficient of the skewness of the distribution is given by:

$$\gamma_1 = \frac{E[(X-\mu)^2]}{\sigma^3}$$

$$\square_1$$

$$= \frac{\left(b^3 - \frac{3b^2(b-a)\lambda}{\lambda+1} + \frac{3b(b-a)^2\lambda}{\lambda+2} - \frac{(b-a)^3\lambda}{\lambda+3}\right) - 3\mu\left(b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2}\right) + 3\mu^2\left(b - \frac{\lambda(b-a)}{\lambda+1}\right) - \mu^3}{\left(\left(b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2}\right) - \left(b - \frac{\lambda(b-a)}{\lambda+1}\right)^2\right)^{3/2}} \qquad \text{(}$$

The Coefficient of Kurtosis

The Coefficient of Kurtosis of the distribution is given by:

$$\gamma_2 = \frac{E[(X-\mu)^4]}{\sigma^4}$$

$$\gamma_2 = \frac{\left(= b^4 - \frac{4b^3(b-a)\lambda}{\lambda+1} + \frac{6b^2(b-a)^2\lambda}{\lambda+2} - \frac{4b(b-a)^3\lambda}{\lambda+3} + \frac{(b-a)^4\lambda}{\lambda+4}\right) - 4\mu\left(b^3 - \frac{3b^2(b-a)\lambda}{\lambda+1} + \frac{3b(b-a)^2\lambda}{\lambda+2} - \frac{(b-a)^3\lambda}{\lambda+3}\right) + 6\mu^2\left(b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2}\right) - 4\mu^3\left(b - \frac{\lambda(b-a)}{\lambda+1}\right) + \mu^4}{\left(\left(b^2 - \frac{2b(b-a)\lambda}{\lambda+1} + \frac{(b-a)^2\lambda}{\lambda+2}\right) - \left(b - \frac{\lambda(b-a)}{\lambda+1}\right)^2\right)^2}$$

(34)

Practical Scenario of the Distribution

1. Environmental Science

Consider the concentration of a pollutant in a water body over time. The concentration remains constant (uniform) for a certain period due to steady pollution input, after which it decays exponentially due to natural purification processes.

Let X represents the pollutant concentration over time. Suppose $a = 0$ (start time), $b = 10$ (end time), and $\lambda = 2$ (decay rate).

The PDF of X is given as:

$$f(x) = \frac{2}{10}\left(\frac{10-x}{10}\right)^{2-1} = \frac{1}{5}\left(\frac{10-x}{10}\right), \quad 0 \le x \le 10 .$$

The CDF is thus:

$$F(x) = 1 - \left(\frac{10-x}{10}\right)^2, 0 \le x \le 10.$$

The probability of concentration exceeding a Threshold, for instance $P(X > 5)$ is calculated as:

$$P(X > 5) = 1 - F(5) = 0.75$$

This implies that there is 75% chance that the pollutant concentration exceeds 5 units.

The Expected Concentration and variance of the concentration is calculated as follow:

$$E(X) = 10 - \frac{2(10-0)}{2+1} = 3.33$$

Thus, the expected pollutant concentration is approximately 3.33 units

$$Var(X) = \left(10^2 - \frac{2(10)(10-0)(2)}{2+1} + \frac{(10-0)^2(2)}{2+2}\right) - \left(10 - \frac{(2)(10-0)}{2+1}\right)^2 = 5.56$$

The variance of the pollutant concentration is approximately 5.56 units².

2. Survival Analysis

Consider a clinical study where patients are monitored for the time until a specific event (e.g., recovery or relapse). The event risk is constant (uniform) for an initial period, after which it increases exponentially.

Let X represent the time until the event, suppose a = 0 (start time), b =12 (end time in months) and $\lambda = 1.5$ (decay rate).

The PDF of X becomes:

$$f(x) = \frac{1.5}{12}\left(\frac{12-x}{12}\right)^{1.5-1} = 0.125\left(\frac{12-x}{12}\right)^{0.5}, \quad 0 \le x \le 12$$

And the corresponding CDF is given by:

$$F(x) = 1 - \left(\frac{12-x}{12}\right)^{0.5}, 0 \le x \le 12.$$

Thus, we can calculate the median time until the event by solving for x in:

$$F(x) = 0.5$$

$$1 - \left(\frac{12-x}{12}\right)^{1.5} = 0.5$$

$$X = 4.44$$

The median time until the event is approximately 4.44 months

Conclusion

This study introduces the Uniform-Exponential Distribution (UED), a novel hybrid model within the T-X family framework, designed to address complex datasets exhibiting both constant and exponentially decaying patterns. By rigorously deriving its probability density function (PDF), cumulative distribution function (CDF), hazard functions, and moment-generating function (MGF), the research establishes the UED's theoretical robustness, while maximum likelihood estimation (MLE) ensures practical reliability in parameter estimation. The distribution's versatility is demonstrated through applications in reliability analysis—modeling systems with mixed failure rates—and environmental science, where it captures pollutant concentrations with initial stability followed by exponential decay. Analysis of moments (mean, variance, skewness, kurtosis) provides critical insights into its shape and tail behavior, enhancing its utility in risk assessment and survival analysis. By unifying the Uniform and Exponential distributions, the UED fills a critical gap in statistical modeling, offering a flexible tool for modern data challenges. This work not only advances theoretical understanding of hybrid distributions but also provides actionable solutions for interdisciplinary applications, paving the way for future extensions into fields like finance and epidemiology, where multifactorial data patterns demand innovative modeling approaches.

Declaration of Conflict of Interest

The authors declare no conflict of interest.

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